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Navier-Stokes Equations with Navier Boundary Conditions in Nearly Flat Domains

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Dedicated to Professor Ciprian Foias on the occasion of his 75th birthday.

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Abstract

We consider the Navier–Stokes equations in a thin domain of which the top and bottom are not flat. The velocity fields are subject to the Navier conditions on those boundaries and the periodicity condition on the other sides of the domain. The model arises from studies of climate and oceanic flows. We show that the strong solutions exist for all time provided the initial data belong to a "large" set in the Sobolev space H^1. Furthermore we show, for both the autonomous and the nonautonomous problems, the existence of a global attractor for the class of all globally-defined strong solutions. This attractor is proved to be also the global attractor for the Leray-Hopf weak solutions of the Navier–Stokes equations. One issue that arises here is a nontrivial contribution due to the boundary terms. We show how the boundary conditions imposed on the velocity fields affect the estimates of the Stokes operator and the (nonlinear) inertial term in the Navier–Stokes equations. This results in a new estimate of the trilinear term, which in turn permits a short and simple proof of the existence of globally-defined strong solutions.

Keywords: Navier-Stokes equations; thin domain; global existence; strong solution; global attractor.

AMS classification numbers: Primary 35Q30, 76D05, 37L05; Secondary: 76D03, 35B65, 35K55, 35K60, 35K50.

Outline

1 Introduction

The main objective of this article is to present a theory of the longtime behavior of strong solutions of a special class of Navier–Stokes equations on thin 3D (three-dimensional) domains. Thin domains are encountered in the study of many problems in fluid mechanics. For example, in ocean or great lakes dynamics, one is dealing with the fluid regions which are thin compared to the horizontal length scales. Other examples include lubrication, meteorology, blood circulation, etc.

Several studies of the behavior of strong solutions of the Navier–Stokes equations on thin 3D domains have been published in the last fifteen years (see the list of references). These works are a part of a broader study of the behavior of various PDEs on thin n-dimensional domains, where $n \geq 2$. (See [31] for an informative treatment of the recent literature on this issue.) In terms of this broader context, the theory of the Navier–Stokes equations is of special interest because of the close connection with the well known Global Regularity Problem for the 3D Navier–Stokes equations and the related dynamical properties of the global attractor in this setting, see Theorem 1.1.

Note that the Navier–Stokes equations on a domain $\Omega \subset \mathbb{R}^3$ are:

$$\partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0, \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $u = u(t, x)$ is the unknown velocity field, $p = p(t, x)$ is the unknown pressure, $f = f(t, x)$ is a given body force. For simplicity, it is assumed in the sequel, that the viscosity ν is fixed, with $\nu = 1$. One can easily obtain the results for a general ν by the standard scaling arguments. Of course the constants, that arise in this case, will depend on ν.

One seeks solutions $u(t, x)$ $(t > 0,\ x \in \Omega)$ of (1.1) that satisfy the given boundary conditions with $u(0, x) = u^0(x)$, where $u^0 \in H$ is a given initial velocity field, and H denotes a suitable space of solenoidal vector fields in $L^2(\Omega)^3$. For strong solutions, we require that $u_0 \in V^1 = H \cap H^1(\Omega)^3$.

Note that with suitable smoothness assumptions on Ω and with homogeneous boundary conditions, the Stokes operator A for (1.1) satisfies

$$Au = \mathbb{P}(-\Delta u), \qquad \text{for } u \in \mathcal{D}(A), \tag{1.2}$$

where $\mathbb{P}$ is the Helmholtz-Leray projection, and $\mathcal{D}(A)$, the domain of A, is a subspace of $H^2(\Omega)^3$ consisting of vector fields u on Ω that satisfy the boundary conditions and $\nabla \cdot u = 0$ in Ω. By applying the Helmholtz-Leray projection $\mathbb{P}$ to (1.1), one obtains

$$\partial_t u + Au + B(u,u) = F(t), \tag{1.3}$$

where $B(u,u) = \mathbb{P}((u\cdot\nabla)u)$ and $F = \mathbb{P} f$ (see [5, 11, 21, 25, 40, 41, 42, 43, 44, 47], for derivations of (1.3)). Let

$$L^\infty(L^2) = L^\infty(0,\infty; L^2(\Omega)^3), \quad C(\mathbb{R}, L^2) = C(\mathbb{R}, L^2(\Omega)^3),$$
$$\text{and } L^\infty(\mathbb{R}, L^2) = L^\infty(\mathbb{R}, L^2(\Omega)^3).$$

