DOUBLE E-POINTS IN RATIONAL GAMES

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Abstract

In this short work we intoduce a new concept of solution for rational games. We allow two different structure functions in the numerator payoff function and in the denominator payoff function. A general existence theorem is obtained. Even though the hyphotesis of the theorem is strong, we present an example of an extensive game with complete information having double E-Points as a large variety.

Key words: Equilibrium, E-Points, complete information.

Introduction

The concept equilibrium point is due to Nash, and in the year 2010 we have its 60^{th} anniversary. Related to this subjet, the author of this paper has introduced the concept of E - points, besides studying rational games. Rational games have appeared in literature many years ago. For example the reader may take a look at the papers by Cohen, Marchi, Oviedo [5] [6]. Some other related work can be consulted in Marchi [4].

Until today, there are few applications of E - Points. However the potentiality is large, especially in extensive games.

In this short paper I consider a generalization of rational games and at the same time rational games having two different structures in the payoff function in the numerator and the denominator.

I establish an existence theorem having sufficient conditions for the existence of Double E - Points in rational games. As two applications we present two extensive games with large sets of Double - E - points, without studying the mentioned sufficient condition. It would be interesting to study the behavioural strategies.

Double - E - Points

Consider two n-person non-cooperative games

$$\Gamma^1 = \left\{\Sigma_i^1; A_i^1; i \in N\right\} \quad \text{and} \quad \Gamma^2 = \left\{\Sigma_i^1; B_i^2; i \in M\right\}$$

with the same pure strategy set Σ^1. From here we define the rational game

$$\Gamma_r = \left\{\Sigma_i^1; A_i \,/ B_i;\ i \in M\right\}$$

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and the "mixed extension"

$$\widetilde{\Gamma}_r = \left\{ \widetilde{\Sigma}_i^1; E_i(x) / F_i(x); \; i \in N \right\}$$

where E_i and F_i are the expected value of A_i and B_i respectively.

On the other hand we know that the E - point of a mixed extension game, say for example $\widetilde{\Gamma}^1$, is a point $\overline{x}$ so that

$$E_i\left(\overline{x}_{d(i)}, \overline{x}_{N-d(i)}\right) \geq E_i\left(x_{d(i)}, \overline{x}_{N-d(i)}\right) \forall_i \forall_{x_{d(i)}} \in X_{d(i)}$$

so that $d(i) \subset N$ is the set of players for $i - th$ player and $x_{d(i)} = (x_{j_1}, ..., x_{j_r})$ where $\{j_1, ..., j_r\} = d(i)$ and $X_{d(i)} = \underset{j \in d(i)}{X} \widetilde{\Sigma}_i^1$.

Generally they do not exist. But it is interesting to study of the existence on the example given in the mixed extension of diagonal game. See [4].

Now finally we consider another game with the structure $\overline{d}$, then the non-cooperative game $\widetilde{\Gamma}^2$with structure $\overline{d}(i), i \in N$ has its definition as a satisfying point $\widetilde{x}$

$$F_i\left(\overline{x}_{\overline{d}(i)}, \overline{x}_{N-d(i)}\right) \geq F_i\left(x_{d(i)}, \overline{x}_{N-d(i)}\right) \forall_i \forall_{x_{\overline{d}(i)}} \in X_{\overline{d}(i)}$$

The same comment given for the game $\widetilde{\Gamma}^1$ is applied to the non-cooperative game $\widetilde{\Gamma}^2$.

Further we will consider the final game $\widetilde{\Gamma}_r$ with two given structure functions, one for the numerator $d(i)$ and another one for the denominator function $\overline{d}(i)$. We define $\overline{x}$ to be a double E - point if it satisfies.

$$\frac{E_i\left(\overline{x}_{d(i)}, \overline{x}_{N-d(i)}\right)}{F_i\left(\overline{x}_{\overline{d}(i)}, \overline{x}_{N-d(i)}\right)} \geq \frac{E_i\left(x_{d(i)}, \overline{x}_{N-d(i)}\right)}{F_i\left(x_{\overline{d}(i)}, \overline{x}_{N-\overline{d}(i)}\right)} \quad \forall_i \forall x \in X \tag{1}$$

Here to make it easier we assume that the payoff function F_i is strictly positive.

