BOUNDARY CONFIGURATIONS SPANNING CONTINUA OF MINIMAL SURFACES

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Dedicated to Herbert Beckert on occasion of his sixty-fifth birthday.

New examples are constructed of one-parameter families of stationary solutions to the Plateau problem with three coaxial circles as boundary, and of non-congruent area-minimizing solutions to certain free and partially-free boundary problems. A similar phenomenon is exhibited for the partition problem.

1. Introduction

It is well known that a closed smooth Jordan curve can bound a large number of minimal surfaces, even of those that are of given topological type. This phenomenon is illustrated by the theorem of Böhme [4], p. 8, who has proved that, for each positive integer N and for each $\varepsilon>0$, there exists a real-analytic Jordan curve in $\mathbb{R}^3$ of total curvature less than $4\pi + \varepsilon$ which bounds at least N minimal surfaces of the type of the disc. On the other hand, Nitsche [18] has shown that a real-analytic Jordan curve in $\mathbb{R}^3$ of total curvature less than 4π bounds exactly one disc-type minimal surface, and Sauvigny [22] proved uniqueness for extreme polygons in $\mathbb{R}^3$ of total curvature less than 4π. The only other known uniqueness theorem for boundaries consisting of a single Jordan curve is due to Radó (cf. [17], pp. 358-359). It states that a Jordan curve will bound exactly one minimal surface if it has a one-to-one (parallel or central) projection onto a planar convex curve. Finally, Meeks and Yau [15], pp. 160-161, have linked uniqueness with the nonexistence of two distinct stable embedded minimal discs.

Böhme's theorem is based on the Böhme-Tromba index theorem [5] which, on the other hand, implies generic finiteness, as has been shown in [5] (generic uniqueness results for area minimizing discs had earlier been found by Morgan and Tromba). The central question for the Plateau problem presently is to decide whether a single smooth (or real-analytic) closed Jordan curve in $\mathbb{R}^3$ can bound only finitely many minimal surfaces (of general type, or of the type of the disc). Tomi [26] has proved that a real analytic curve in $\mathbb{R}^3$ bounds only finitely many minimal surfaces which are absolutely area minimizing among disc-type surfaces and Hardt-Simon [13] have established that each $C^{4,\alpha}$-curve in $\mathbb{R}^3$ can hold only finitely many oriented area minimizing surfaces. Further finiteness results are due to Tomi [27], Nitsche [19], and Beeson [2]. A generalization of Tomi's result to minimal surfaces in Riemannian manifolds has been given by Quien [21].

On the other hand, boundary configurations consisting of several smooth closed curves can certainly bound infinitely many minimal surfaces. Morgan [16] has constructed a rotationally symmetric configuration of four circles holding continua of congruent but nonsymmetric minimal surfaces of arbitrarily high genus. We shall show that the same phenomenon can be found with configurations consisting of three coaxial circles in parallel planes, but not with two circles, by virtue of a result due to Schoen. It is unknown if there are configurations of several smooth curves which bound infinitely many noncongruent minimal surfaces or fixed topological type, or of continua of those.

It is fairly easy to find free boundary problems which have families of congruent minimal surfaces as solutions. We shall, however, exhibit free as well as partially free boundary configurations which have 1-parameter families of noncongruent area minimizing minimal surfaces (of fixed topological type) as solutions. This might be of some interest because of the following recent results:

1. *If a compact analytic* H-*convex body* M *in* $\mathbb{R}^3$ *has the properties that there is a closed Jordan curve in* M *which cannot be contracted in* M , *and, secondly, that the free boundary problem for* M *admits infinitely many minimizing solutions of disc-type, then* M *must be homeomorphic to a solid torus* (Tomi [28]).

2. *If the torus* M *is foliated by a smooth* S^1*-family of plane, disc-type minimal surfaces, being orthogonal to* M, *then all surfaces in the family are congruent* (Tomi [28]).

3. *Let* S *be a compact, embedded, real-analytic surface in* $\mathbb{R}^3$ *, and* Γ *be a homotopically non-trivial Jordan curve in the unbounded component of* $\mathbb{R}^3$-S. *Then there are only finitely many geometrically different surfaces of the type of the disc and with in* $\mathbb{R}^3 - \Gamma$ *noncontractible boundary which minimize area within the configuration* $\langle\Gamma,S\rangle$ (Alt-Tomi [1]).

In particular, our examples provide an answer to two questions raised by Tomi [28]:

There exist (topological) tori admitting families of non-flat disc-type minimal surfaces which intersect the the tori at a right angle, and secondly, the surfaces in such a family need not be congruent (nor isometric).

The authors would like to thank A. Kuster for preparing figures 4,6,7,8,9; O. Wohlrab for providing figure 2; and F. Tomi for making his paper [28] available (March 1985). The second author gratefully acknowledges the hospitality and the support by the Institute for Mathematics and its Applications of the University of Minnesota at Minneapolis during September 1984.

2. The Plateau problem with three boundary components

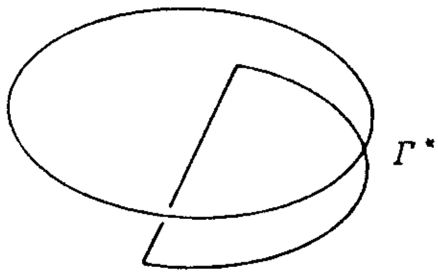
The Plateau problem in the general form as stated by Jesse Douglas asks for a minimal surface in Euclidean space of given topological type and with a prescribed configuration Γ of one or several Jordan curves as its boundary (cf. [8], [17], and [29] for references to the literature).

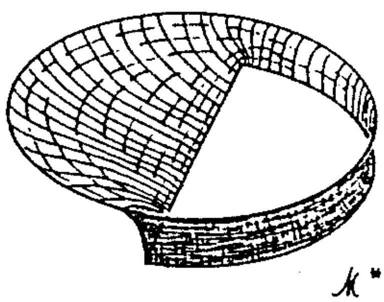
Frank Morgan [16] has given an example of a boundary configuration Γ, consisting of four coaxial circles in parallel planes, and a sequence of minimal surfaces M_n spanning Γ, of genus greater than or equal to n, which are not rotationally symmetric. Therefore each surface M_n is a member of a one-parameter family of congruent minimal surfaces, having the same boundary, that are obtained by rotation from M_n.