The body force $f = f(t,x)$ is assumed to be in $L^\infty(L^2)$ with the norm:

$$\|f\|_{L^\infty(L^2)} = \|f\|_\infty \stackrel{\text{def}}{=\!=} \operatorname{ess\,sup}\{\|f(t)\|_{L^2(\Omega)} : t \geq 0\}. \tag{1.4}$$

In the initial papers on the Navier–Stokes equations on thin 3D domains, Raugel and Sell [33, 34, 35] considered the case where the domain satisfies:

$$\Omega = \Omega_1 = Q \times (0, \varepsilon),$$

with Q being a suitable bounded domain in $\mathbb{R}^2$ and $0 < \varepsilon \leq 1$. The boundary conditions are: spatial periodicity in x_3, where $0 < x_3 < \varepsilon$, and either spatial periodicity in $(x_1, x_2) \in Q$ when Q is a rectangle, or homogeneous Dirichlet boundary conditions when Q is arbitrary.

The main results of [33, 34, 35] can be summarized as follows:

Theorem 1.1. *For the Navier–Stokes equations on $\Omega = \Omega_1$, there is an ε_0, with $0 < \varepsilon_0 \leq 1$, such that for each ε with $0 < \varepsilon < \varepsilon_0$, there exist "very large" sets $N_1(\varepsilon) \subset L^\infty(L^2)$ and $N_2(\varepsilon) \subset V^1$, such that: Whenever the data (f, u_0) belong to $N_1(\varepsilon) \times N_2(\varepsilon)$, there exists a unique strong solution $u(t) = S(f,t)u_0$ of* (1.3) *satisfying $u(0) = u_0$, and $u(t)$ remains in V^1, for all $t \geq 0$.*

Moreover, when $f \in \mathcal{H}_\varepsilon$, where $\mathcal{H}_\varepsilon$ is a time-translation invariant set in $C(\mathbb{R}, L^2) \cap L^\infty(\mathbb{R}, L^2)$ that is compact in the topology of uniform convergence on bounded sets in $\mathbb{R}$, then the following hold:

(i) There exists a "robust" family of compact attractors $\mathfrak{A}_\varepsilon$ for the strong solutions of (1.3)*, which satisfies the Little Regularity Property* [40]*:*

$$\mathfrak{A}_\varepsilon \subset \mathcal{H}_\varepsilon \times N_2(\varepsilon) \subset \mathcal{H}_\varepsilon \times H^1(\Omega)^3, \qquad \text{for } 0 < \varepsilon < \varepsilon_0.$$

(ii) Each $\mathfrak{A}_\varepsilon$ is the global attractor for the Leray-Hopf weak solutions of the Navier–Stokes equations (see [39, 9]).

(iii) For each data (f,u_0) *in* $\mathfrak{A}_\varepsilon$*, the strong solution* $S(f,t)u_0$ *satisfies various regularity properties, including* $S(f,t)u_0 \in H^2(\Omega)^3$*, for all* $t \in \mathbb{R}$.

(iv) The family $\mathfrak{A}_\varepsilon$*, for* $\varepsilon \in [0,\varepsilon_0)$*, is upper semicontinuous at* $\varepsilon = 0$*, where* $\mathfrak{A}_0$ *is the global attractor for the reduced problem, the 2D Navier–Stokes equations* (1.7).

The argument used to obtain Theorem 1.1 is based on an averaging of the Navier–Stokes equations in the thin direction. That is, set

$$M_1 u = v, \quad \text{where} \quad v(x_1,x_2) = \frac{1}{\varepsilon}\int_0^\varepsilon u(x_1,x_2,s)\,ds \tag{1.5}$$

and let $w = u - v$. With the periodic boundary conditions on Ω_1, the mapping M_1 is a projection on $L^2(\Omega_1)^3$, and it commutes with the Stokes operator A. Therefore, one can rewrite the equation (1.3) in terms of a system of the (v,w)-equations:

$$\begin{cases} \partial_t v + Av + B(v,v) &= M_1\,G(t), \\ \partial_t w + Aw &= (I - M_1)\,G(t), \end{cases} \tag{1.6}$$

where $R(v,w) = B(v,w) + B(w,v) + B(w,w)$ and $G(t) = F(t) - R(v(t),w(t))$.

When one has $(I - M_1)F(t) \equiv 0$, then $\{w \equiv 0\}$ is an invariant set for (1.6) and $v = v(t)$ is a solution of a related 2D-problem:

$$\partial_t v + Av + B(v,v) = M_1\,F(t) = F(t). \tag{1.7}$$

The two key features underlying the proofs are:

(i) When $w(t)$ is small, then the v-equation in (1.6) is a small perturbation of the 2D Navier–Stokes equations (1.7). This means that one can exploit 2D Sobolev-like inequalities (see [1, 5, 41]) to estimate $\|v(t)\|_{H^1(\Omega)}$, for example.