From (1), since $F_i > 0$ then we have an analogous inequality which is equivalent to

$$E_i(\overline{x}) \; F_i\left(\overline{x}_{\overline{d}(i)}, \overline{x}_{N-\overline{d}(i)}\right) - E_i\left(x_{d(i)}, \overline{x}_{N-d(i)}\right) \; F_i(\overline{x}) \geq 0 \quad \forall_i \forall x$$

Next we have

Theorem 1: *Given the "mixed extension" $\widetilde{\Gamma}$, with $F_i > 0$ if for any $y \in X$ there is an $x \in X$ so that*

$$G_i(x, y) = E_i(y) \, F_i\left(x_{d(i)}, y_{N-d(i)}\right) - F_i(y) \, E_i\left(x_{\overline{d}(i)}, y_{N-d(i)}\right) \geq 0 \quad \forall_i$$

and are concave in $x_{d(i)} \subset X_{d(i)}$ and convex in $x_{a(i)} \subset X_{a(i)}$ there then $\widetilde{\Gamma}_r$ has a double E-point.

Proof: Consider the game $\widetilde{\Gamma}_r$ and the function $G_i(x, y)$

By hypothesis for each $y \in X$ there is a $x \in X$ such that $-G_i(x, y) = 0$. Consider two points $x, z \in X$ then we have

$$- G_i (\lambda x + (1-\lambda) z, y) =$$
$$= E_i (y) F_i \left(\lambda x_{\overline{d}(i)} + (1-\lambda) z_{\overline{d}(i)}, y \; N - \overline{d}(i)\right) - E_i \left(\lambda x_{\overline{d}(i)} + (1-\lambda) z_{\overline{d}(i)}, y \; N - \overline{d}(i)\right) \; F_i (y) \geq 0$$

for all $\lambda \in (0,1)$

$$\geq \left[\lambda E_i (y) F_i \left(x_{d(i)}, y_{N-d(i)}\right) + (1-\lambda) E_i (y) F_i \left(z_{\overline{d}(i)}, y_{N-d(i)}\right)\right] -$$
$$- \left[\lambda E_i \left(x_{\overline{d}(i)}, y_{N-d(i)}\right) F_i (y) + (1-\lambda) E_i z_{\overline{d}(i)}, y\right] = \lambda G_i (x,y) + (1-\lambda) G_i (z,y)$$

which says that for each y the continuous function $G_i (y)$ is concave in x.

Take now the multivalued function defined as follows $\varphi_i (y)$

$$\varphi_i (y) = \left\{\overline{x} \in X : \max_x G_i (x,y) = G_i (\overline{x}, y)\right\}$$

Then the mapping defined by the product $\varphi (y) = \underset{i \in M}{\cap} \varphi_i (y)$ fulfills the Kakutani‘s hypothesis of the fixed point theorem. Then there exists a point $\overline{x}$ so that for each $i \in N : \overline{x} \in \varphi (\overline{x})$ or $G_i (\overline{x}, \overline{x}) \geq G_i (x, \overline{x})$ for each $i \in M$ and each $x \in X$: Therefore

$$-G_i (\overline{x}, \overline{x}) = E_i (\overline{x}) F_i \left(x_{\overline{d}(i)}, \overline{x}_{N-\overline{d}(i)}\right) - F_i (\overline{x}) E_i \left(x_{d(i)}, \overline{x}_{N-d(i)}\right)$$

or

$$\frac{E_i (\overline{x})}{F_i (\overline{x})} \geq \frac{E_i \left(x_{d(i)}, \overline{x}_{N-d(i)}\right)}{F_i \left(x_{\overline{d}(i)}, \overline{x}_{N-\overline{d}(i)}\right)} \forall_i \quad \forall x \in X$$

and then the existence of a Doble E - point is proved.

□

From this theorem of existence we might object that in the hypothesis there are strong mathematical conditions, however, we will present two examples of extensive games where the existence of Double E - Points appear naturally. Moreover, there are a variety of dimensions of them.