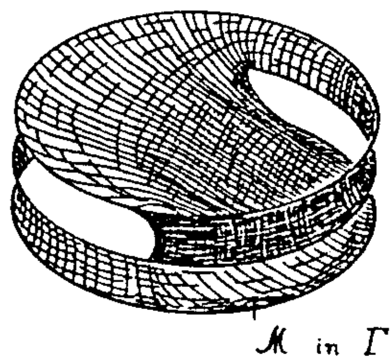
If, on the other hand, Γ consists of only one circle, then the maximum principle implies that any minimal surface bounded by Γ has to be a disc. Similarly each immersed minimal surface M bounded by two coaxial circles that lie in parallel planes must be rotationally symmetric, that is, M has either to be a catenoid or a pair of plane discs (see Schoen [23], Corollary 3, p. 796).

This leaves the question whether a rotationally symmetric configuration of three closed curves can only bound rotationally symmetric minimal surfaces. *We shall construct a rotationally nonsymmetric minimal surface of genus zero, bounded by three coaxial circles which lie in parallel planes.*

To this end, we consider a configuration Γ consisting of three circles Γ_0, Γ_1, Γ_{-1}, described by the equations $x^2 + y^2 = 1$ and $z = 0, \lambda$, and $-\lambda$ respectively, $\lambda>0$, and a second configuration Γ^* which consists of the circle Γ_1 and another closed curve γ that lies in the same plane as Γ_0, and is formed by the semicircle $\Gamma' = \Gamma_0 \cap \{x > 0\}$ and by the interval $I = \{x = 0, z = 0, -1 < y < 1\}$ on the y-axis. For small enough λ there is a minimal surface M^* of the type of an annulus bounded by Γ^*. By Schwarz's reflection principle, we can extend M^* as a minimal surface across the straight segment I. For this purpose, we rotate M^* by 180° about the y-axis to form a second minimal surface M^{**}. Their union $M = M^* \cup M^{**}$ is a minimal surface with boundary Γ and has genus zero. The segment I has become part of the interior of M, and the surface M can be described by a harmonic mapping $x : B \to \mathbb{R}^3$, given in conformal coordinates of a triply connected planar domain B (cf. [8], p. 119). Since M^* is not symmetric under rotations about the z-axis, also M







has to be rotationally nonsymmetric.

It remains to find a connected minimal surface M^* bounded by the configuration Γ^*. By virtue of J. Douglas' theorem ([9]; cf. also the remarks of Tromba [29]), there exists an area minimizing minimal surface M^* which is defined on an annulus and has Γ^* as boundary, provided that λ is small enough. In fact, the existence of Douglas' solution is ascertained under the hypothesis that

$$(2.1) \qquad m(\Gamma^*) < m(\gamma) + m(\Gamma_1) ,$$

where $m(\Gamma^*)$ is the greatest lower bound of area for surfaces of the type of the annulus with boundary $\Gamma^* = \gamma \cup \Gamma_1$, and where $m(\gamma)$ and $m(\Gamma_1)$ are the corresponding lower bounds for disc-type surfaces bounded by γ and Γ_1, respectively. Clearly,

$$m(\gamma) = \pi/2 \quad , \quad m(\Gamma_1) = \pi \quad ,$$

and $m(\Gamma^*)$ is smaller than the area $A(S)$ of the surface S that consists of the cylinder surface between Γ_0 and Γ_1, and of the half-disc $\{x^2 + y^2 < 1, x < 0, z = 0\}$; that is

$$m(\Gamma^*) < 2\pi\lambda + \pi/2 \ .$$

Thus Douglas' condition (2.1) is satisfied for $\lambda < 1/2$. A somewhat more complicated comparison surface S, consisting of half of a catenoid, half of a cone and two triangles shows that even the condition $\lambda < 0.7$ suffices to ensure the existence of a Douglas solution M^* within the frame Γ^*. Moreover, the hypothesis (2.1) implies that the surface M^* is an immersion ([12], Theorem 10.5). By the maximum principle, the interior of M^* lies between the planes $z = 0$ and $z = \lambda$. Therefore the interior of M^* does not meet the interior of M^{**} where M^{**} is the reflection of M^* at the y-axis. Thus also $M = M^* \cup M^{**}$ is immersed. Since M is rotationally nonsymmetric, we have shown:

The configuration consisting of three coaxial unit circles in parallel planes at distance $\lambda \leq 0.7$ *bounds a continuum of congruent immersed minimal surfaces of genus zero.*

We note that M cannot have boundary branch points because its boundary lies on a strictly convex set, a cylinder (see [17], p. 331). F. Tomi has kindly pointed out to us that, by virtue of a result of Almgren and Simon[1)], there is an embedded minimal surface M^* spanning Γ^*, and therefore also $M = M^* \cup M^{**}$ is embedded.

1) Ann. Scuola Norm. Sup. Pisa 6, 447-495 (1979)

We finally would like to emphasize that for boundary configurations consisting of one or several closed Jordan curves, the only currently known examples of continua of stationary solutions to Plateau's problem are *families of congruent surfaces*. It appears rather improbable that all such continua are of this form, although no other examples have yet been found. Of particular interest is the question of a continuum or at least denumerably many minimal surfaces, whether congruent or not, within a single Jordan curve (cf. [8], p. 122). If the proof of the general bridge theorem, stated in [15], p. 167, could be supplied, the existence of closed rectifiable but non-smooth Jordan curves bounding an uncountable set of stable minimal surfaces would be ensured. It is, however, not at all clear whether closed Jordan curves can bound continuous 1-parameter families of minimal surfaces. We, moreover, do not know if real-analytic or smooth Jordan curves can bound infinitely many minimal surfaces.

3. Free boundary problems

We now consider boundary configurations $\langle \Gamma, S \rangle$ in $\mathbb{R}^3$ consisting of a system Γ of Jordan curves $\Gamma_1, \dots, \Gamma_m$, and of a system S of surfaces $S_1, \dots, S_n$. Each of the curves Γ_i either is a closed curve, or else a Jordan arc with end points on S. We shall call S the *free part* of the configuration $\langle \Gamma, S \rangle$.

A minimal surface M is said to be *stationary within the configuration* $\langle \Gamma, S \rangle$ if the boundary of M lies on $\Gamma \cup S$ and, moreover, if M meets S orthogonally at the part $\Sigma = \partial M \cap S$ of its boundary.

We shall call Σ the *free trace* of M on S.

This definition requires a certain degree of smoothness for S, but it is not difficult to define stationary minimal surfaces for fairly general *supporting surfaces* S.

In the following we shall assume that S contains at least one surface whereas Γ may be empty.

The *free-boundary problem* of a configuration $\langle\Gamma,S\rangle$ asks for a minimal surface that is stationary within $\langle\Gamma,S\rangle$. Such a problem is said to be *partially free* if Γ is nonvoid, otherwise *completely free* or simply *free*.