(ii) When ε is small, then the w-equation in (1.6) is "super-stable" in the sense that, for the simplified problem where $R(v,w) = 0$, one has

$$\|A^{\frac12} w(t)\|^2_{L^2(\Omega)} \le \exp(-C\varepsilon^{-2}\,t)\|A^{\frac12} w(0)\|^2_{L^2(\Omega)} + C\,\varepsilon^2\,\|(I - M_1)F\|^2_\infty, \tag{1.8}$$

for $t \ge 0$. Thus $w(t)$ becomes small "very" fast. Since $w(0)$ and F are to be large, one needs to exploit the fact that the exponential term in (1.8) decays very rapidly when ε is small.

The (v,w) equations in (1.6) and the commutativity of the averaging operator M_1 with the Stokes operator A play a central role in establishing Theorem 1.1.

There have been several generalizations of Theorem 1.1 to cover a variety of boundary conditions on Ω_1, see Temam and Ziane [45, 46]. Of special note is the use of the free boundary conditions

$$u \cdot N = 0 \quad \text{and} \quad (\text{curl}\ u) \times N = 0, \tag{1.9}$$

which are equivalent the following joint Dirichlet-Neumann boundary conditions on Ω_1:

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0, \qquad \text{when } x_3 = 0 \text{ and } x_3 = \varepsilon. \tag{1.10}$$

Also see [22, 27, 28] for related results. More recently, Iftimie [19] and Iftimie and Raugel [20] obtained sharper results by using joint spatial-temporal Sobolev-like inequalities. In addition, the theory of thin domain dynamics was extended to a 2-fluids problem, such as air and water in a coupled atmospheric-oceanic problem, in Chueshov, Raugel, and Rekalo, [6]. Another analytically noteworthy step was made by Temam and Ziane in [46], where the physical domain is a spherical annulus and one uses the free boundary conditions (1.9) to obtain similar results.

In our study, we are concerned primarily with the Navier boundary conditions on $\partial\Omega$:

$$u \cdot N = 0 \quad \text{and} \quad [D(u)N]_{\tan} = 0, \tag{1.11}$$

where N denotes the unit outward normal to $\partial\Omega$, the deformation tensor is given by $Du = D(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{t}})$, where the superscript "t" denotes the transpose and the subscript "tan" refers to the tangential part of a vector. The conditions were proposed by Navier [29] as an alternative to the no-slip boundary condition for viscous fluids. Note that, for $\Omega = \Omega_1$, the conditions (1.11) on the flat top and bottom boundaries are the same as (1.10). We will have need to refer to the two conditions in (1.11) separately. So we will use:

$$u \cdot N = 0 \tag{1.12}$$

and

$$[D(u)\, N]_{\tan} = 0. \tag{1.13}$$

In the recent paper [21], Iftimie, Raugel, and Sell study the Navier–Stokes equations on the domains:

$$\Omega = \Omega_2 = \{(x_1, x_2, x_3) \in \mathbb{T}^2 \times \mathbb{R} : 0 < x_3 < \varepsilon\, g(x_1, x_2)\}, \tag{1.14}$$

where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2D torus. The solutions, in addition to being spatially periodic in (x_1, x_2), satisfy the Navier boundary conditions (1.11) on the top and bottom boundaries.

Notably, by using the Navier boundary conditions in this more complicated geometry, one encounters a problem of significantly higher complexity than seen in the references cited above. For instance, one uses a suitably modified averaging operator $\widehat{M_1}$ in place of M_1 in (1.5). When $g \neq$ constant, one finds that, in general, the terms $v = \widehat{M_1}u$ and $w = (Id - \widehat{M_1})u$ do not satisfy the second Navier boundary condition (1.13), even when u does. Hence the (v, w) equations (1.6), which play crucial roles in the earlier studies, are not available for the problem on Ω_2.

In this article, we study the strong solutions of the Navier–Stokes equations (1.1) on a more general domain than Ω_2, namely, the domain $\Omega = \Omega_3$, where:

$$\Omega_3 = \Omega^\varepsilon = \{(x_1, x_2, x_3) \,:\, (x_1, x_2) \in \mathbb{T}^2,\, h_0^\varepsilon(x_1, x_2) < x_3 < h_1^\varepsilon(x_1, x_2)\}, \tag{1.15}$$

with $\varepsilon \in (0, 1]$, $h_0^\varepsilon = h_0 = \varepsilon g_0$, $h_1^\varepsilon = h_1 = \varepsilon g_1$, and where g_0, g_1 are given C^4 functions defined on $\mathbb{T}^2$. The Navier boundary conditions (1.11) are assumed to hold on $\Gamma^\varepsilon = \Gamma = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$, where