<u>*Examples*</u>

The first example that we present is an extensive game with complete information given by the following tree

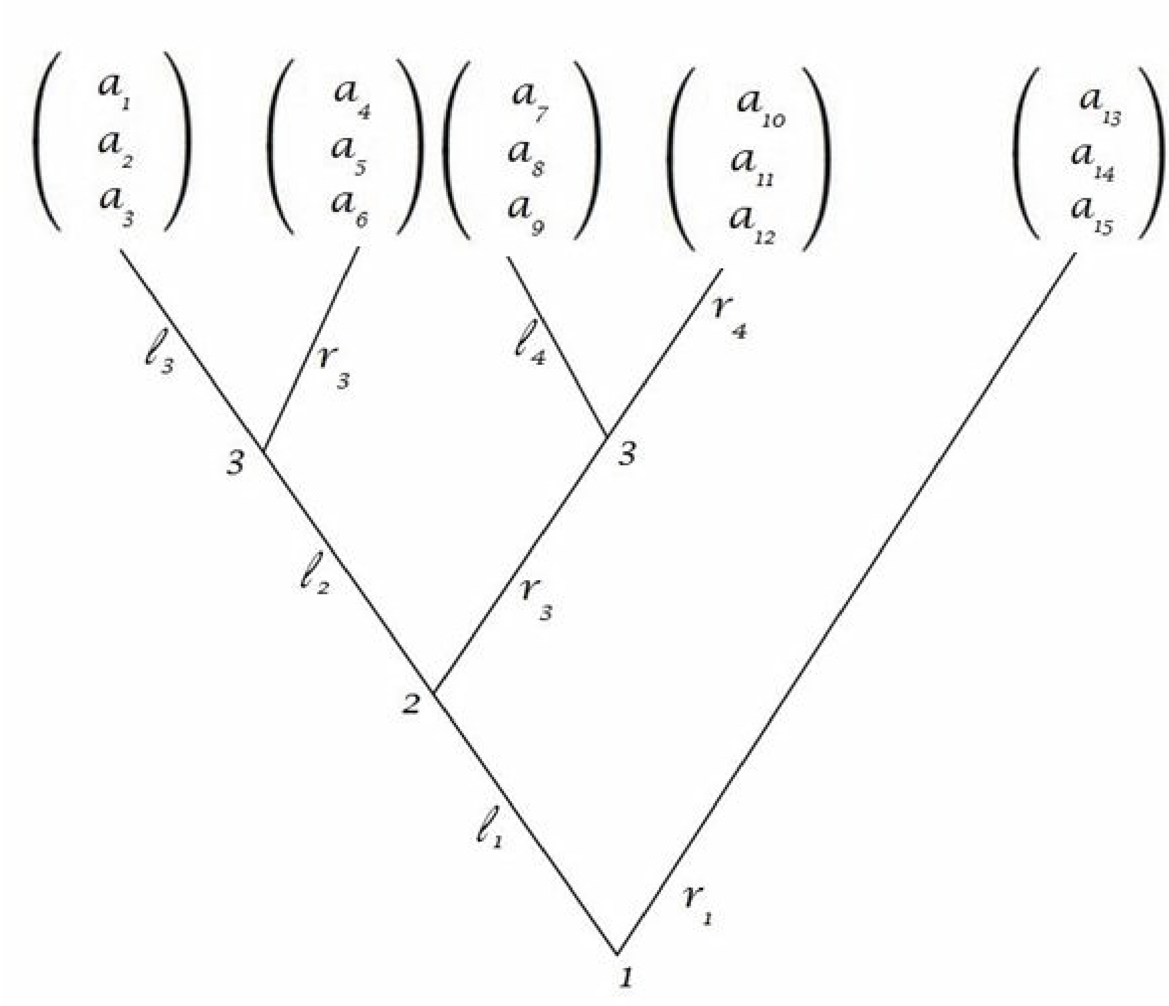


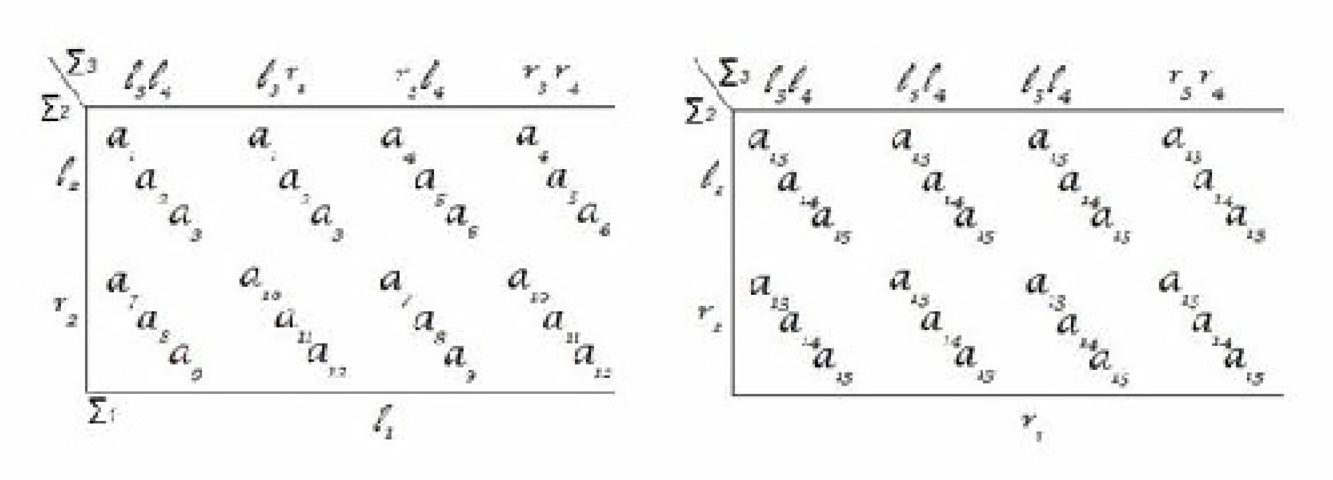
Fig. 1

where a_r with $r \equiv 1$ *mod* 3, $r \equiv 2$ *mod* 3 and $r \equiv 3$ *mod* 3 are the payoff functions of the respective players 1, 2 and 3. The other things are universals.

The corresponding pure strategies are

$$\Sigma_1 = \{l_1, r_1\} \quad \Sigma_2 = \{l_2, r_2\} \text{ and } \Sigma_3 = \{l_3l_4, l_3l_4, r_3l_4, r_3r_4\}$$

The respective payoff functions are



We present an equilibrium point, namely $\{l_1, l_2, l_3, l_4\}$ when

$$a_1 \geq a_{13} \quad a_2 \geq a_8 \quad a_3 \geq a_6$$

and the numerous variety is highly dimensional.
Now if the structure functions are

$$d(1) = \{1, 2\} \quad d(2) = \{2, 3\} \quad d(3) = \{1, 3\}$$

with

$$\begin{array}{ccc} a_1 \geq a_{13} & a_2 \geq a_8 & a_3 \geq a_{15} \\ a_1 \geq a_7 & a_2 \geq a_5 & a_3 \geq a_6 \end{array}$$

There will exist an E - point, namely $\{l_1, l_2, l_3, l_4\}$. Moreover, if we consider the same tree with the payoff functions given by b‘s instead of a‘s, and the structure function written as

$$\overline{d}(1) = \{1, 2\}, \ \overline{d}(2) = \{2\} \quad \overline{d}(3) = \{2, 3\}$$

with

$$b_1 \leq b_{13} \quad b_2 \leq b_8 \quad a_3 \leq b_8 \quad b_3 \leq b_9 \quad b_1 \leq b_7 \quad \text{and} \quad d_3 \leq a_{12}$$

We have a Double E-Point, and again a variety the Double E-Point is of high dimension.

A good mathematical topic would be to see the relation of E-Points and Double E-Points with the corresponding generalization of the correlated equilibrium due to the Nobel Prize R. Aumann[]. By the way, it is interesting to see the corresponding and proper generalization of the correlated in the sense of E-Points. Moreover, it is significant to say that the set of equilibrium correlated is a convex polihedron. The E-Points are not.

Some more interesting results and applications can be given in the line made in Marchi [7].

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