It is trivial to find supporting surfaces S which bound continua of stationary minimal surfaces. For instance, the sphere, the cylinder, or the torus furnish simple examples. In these cases, however, all minimal surfaces belonging to the same continuum are congruent to each other. Therefore it might be of interest to see that there are free or even partially free boundary problems which possess denumerably many noncongruent solutions, or even continua of noncongruent solutions.

We first mention that, in 1872, H.A. Schwarz has described two boundary configurations $\langle\Gamma,S\rangle$ that bound denumerably many noncongruent stationary minimal surfaces (see [25], pp. 126-148). His first configuration $\langle\Gamma_1,\Gamma_2,S\rangle$ consists of a cylinder surface S and two straight arcs which are perpendicular to each other

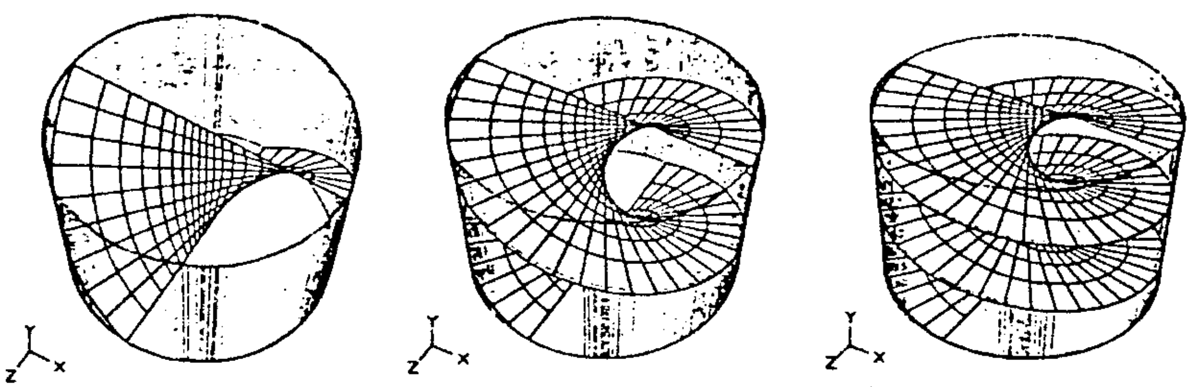


Figure 2: Three of infinitely many noncongruent minimal surfaces that are stationary within $\langle\Gamma_1,\Gamma_2,S\rangle$

as well as to the cylinder axis and pass through the axis at different heights. This configuration bounds denumerably many left and right winding helicoids which meet the cylinder S at a right angle (Figure 2). Only two of these helicoids are area minimizing, the others are but stationary.

The other boundary frame considered by Schwarz consists of two parallel faces S_1 and S_2 of some cube which are connected by two perpendicular diagonals lying on opposite faces (Figure 3).

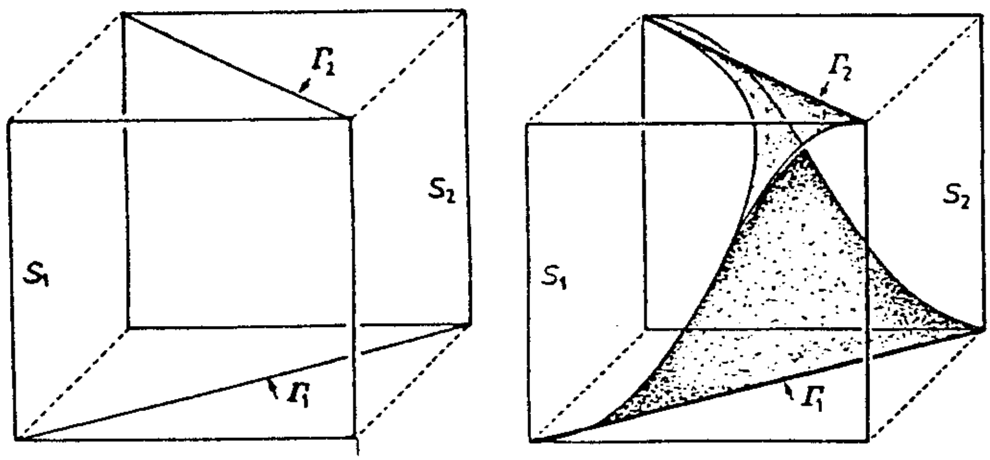


Figure 3: Gergonne's surface

This configuration $<\Gamma_1,\Gamma_2,S_1,S_2>$ bounds one area minimizing surface, depicted in Figure 3, and denumerably many other, noncongruent stationary minimal surfaces.

It might be interesting to learn of a configuration $<\Gamma,S>$, consisting of a circle Γ and of a supporting surface, which bounds a continuum of noncongruent and area minimizing (hence stationary) minimal surfaces. Yet such an example can easily be derived from classical results on minimal surfaces of revolution; see Bolza [6] (Beispiel I), Bliss [3], pp. 85-127, and Carathéodory [7]. §§273, 340-341, 360-367.

Let $x(t)$, $y(t)$, $t_1 < t < t_2$, be the parameter representation of a curve contained in the upper half plane $\{y>0\}$. The surface area of its surface of revolution about the x-axis is given by the integral $2\pi \int_{t_1}^{t_2} y\sqrt{dx^2 + dy^2}$. Thus the minimal surfaces of revolution are described by the extremals of the functional $\int y\sqrt{dx^2 + dy^2}$, $y > 0$, which are the parallels to the positive y-axis,

(3.1) $$x = x_0 \quad , y > 0 \quad ,$$

and the catenaries

(3.2) $$y = a \cosh \left(\frac{x-x_0}{a}\right) , \ -\infty < x < \infty \quad ,$$

which form a 2-parameter family of nonparametric curves, $a > 0$, $-\infty < x_0 < \infty$. The point (x_0,a) is the vertex of the catenary (3.2).

Let us consider all catenaries passing through some fixed point $P = (0,b)$, $b>0$, on the y-axis. They must satisfy $b = a \cosh \frac{x_0}{a}$ or $b = a \cosh \lambda$, if we introduce the new parameter $\lambda = -\frac{x_0}{a}$. Then there is a 1-1 correspondence between all real values of the parameter λ and all catenaries passing through $(0,b)$ which is given by

(3.3)
$$y = g(x,\lambda) := a(\lambda) \cosh\left(\lambda + \frac{x}{a(\lambda)}\right) , \quad x \in \mathbb{R} ,$$

$$a(\lambda) := \frac{b}{\cosh \lambda} , \ \lambda \in \mathbb{R} .$$

We can also write

(3.3') $$g(x,\lambda) = b \cosh \frac{x}{a(\lambda)} + \sinh \lambda \sinh \frac{x}{a(\lambda)} \quad ,$$

and $\sinh \lambda = \pm\sqrt{b^2 - a^2(\lambda)}$.