$$\Gamma_j^\varepsilon = \Gamma_j = \{(x_1, x_2, x_3) \,:\, (x_1, x_2) \in \mathbb{T}^2,\, x_3 = h_j^\varepsilon(x_1, x_2)\} \quad j = 0, 1, \tag{1.16}$$

are the bottom and the top of Ω_3. To avoid irregular domains, we require that

$$g(x') \stackrel{\text{def}}{=\!=} (g_1 - g_0)(x') \geq c_0 > 0, \qquad \text{for all } x' \in \mathbb{T}^2. \tag{1.17}$$

While our general approach to the problem is based on the one used in [21], we also exploit an important property of the boundary conditions, see Proposition 4.6. As a result, the methodology developed here, can therefore be readily adapted for domains with different geometries (see [18]), or for variations in the Navier boundary conditions (see [16]). For instance, it is no longer the case that one is able to exploit the relationship (1.10), which holds on the bottom where $h_0^\varepsilon = 0$, as in [21]. Instead, by using Proposition 4.6, one establishes basic properties on the general domain Ω^ε, see for example, Lemmas 4.7, 4.8, 4.27, and 4.28. In addition to such generalizations, we need to make a more indepth study of the connections between the Navier boundary conditions and the Stokes operator (see the proof of Proposition 4.21), and the inertial term $(u \cdot \nabla)u$ in the Navier–Stokes equations (see the proof of Proposition 5.1). This effort leads to a short and more transparent proof of the global regularity of the strong solutions. There are two Key Features in our proof of the global regularity:

(i) A good estimate of the nonlinear term

$$\langle (u \cdot \nabla)u, Au \rangle_{L^2(\Omega^\varepsilon)}, \qquad \text{see Proposition 5.1.} \tag{1.18}$$

What underlies the derivation of this estimate is a basic inequality of the L^2-norm of $(Au + \Delta u)$, where $u \in \mathcal{D}(A)$ and A is the Stokes operator, see Lemma 4.19 on page 26 and [15]. This results in an unexpected <u>linear</u> differential inequality for the norm $\|A^{\frac{1}{2}} u(t)\|^2_{L^2(\Omega^\varepsilon)}$, see (6.32).

(ii) The estimate of the term (1.18) includes the expression $\varepsilon^{-1} \|A^{\frac{1}{2}} u(t)\|^2_{L^2(\Omega^\varepsilon)}$. Applying the Gronwall inequality directly will not yield good estimates, due to the negative power of ε. We use instead the Uniform Gronwall inequality in this case to obtain sharper estimates for $\|A^{\frac{1}{2}} u(t)\|^2_{L^2(\Omega^\varepsilon)}$ for $t > 1$. This enables one to complete the proof of global regularity, as we will see. This approach avoids the use of an exponential factor that arises in the argument in [21].

The paper is organized as follows. In Section 2 we present the functional settings suitable for the Navier conditions and our thin domains. Section 3 consists of statements of the main theorems concerning the global existence of strong solutions and the global attractors for both the strong solutions and weak solutions of the Navier–Stokes equations. The proofs of these theorems and a general discussion of the longtime dynamics of the Navier–Stokes equations, when the parameter ε is small, are given in Sections 6. We present in Section 4 the key mathematical properties required for these proofs. (Some of the technical arguments used to verify these properties are presented in the Appendix, Section 7.) In particular, we prove fundamental inequalities of the Poincaré-Trace class and variations of the classical Ladyzhenskaya and Agmon inequalities. Also we develop here the theory of the boundary behavior of strong solutions in the setting of the Navier boundary conditions. This theory is used to study the Stokes operator, in particular, the uniform estimate for $\|u\|_{H^1}$ in terms of $\|A^{\frac{1}{2}}u\|_{L^2}$ (Uniform Korn Inequality), and the uniform estimate for $\|u\|_{H^2}$ in terms of $\|Au\|_{L^2}$. The averaging operators needed in our analysis and their properties, such as the generalized Poincaré inequality (4.82), are also presented. The main nonlinear estimate is obtained in Section 5. In the proof, we make use of the geometric interpretation of the Navier boundary conditions found in Section 4.

2 Preliminaries

In this section, we present the functional spaces and operators which are important to our study. This includes both the setting of general bounded, smooth domains Ω in $\mathbb{R}^3$ and the nearly flat domains $\Omega = \Omega_3$, which are under consideration in this article.

In this article, we make use of the following Hilbert spaces:

$$L^2(X)^k = L^2(X, \mathbb{R}^k) \quad \text{and} \quad H^m(X)^k = H^m(X, \mathbb{R}^k) = W^{m,2}(X, \mathbb{R}^k),$$

where m and k are positive integers, X can be Ω, Ω^ε, Γ, $\mathbb{T}^2$, etc.

Since we will focus on the domain Ω^ε, and since the constant k is understood by the context, we will sometimes use the abbreviated notation:

$$L^2 = L^2(\Omega^\varepsilon)^k \quad \text{and} \quad H^m = H^m(\Omega^\varepsilon)^k. \tag{2.1}$$

Thus the norms and inner products

$$\|\cdot\|_{L^2} = \|\cdot\|, \quad \|\cdot\|_{H^m}, \quad \langle\cdot,\cdot\rangle_{L^2} = \langle\cdot,\cdot\rangle, \quad \text{and} \quad \langle\cdot,\cdot\rangle_{H^m} \tag{2.2}$$

refer to the convention described in (2.1).

2.1 General domains

We consider in this section an open, bounded, connected domain $\Omega \subset \mathbb{R}^3$ with C^4 boundary $\partial\Omega$. We assume that the Navier boundary conditions (1.11) are satisfied on $\partial\Omega$. Let

$$H_1 = H_1(\Omega) \stackrel{\text{def}}{=\!=} \{u \in L^2(\Omega)^3 \,:\, \nabla \cdot u = 0 \text{ on } \Omega,\ u \cdot N = 0 \text{ on } \partial\Omega\}. \tag{2.3}$$

Recall the classical decomposition

$$L^2(\Omega)^3 = H_1 \oplus H_1^\perp, \qquad \text{where } H_1^\perp = \{\nabla\phi : \phi \in H^1(\Omega)\}.$$

We treat a variation of this decomposition below.

The Green formula: We use here the Euclidean structure on $\mathbb{R}^3$, where $u \cdot v$ and $Du : Dv$ is the scalar product of the matrices: Du and Dv. For $u \in H^2(\Omega^\varepsilon)^3$ and $v \in H^1(\Omega^\varepsilon)^3$, one has

$$\begin{aligned} \int_\Omega \Delta u \cdot v \, dx = & \int_\Omega [-2(Du : Dv) + (\nabla \cdot u)(\nabla \cdot v)] \; dx \\ & + \int_{\partial\Omega} \{2((Du)N) \cdot v - (\nabla \cdot u)(v \cdot N)\} \, d\sigma. \end{aligned} \tag{2.4}$$

Equation (2.4) is proved by means of a straightforward integration by parts, see [21, 42, 47], for example. The variational form of the Navier–Stokes equations under the Navier boundary conditions, is based on (2.4).

When u and v are divergence-free, u satisfies the Navier boundary condition (1.11), and v satisfies the slip condition (1.12), the boundary integral in (2.4) vanishes, and one has

$$-\int_\Omega \Delta u \cdot v \, dx = 2 \int_\Omega (Du : Dv) \, dx.$$

One defines the bilinear form

$$E(u, v) = 2 \int_\Omega Du : Dv \, dx, \qquad \text{for } u, v \in H^1(\Omega)^3. \tag{2.5}$$

Note that $E(\cdot, \cdot)$ is bounded in $H^1(\Omega)^3$ and

$$0 \le E(u, u) = 2\|D(u)\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)}, \qquad \text{for } u \in H^1(\Omega)^3. \tag{2.6}$$

Furthermore, $E(u, u) = 0$ if and only if $u = a + b \times x$ for some $a, b \in \mathbb{R}^3$. In this connection, we define

$$Z_0 \stackrel{\text{def}}{=\!=} \{u = a + b \times x \ \text{ satisfying } u \cdot N = 0 \text{ on } \partial\Omega, \text{ for some } a, b \in \mathbb{R}^3\,\}. \tag{2.7}$$

By restricting to a subspace V of $H^1(\Omega)^3$ that is orthogonal in $L^2(\Omega)^3$ to Z_0, the Korn inequality (see e.g. [42]) implies that there is a constant $\widehat{C} > 0$ such that

$$\|u\|^2_{H^1(\Omega)} \le \widehat{C}\, \|E(u, u)\|^2_{L^2(\Omega)}, \qquad \text{for all } u \in V, \tag{2.8}$$

which shows that $E(\cdot,\cdot)$ is coercive in V. For the Navier–Stokes equations, we also require that $V \subset H_1$, where H_1 is defined in (2.3). In particular we set

$$V = V^1 \stackrel{\text{def}}{=\!=} H^1(\Omega)^3 \cap H_1 \cap Z_0^\perp = H^1(\Omega)^3 \cap H, \tag{2.9}$$

where $Z_0^\perp$ denotes the orthogonal complement of Z_0 in $L^2(\Omega)^3$, and

$$H \stackrel{\text{def}}{=\!=} H_1 \cap Z_0^\perp. \tag{2.10}$$

For the vector fields satisfying both Navier boundary conditions, we define the space

$$V^2 \stackrel{\text{def}}{=\!=} \{u \in H^2(\Omega)^3 \cap V^1 : (1.11) \text{ holds on } \partial\Omega\}. \tag{2.11}$$