We now consider the branches $y = g(x,\lambda)$, $x > 0$ lying in the upper quadrant of the x,y-plane. There exists exactly one conjugate point

$Q(\lambda) = (\xi(\lambda), \eta(\lambda))$ with respect to P on each catenary (3.3) with $\lambda < 0$. The points $Q(\lambda)$, $\lambda \in \mathbb{R}$, form a real-analytic curve E that resembles a branch of a parabola extending from the origin to infinity. The curve E is given by the condition

$$\frac{\partial}{\partial\lambda} g(x,\lambda) = 0$$

and describes the envelope of the catenary arcs

$$C_\lambda = \{(x,g(x,\lambda)) : 0 \leq x \leq \xi(\lambda)\} , \quad \lambda \in \mathbb{R} .$$

The domain $\Omega = \{(x,y): 0 < x < \xi(\lambda),\ y > \eta(\lambda)$ for some $\lambda\}$ is simply covered by the open arcs $\dot{C}_\lambda = C_\lambda - \{P,Q(\lambda)\}$.

Consider the *wavefronts* W_c , $c>0$, emanating from P. The curves W_c are real-analytic level lines $\{S(x,y) = c\}$ of the wave function $S(x,y)$ that satisfies the Hamilton-Jacobi equation

$$S_x^2 + S_y^2 = y^2$$

and is given by

$$S(x,g(x,\lambda)) = J(x,\lambda) , \quad 0 \leq x \leq \xi(\lambda) ,$$

where the right hand side is defined by

$$J(x,\lambda) = \int_0^x g(u,\lambda)\sqrt{1+g'(u,\lambda)^2}\, du ,$$

and $g'(u,\lambda) = \frac{\partial}{\partial u} g(u,\lambda)$.

The two families of curves C_λ , $\lambda \in \mathbb{R}$, and W_c , $c>0$, form the *complete figure* (in sense of Carathéodory) associated with the varational problem

$$\int y \sqrt{dx^2 + dy^2} \rightarrow \text{Extr.} , \quad y(0) = b ,$$

in $x \geq 0$, $y > 0$, see Figure 4.

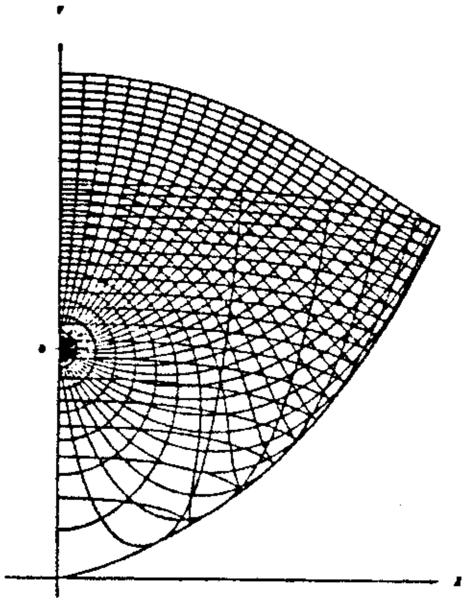


Figure 4a

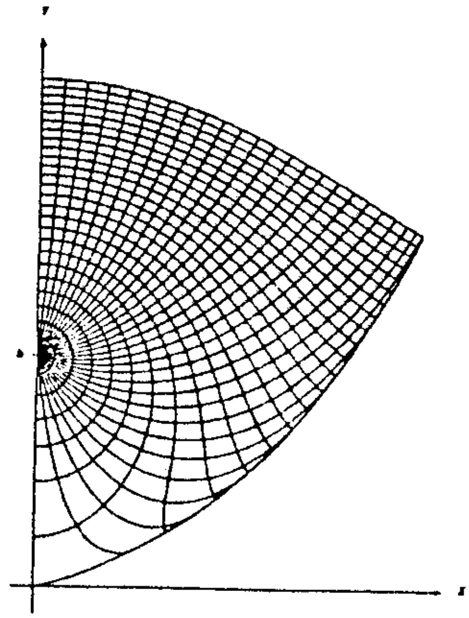


Figure 4b

(a) The catenaries passing through P, and their wavefronts.

(b) A complete figure: the stable catenary arcs passing through P and terminating at their envelope E , together with their wavefronts.

By Adolf Kneser's tranversality theorem, the curves W_c intersect the catenaries C_λ orthogonally. Two curves W_{c_1} and W_{c_2} , $c_1 < c_2$, cut out of each curve C_λ a piece $C_\lambda(c_1,c_2)$ such that

$$\int_{C_\lambda(c_1,c_2)} y\sqrt{dx^2+dy^2} = c_2 - c_1 \quad ,$$

and $c_2 - c_1$ is the infimum of the integral $\int y\sqrt{dx^2 + dy^2}$ along all paths joining W_{c_1} and W_{c_2} within Ω . In particular, if $C_{\lambda,c} = \{(x,g(x,\lambda)): 0 \leq x \leq x_0(\lambda,c)\}$ denotes the subarc of the catenary that connects P with W_c , then $J(x_0(\lambda,c),\lambda)$ is the infimum of the integral $\int y\sqrt{dx^2 + dy^2}$ taken along all curves joining P and W_c within Ω .

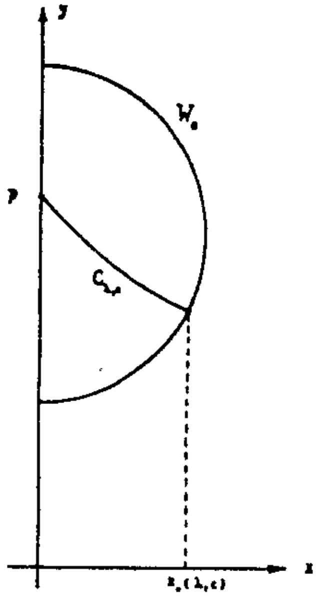


Fig. 5

If we now rotate the whole configuration drawn in Figure 5 about the x-axis, the wave front W_c generates a surface of revolution S_c, and each catenary $C_{\lambda,c}$ produces a minimal catenoid $K_{\lambda,c}$ with the area $2\pi c$. The catenoid $K_{\lambda,c}$ is bounded by two parallel coaxial circles Γ and $\Sigma_{\lambda,c}$ centered at the x-axis; Γ is generated by the rotation of P, and $\Sigma_{\lambda,c}$ by the rotation of the intersection point of C_λ with W_c. Each catenoid $K_{\lambda,c}$ intersects S_c orthogonally and, therefore, is a stationary minimal surface within the configuration $\langle\Gamma,S_c\rangle$. All catenoids $K_{\lambda,c}$, c fixed, have the same area and minimize area among all surfaces of revolution bounded by $\langle\Gamma,S\rangle$ which lie in the open set H generated by rotating $\Omega \cup \Omega^* \cup \{x = 0 , y > 0\}$ about the x-axis. Here Ω^* is the mirror image of Ω at the y-axis in the x,y-plane.