The (related) **Helmholtz-Leray Projection** associated with the Navier boundary conditions is denoted by $\mathbb{P}$, and it is the orthogonal projection of $L^2(\Omega)^3$ onto the space H.

This brings us to the Stokes operator A. Since the bilinear form $E(\cdot,\cdot)$ satisfies (2.8), one can use the compact imbeddings

$$V^1 \hookrightarrow H \hookrightarrow V^{-1},$$

where V^{-1} is the collection of all bounded linear functionals on V^1, with the Lax-Milgram theory, to define the domain $\mathcal{D}(A)$ and the Stokes operator A. For a functional $w \in V^{-1}$, we denote the value of w at $v \in V^1$ by the nonsymmetric form $w(v) = \langle\langle v, w\rangle\rangle$. From the Riesz Representation Theorem, it follows that there is a bounded, linear, one-to-one mapping $B \in \mathcal{L}(V^1, V^{-1})$ of V^1 onto V^{-1}, such that

$$E(v, \widehat{v}) = \langle\langle v, B\,\widehat{v}\rangle\rangle, \qquad \text{for all } v,\, \widehat{v} \in V^1. \tag{2.12}$$

One then uses the identity (2.12) to define

$$\mathcal{D}(A) = \{\widehat{v} \in V^1 : B\,\widehat{v} \in H\}.$$

The Stokes operator A is the restriction of B to $\mathcal{D}(A)$, i.e., $A\,\widehat{u} = B\,\widehat{u}$, for all $\widehat{u} \in \mathcal{D}(A)$. One also obtains the compact imbeddings:

$$\mathcal{D}(A) \hookrightarrow V^1 \hookrightarrow H \hookrightarrow V^{-1}. \tag{2.13}$$

As a result of the Lax-Milgram theory, see [41], for example, one knows that the operator A is a positive, selfadjoint operator on H, and that A has compact resolvent. By using (2.4), one readily obtains

$$E(u, v) = -\langle \Delta u, v\rangle_{L^2(\Omega)}, \quad \text{for all } u \in V^2 \text{ and } v \in V^1.$$

With $v \in V^1$ one has $\mathbb{P}\,v = v$, and

$$E(u, v) = \langle \mathbb{P}\,(-\Delta u), v\rangle_{L^2(\Omega)}, \quad \text{for all } u \in V^2 \text{ and } v \in V^1.$$

From the regularity theory for the Stokes operator, see [42], one finds that

$$V^2 = \mathcal{D}(A) \quad \text{and} \quad Au = \mathbb{P}(-\Delta u), \qquad \text{for } u \in \mathcal{D}(A). \tag{2.14}$$

Moreover, the fractional powers A^r are well-defined, for all $r \in \mathbb{R}$. We let $V^{2r} = \mathcal{D}(A^r)$, for $r \geq 0$. One has $V^1 = \mathcal{D}(A^{\frac{1}{2}})$ and

$$\|A^{\frac{1}{2}} u\|^2_{L^2(\Omega)} = E(u,u) = 2\,\|Du\|^2_{L^2(\Omega)}, \quad \text{for } u \in V^1. \tag{2.15}$$

2.2 The Domain $\Omega_3 = \Omega^\varepsilon$

The considerations of the last section apply to the Navier–Stokes equations on the nearly flat domain $\Omega_3 = \Omega^\varepsilon$, for each $\varepsilon \in (0,1]$. One of the main problems we face in this article is to show that various constants, such as the constant $\widehat{C}$ appearing in (2.8), can be chosen to be independent of ε, see (4.31), for example. Because of this and the periodicity condition, necessary modifications are made for the definitions of the spaces V^1 and H described in (2.9) and (2.10). First of all, we characterize the space Z_0 defined by (2.7).

Lemma 2.1. *A vector field u belongs to Z_0 if and only if $u = (a_1, a_2, 0)$ satisfying*

$$(a_1, a_2) \cdot \nabla_2\, g_j(x') = 0 \qquad \textit{for all } x' \in \mathbb{T}^2, \ \textit{ and } j = 0, 1. \tag{2.16}$$

Proof. Let $u = a + b \times x$ be in Z_0, where a, $b \in \mathbb{R}^3$. Because of the periodicity in $x' \in \mathbb{T}^2$, one finds that $b = 0$. Let $a = (a_1, a_2, a_3)$ and set $\overline{a} = (a_1, a_2)$. Since $a \cdot N = 0$ on Γ, it follows that, see (4.21),

$$a_3 = \varepsilon(\overline{a} \cdot \nabla_2\, g_j) \text{ on } \Gamma_j, \qquad \text{for both } j = 0, 1. \tag{2.17}$$

We now integrate the last equation over the torus $\mathbb{T}^2$ to obtain

$$a_3 = \int_{\mathbb{T}^2} a_3\, dx' = \varepsilon \int_{\mathbb{T}^2} (\overline{a} \cdot \nabla_2\, g_j)\, dx'.$$

Since $\int_{\mathbb{T}^2} \partial_i\, g_j\, dx' = 0$, for $i = 1, 2$ and $j = 0, 1$, one finds that $a_3 = 0$. Furthermore, relation (2.16) is valid.

Conversely, assume that $u = (a_1, a_2, a_3)$ with $a_3 = 0$ and (2.16) holds. Then (2.17) is valid and one has $u \cdot N = 0$ on Γ. Hence, $u \in Z_0$. □

The above lemma gives

$$Z_0 = \{u = (a_1, a_2, 0) \in \mathbb{R}^3 \,:\, (a_1, a_2) \cdot \nabla_2 g_j = 0 \text{ on } \mathbb{T}^2,\ j = 0, 1\}. \tag{2.18}$$

If $u = (a_1, a_2, 0)$ is in Z_0, then by subtracting one equation from the other in (2.16), one obtains

$$(a_1, a_2) \cdot \nabla_2\, g(x') = 0, \quad \text{for all } x' \in \mathbb{T}^2.$$

Therefore $\boxed{Z_0 \subset Z_1}$ where

$$Z_1 \stackrel{\text{def}}{=} \{u = (a_1, a_2, 0) \in \mathbb{R}^3 \,:\, (a_1, a_2) \cdot \nabla_2\, g(x') \,=\, 0, \text{ for all } x' \in \mathbb{T}^2\}. \tag{2.19}$$

Note that we can consider Z_0 and Z_1 as subspaces of $L^2(\Omega^\varepsilon)^3$, for any $\varepsilon \in (0, 1]$.

By virtue of the Uniform Korn Inequality (see Proposition 4.12 and Remark 4.15 below), it turns out that the space Z_1 is more appropriate for our study as $\varepsilon \to 0$. Therefore, we make the following important assumption:

Domain Assumption: We assume throughout that

$$Z_0 = Z_1. \tag{2.20}$$

With this Domain Assumption, the spaces H, V^1 and V^2 for the domain $\Omega_3 = \Omega^\varepsilon$ are defined by (2.10), (2.9) and (2.11), the same way as for the domain Ω_2.

For the discussion on the Domain Assumption, see Remark 4.16.

Remark 2.2. For domain Ω_2, see (1.14), which is studied in [21], the Domain Assumption is satisfied.

Remark 2.3. When $Z_1 \neq \{0\}$ there is a scalar field $\phi = \phi(x')$ on $\mathbb{T}^2$ such that

$$g(x_1, x_2) \,=\, \phi(a_2\, x_1 - a_1\, x_2), \qquad \text{for all } x' = (x_1, x_2) \in \mathbb{T}^2.$$

If $\phi \,=\,$ const, then dim $Z_1 \,=\, 2$. Otherwise, one has dim $Z_1 \,=\, 1$.

We note that there is an alternate, but fully equivalent, way to view functions or vector fields on the domain Ω_3. One begins with the unbounded domain

$$\Omega_4 \,=\, \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, h_0(x_1, x_2) < x_3 < h_1(x_1, x_2)\}.$$

A vector field $u \,=\, u(x', x_3)$ on Ω_4 is then restricted to be periodic in $x' = (x_1, x_2) \in \mathbb{R}^2$, i.e., $u(x' + n', x_3) = u(x', x_3)$, for all $n' \in \mathbb{Z}^2$. Thus the vector field is defined on Ω_3. Our choice of the factor group $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \cong (0, 1)^2$ is made only to simplify the notation. Other geometries, such as rectangles, parallelograms, or equilateral triangles also arise in this way, when one replaces $\mathbb{Z}^2$ with other tilings of the plane $\mathbb{R}^2$, see for example, [33, 23, 30].