In fact, it turns out that the catenoids $K_{\lambda,c}$ even minimize area among all orientable surfaces F bounded by $\langle\Gamma,S_c\rangle$ that are contained in H. A well known projection argument shows that it suffices to prove $\text{Area}(K_{\lambda,c}) < \text{Area}(F)$ for all oriented surfaces F with boundary on $\Gamma \cup S_c$ that are contained in $H^+ = H \cap \{x > 0\}$.

Let now F be such a surface with $\gamma = \partial F \cap S_c$. Then there is a region T in the surface S_c with integer multiplicities, the boundary of which equals $\gamma - \Sigma_{\lambda,c}$. Therefore $K_{\lambda,c} - F + T$ is a cycle, and it follows that there is a three-dimensional region R with integer multiplicities such that the boundary of R is $K_{\lambda,c} - F + T$. Gauss' theorem yields

$$\int_R \operatorname{div} X \, dvol = \int_{\partial R} X \cdot N_{\partial R} \, dA \,, \tag{3.4}$$

where $N_{\partial R}$ is the oriented unit normal vector to ∂R. We infer from the minimal surface equation that the unit normal vectors to the catenoids $K_{\lambda,c}$ form a divergence-free vector field $X = X(x,y,z)$ on H^+ which is tangent to S_c, i.e., $X \cdot N_T = 0$, and can be chosen in such a way that $X \cdot N_{K_{\lambda,c}} = 1$. Hence we obtain from (3.4) that

$$\text{Area}(K_{\lambda,c}) = \int_{K_{\lambda,c}} X \cdot N_{K_{\lambda,c}} \, dA = \int_F X \cdot N_F dA ,$$

and the term on the right-hand side is estimated from above by Area(F), because $X \cdot N_F \le 1$.

Thus we have proved:

There exists a configuration $\langle \Gamma, S_c \rangle$ *consisting of a circle and a real-analytic surface of revolution* S_c *that bounds a family* $\{K_{\lambda,c}\}$ *of*

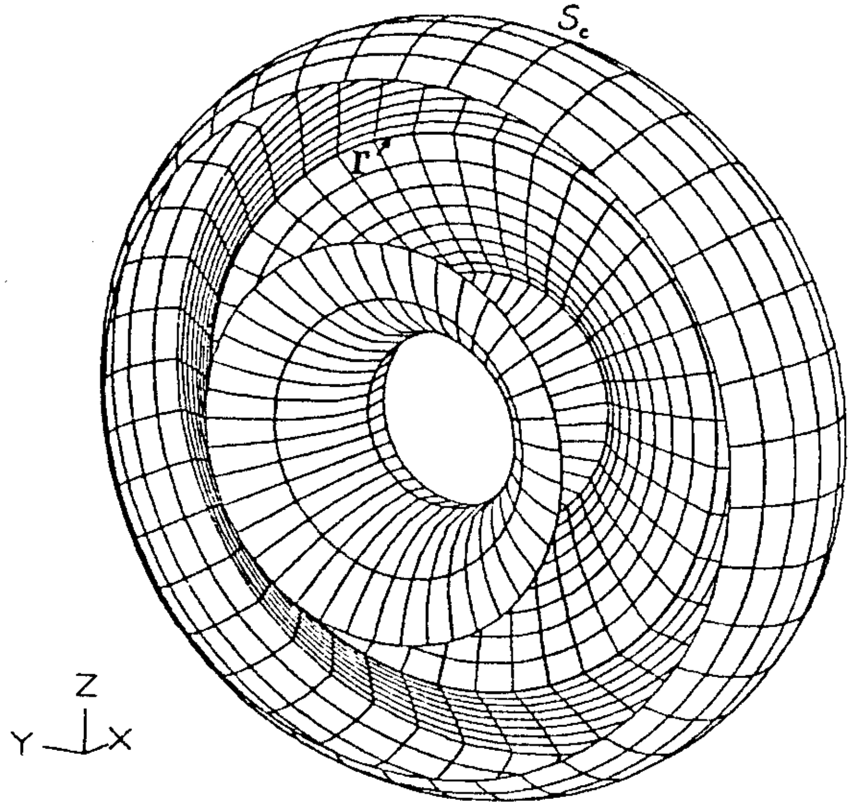


Figure 6

stationary and even area-minimizing minimal surfaces of annulus-type that are really distinct in the sense that, for two different values λ_1 , λ_2 , *the surfaces* $K_{\lambda_1,c}$ *and* $K_{\lambda_2,c}$ *are not congruent.*

A simple modification of the previous example leads to *boundary configurations as shown in* Figure 7 *that bound continua* $\mathfrak{C}$ *of noncongruent stationary surfaces of annulus type which have a completely free boundary on* S. *The surfaces of* $\mathfrak{C}$ *are even area minimizing within the class* $\mathfrak{C}^*$ *of annulus type surfaces whose free boundaries are homologous to those of the surfaces of* $\mathfrak{C}$.

For this purpose, we take two wavefront curves W_{c_1} and W_{c_2} , $c_1,c_2 > 0$, contained in $x>0$, $y > 0$. If c_1 and c_2 are chosen less than $b^2/2$ both curves terminate at the positive y-axis and meet this axis orthogonally. Reflecting both arcs at the y-axis, we obtain two closed real analytic curves

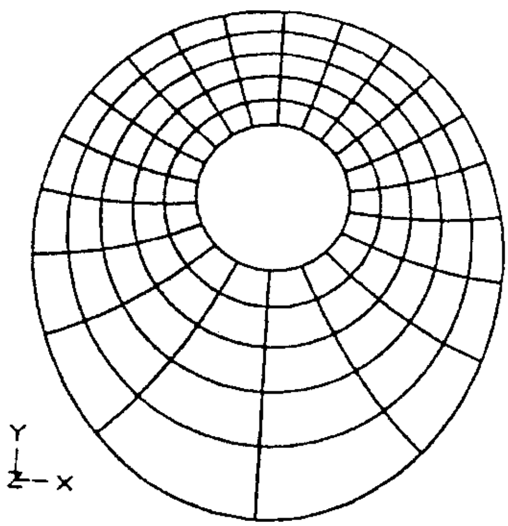


Figure 7a: The family of curves Γ_c .

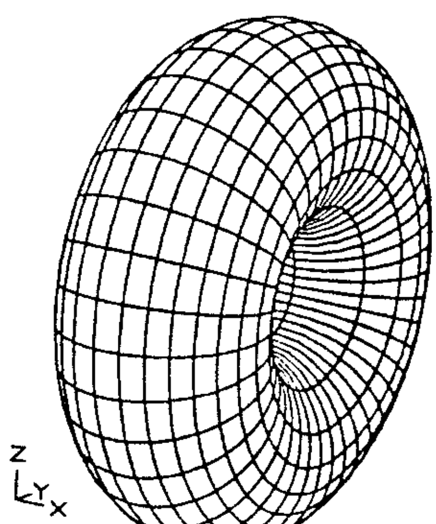


Figure 7b: Exterior view of the configuration $\langle S_1,S_2\rangle$.

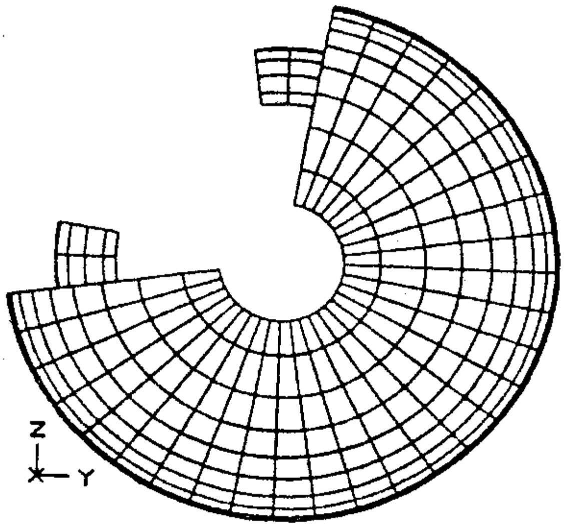


Figure 7c: Part of configuration $\langle S_1, S_2 \rangle$.

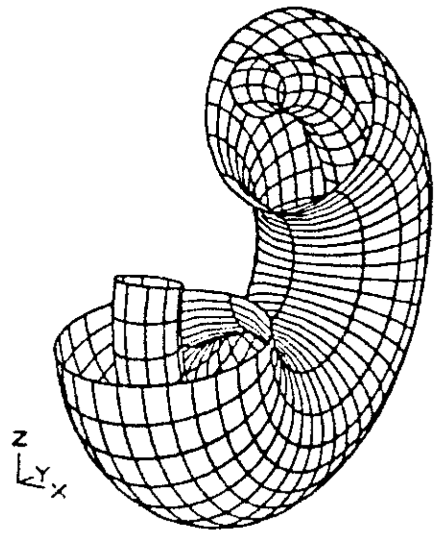


Figure 7d: Another part of $\langle S_1, S_2 \rangle$.

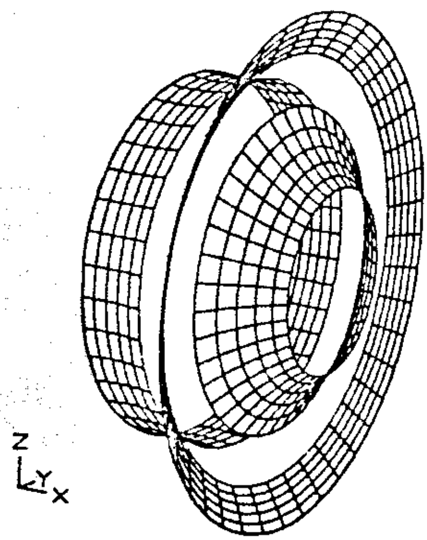


Figure 7e: Three surfaces of the family $\mathcal{C}$.

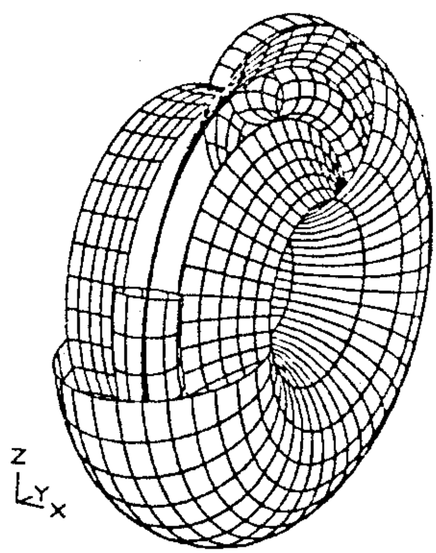


Figure 7f: Three surfaces of $\mathcal{C}$ within $\langle S_1, S_2 \rangle$

Γ_{c_1} and Γ_{c_2}, and their rotation about the x-axis leads to two closed torus-type surfaces S_1 and S_2 that are orthogonally met by a family of catenoids, generated by the catenary arcs $C_\lambda(c_1,c_2)$. These catenoids are stationary annulus-type minimal surfaces within the configuration $\langle S_1,S_2\rangle$, and a reasoning similar to the previous one shows that they even minimize area within $\mathcal{C}^*$ (cf. Figure 7).

A somewhat different example, which is not rotationally symmetric, leads to a free-boundary problem for minimal surfaces of the type of the disc, with boundary lying on a given toruslike surface. Let K_λ, $\lambda \in \mathbb{R}$, be the catenoids obtained by rotating the arc C_λ about the x-axis, and let K^*_λ be the surface obtained from K_λ by reflection at the y,z-plane. Let, moreover, $K_{-\infty}$ be the disc interior to the circle Γ in the y,z-plane, and let K_∞ be the plane domain exterior to Γ. We may think of $K_{\pm\infty}$ as degenerate catenoids obtained for $\lambda \to \pm\infty$. Then the surfaces K_λ, K^*_λ, $-\infty < \lambda < \infty$, describe a minimal foliation, singular at Γ, of the rotationally symmetric domain H.

We now introduce cylindrical coordinates (x,r,θ), where $y = r\cos\theta$, $z = r\sin\theta$. For each $r \in (0,b)$, there exists exactly one value $c(r) > 0$ such that the closed real-analytic curve $\Gamma_{c(r)}$ in the plane $\theta = 0$, obtained from the wavefront $W_{c(r)}$ as described before, passes through $(0,r,0)$.

Denote by $L_{r,\theta}$ the closed curve that is obtained by rotating $\Gamma_{c(r)}$ about the angle θ around the x-axis. The curves $L_{r,\theta}$, $0 < r < b$, $0 \le \theta < 2\pi$, meet the plane $x = 0$ orthogonally at the points $(0,r,\theta)$ and sweep an open subdomain H_0 of H.

Let γ_0 be a real-analytic Jordan curve in the plane $x = 0$, say, a circle, which is contained in the open disc $K_{-\infty}$ (the interior of Γ) and does not wind about the origin. As the point $(0,r,\theta)$ traverses the curve γ_0, the curves $L_{r,\theta}$ sweep out a toruslike surface S which bounds a tube G. This tube is foliated by a family M_λ, M^*_λ, $-\infty < \lambda < \infty$, of minimal surfaces that are cut by S out of the catenoids K_λ , K^*_λ . The surfaces M_λ , M^*_λ are of the type of the disc and meet S perpendicularly; hence they are stationary within S . Moreover, the unit normal vectors to M_λ , M^*_λ form a divergence free vector field on the set $H - \Gamma$ containing G which is tangent to S. Then, by an argument parallel to the previous reasoning, all surfaces M_λ , $M^*{}_\lambda$

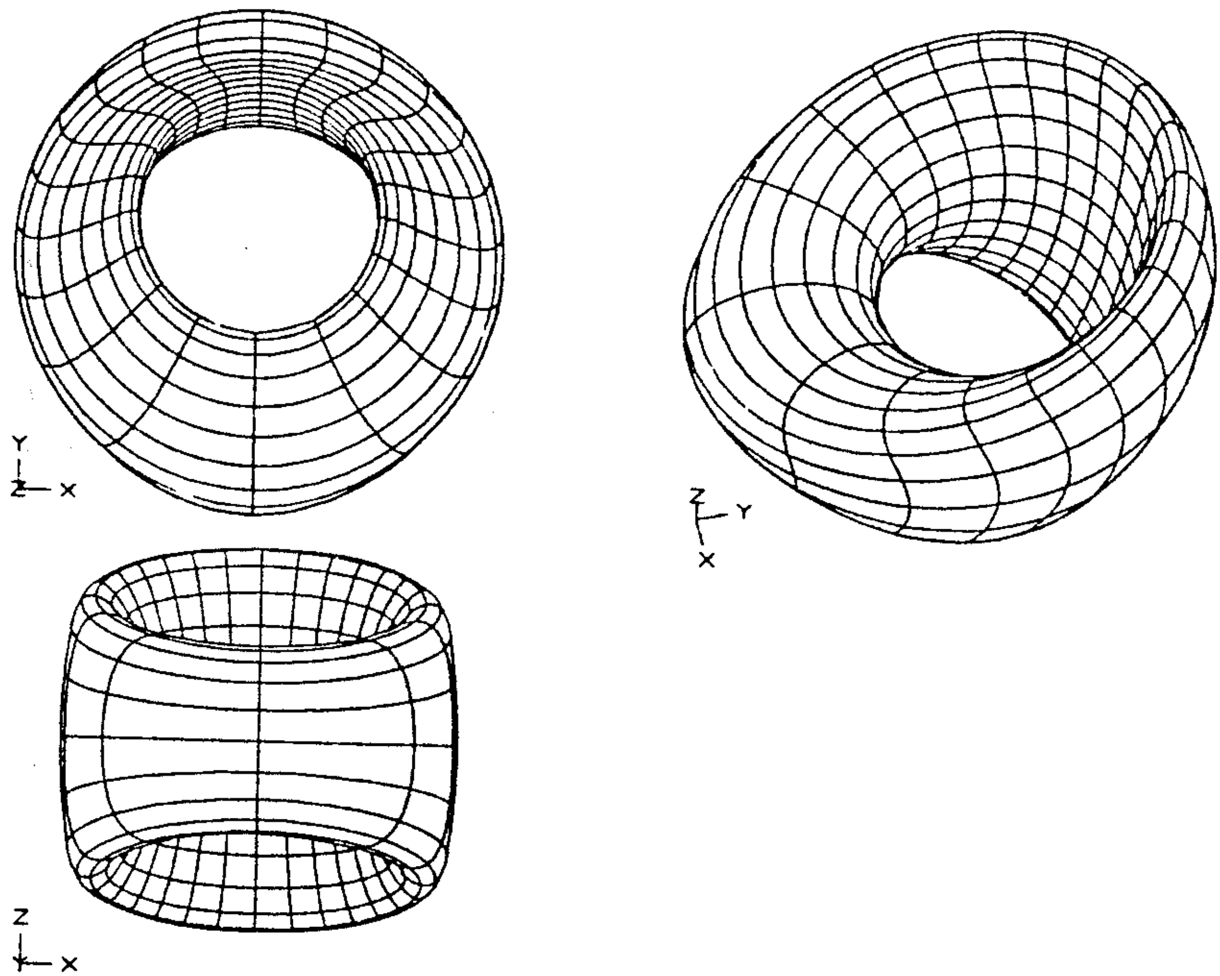


Figure 8: Three views of the surface S which also show the traces of the minimal surfaces M_λ , M^*_λ on S.

have equal area, and each oriented surface F contained in $H-\Gamma$ and with a boundary γ homologous in S to γ_0, has area larger than the leaves M_λ, M_λ^* unless it coincides with one of these surfaces. Thus we have shown:

There exists a real-analytic, embedded surface S *of the type of the torus, and a homology class* $[\gamma_0]$ *in* $H_1(S;\mathbb{Z})$, *so that* S *bounds a family of noncongruent stationary minimal surfaces of the type of the disc which have smallest area among all oriented surfaces in* $H-\Gamma$ *having boundary lying on* S *and homolgous in* S *to* γ_0.

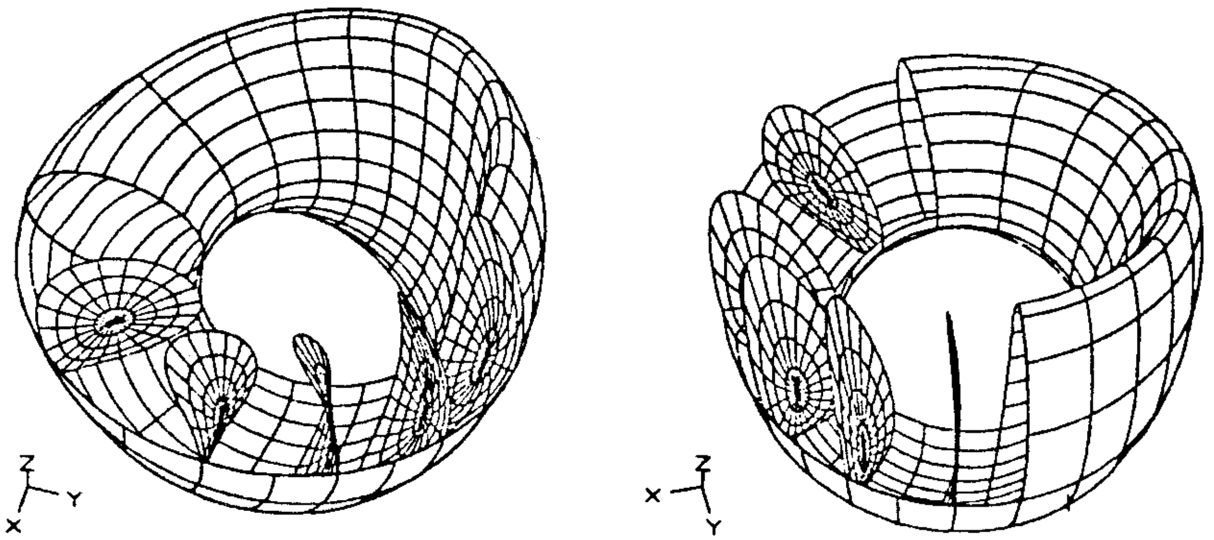


Figure 9: A part of surface S with various samples of surfaces M_λ, M_λ^* (two different views).

4. Isoperimetric problems

Let us consider the *partition* or *relative isoperimetric problem*. In the simplest case, this is the following question:

Given a convex body B with smooth boundary S, find an embedded surface M of given topological type, contained in B and with boundary on S, that divides B into two regions of prescribed volume and has smallest or at least stationary area among surfaces satisfying these conditions.

A surface M of admissible type will be stationary for this problem if and only if M has constant mean curvature and meets S orthogonally (cf. [11], [20]). If, for instance, B is the Euclidean ball $B_1(0)$ in $\mathbb{R}^3$, the spherical caps $M = B \cap \partial B_R(x_0$ are solutions for any center x_0 at distance $\sqrt{1+R^2}$ from the origin. These surfaces are the only stationary surfaces of the type of the disc, as Nitsche [20] has recently shown.

One might observe that, for each prescribed volume $V < \frac{4\pi}{3}$, there is a *two-parameter family of congruent solutions* with radius $R = R(V)$, depending on the parameter x_0, and this family is obtained through the action of $O(3)$, the group of rotations of $\mathbb{R}^3$, which leaves B invariant.

A more interesting example for the partition problem leads to a *one-parameter family of noncongruent solutions* (for a fixed value of the ratio of the dividing volumina). For this purpose, we consider a rotationally symmetric body in euclidean space,

$$B = \{(x,y,z) : x^2 + y^2 < f(z)^2 \quad , \quad z_1 < z < z_2 \} .$$

Then through each point (x,y,z) of $S = \partial B$ there passes a surface $M(z)$ in the form of a *spherical cap*, the center of which is the intersection with the z-axis of the tangent plane to S at the point (x,y,z). The sphere therefore has radius $r(z) = f(z)w(z)/f'(z)$, where $w(z) = \sqrt{1+(f'(z))^2}$. Then the cap $M(z)$ is stationary for the partition with the value $V = v(z)$ for the volume parameter. Here $v(z)$ is the volume of the subregion inside B which is bounded above (i.e., on the side of increasing z) by $M(z)$. Since

$$v = \frac{\pi f^3}{3f'}\left(\frac{2w^2}{w+1} - 1\right) + \pi \int f^2 dz + \text{const} \quad ,$$

it follows that $v(z)$ will be constant on any interval in which

$$ff'' + 2w^2(w+1) = 0 \ . \tag{4.1}$$

Suppose now that $z_1 < 0 < z_2$, and let any initial values $r_0 > 0$, p_0 be given. We consider the solution of (4.1) which satisfies

$$f(0) = r_0 \quad , \quad f'(0) = p_0 \ .$$

The solution may be continued as long as $f'(z)$ remains bounded, which may be the case for $\zeta_1 < z < \zeta_2$, where $\zeta_1 < 0 < \zeta_2$. Then (4.1) implies that

$$f''(z) < 0 \quad \text{in} \quad [\zeta_1, \zeta_2] \ ,$$

i.e., f is strictly concave, and the mean curvature $H(z)$ of the cap $M(z)$ is strictly decreasing, since

$$H' = \left(\frac{f'}{fw}\right)' < 0 \quad \text{on} \quad [\zeta_1, \zeta_2] \ .$$

Hence the caps are noncongruent.

For suitably chosen z_1 and z_2 with $z_1 < \zeta_1 < \zeta_2 < z_2$, let $f(z)$ be extended in such a way that $f > 0$; B is a smooth convex, and rotationally symmetric body; and so that the spherical caps $M(z)$ lie inside B for $\zeta_1 < z < \zeta_2$. Then *the partition problem in* B *with prescribed volume* $V = v(0)$ *has a one - parameter family of noncongruent stationary solutions* $M(z)$, $\zeta_1 < z < \zeta_2$.

We note that this discussion could be made more precise since the equation (4.1) for $r = f(z)$ could be explicitly integrated for the inverse function $z = g(r)$. In fact, we obtain that $f^2w/(w+1)$ is a constant, say $a^2/2$ whence

$$g'(r) = \pm \frac{2r^2-a^2}{2r\sqrt{a^2-r^2}} \quad ,$$

and

$$g(r) = v - \frac{a}{4} \log \left(\frac{a+v}{a-v}\right) + b \quad , \quad v = \pm\sqrt{a^2-r^2} \quad .$$

Allowing v to assume both signs, we obtain a complete curve in the z,r-plane. The resulting surface of revolution $S_{a,b}$ is a real-analytic, complete, immersed surface which intersects itself along a circle in its plane of symmetry $z = b$, and approaches the z-axis as $z \to \pm\infty$. We leave details to the reader.

In the previous cases, the explicit knowledge of the solution greatly simplified the construction of special boundaries S leading to "blocks" of noncongruent solutions. Related and equally interesting variational problems would require significantly more extensive analysis. For example, the *capillary problem* (in the presence of a constant gravitational field) asks for a surface bounding a given volume of liquid in a body B, and, in equilibrium, the liquid surface meets $S = \partial B$ at a given constant angle and has prescribed mean curvature

$$H(x,y,z) = \kappa z + \lambda \ ,$$

where κ is a given constant and λ is a Lagrange parameter.

In analogy with the partition problem, we might try to find a special body B, so that the capillary problem has a one-parameter family of solutions bounding the same liquid volume.

Similar questions should be asked for the classical isoperimetric problem in a Riemannian manifold (see [10] for applications). It is fairly easy to construct interesting examples of families of noncongruent solutions if we admit Lipschitz-continuous metrics. This can, in fact, be carried out by piecing together two copies of suitably chosen bodies B which appeared in the partition problem and by defining the Riemann metric via the Euclidean metric, employing normal coordinates on the boundary of the two copies of B. The case of smooth metrics seems to be more complicated.

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