Remark 2.4. For $u \in H^1(\Omega_3)^3$, with $D(u) = 0$ one has $u = a + b \times x$, $x \in \Omega_4$, as a vector field on Ω_4. Due to the periodicity of u, the vector b must be zero.

The Ubiquitous C. We will use the symbol C to denote a local variable in the sense that it may change from line to line, sometimes more often. We interpret C as a positive "constant", independent of ε.

3 Main Results

The Main Results, which we present here, consist of The Main Theorem on global existence, with two corollaries, along with two theorems on the global attractors for the problem. For the remainder of this article we let (f, u_0) denote the data of the Navier–Stokes equations (1.1) on $\Omega_3 = \Omega^\varepsilon$. We assume, in the following Theorem 3.1, Corollary 3.2 and Corollary 3.3, that that f satisfies

Forcing Function Assumption I: $f(t)$ is orthogonal to Z_1 in $L^2(\Omega^\varepsilon)^3$, i.e. $f(t) \in Z_1^\perp$, for all $t \geq 0$.

Next we let

$$\mathbb{K} = \mathbb{K}(p, q, r, s) = (\, k_{0,p},\, k_{1,q},\, K_{0,r},\, K_{1,s}\,), \tag{3.1}$$

where $k_{0,p}$, $k_{1,q}$, $K_{0,r}$, and $K_{1,s}$ are positive parameters, and p, q, r, s denote nonnegative numbers. In the Main Theorem stated below, we assume that the data (f, u_0) satisfy:

$$\begin{aligned} \|M_2\, u_0\|_{L^2}^2 \,&\leq\, k_{0,p}^2\, \varepsilon^p\ , \quad \|u_0\|_{H^1}^2 \,\leq\, k_{1,q}^2\, \varepsilon^{-1+q}, \\ \|M_2\, \mathbb{P}\, f\|_\infty^2 \,&\leq\, K_{0,r}^2\, \varepsilon^r\ , \quad \|(I - M_2)(\mathbb{P}\, f)\|_\infty^2 \,\leq\, K_{1,s}^2\, \varepsilon^{-1+s}, \end{aligned} \tag{3.2}$$

where M_2 is an orthogonal projection on $L^2(\Omega^\varepsilon)^3$, see (4.67). Also, we define

$$|m|_\varepsilon^2 \stackrel{\text{def}}{=\!=} (k_{0,p}^2\, \varepsilon^p + k_{1,q}^2\, \varepsilon^{1+q}), \quad |\widehat{m}|_\varepsilon^2 \stackrel{\text{def}}{=\!=} (k_{0,p}^2\, \varepsilon^p + k_{1,q}^2\, \varepsilon^q), \tag{3.3}$$

$$|\ell|_\varepsilon^2 \stackrel{\text{def}}{=\!=} (K_{0,r}^2\, \varepsilon^r + K_{1,s}^2\, \varepsilon^s), \tag{3.4}$$

The quantities $|\widehat{m}|_\varepsilon^2$ and $|\ell|_\varepsilon^2$ are used later as ε-dependent measures of the size of u_0 and f, respectively. With $c_1 > 0$ given by the Uniform Korn Inequality (4.12), we define α by

$$4\,\alpha = c_1. \tag{3.5}$$

Theorem 3.1 (**Main Theorem**)**.** *Let* $\mathbb{K}$, $|m|_\varepsilon^2$, $|\widehat{m}|_\varepsilon^2$, *and* $|\ell|_\varepsilon^2$ *be given as above, where* p, q, r, s *are arbitrary nonnegative numbers. Then there are* $\varepsilon_1 \in (0, 1]$ *and* $R_0^2 > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_1$, *and*

$$|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2 \,\leq\, R_0^2, \tag{3.6}$$

and the data $(f, u_0) \in L^\infty(L^2) \times V^1$ *satisfy* (3.2)*, then the Navier-Stokes equations* (1.1) *has a unique, strong solution* $u(t)$ *on* $[0, \infty)$ *satisfying* $u(0) = u_0$. *Moreover, there are positive constants* D_1^2 *and* D_2^2, *which do not depend on* ε, *such that one has*

$$\|u(t)\|_{H^1}^2 \,\leq\, \varepsilon^{-1}\, D_1^2\, \big(|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2\big), \quad \textit{for all } 0 \leq t < 1, \tag{3.7}$$

$$\|u(t)\|_{H^1}^2 \,\leq\, \varepsilon^{-1}\, D_1^2\, \big(e^{-2\alpha t}\, |m|_\varepsilon^2 + |\ell|_\varepsilon^2\big), \quad \textit{for all } t \geq 1, \tag{3.8}$$

$$\int_{t-1}^{t} \|u(\tau)\|_{H^2}^2\, d\tau \,\leq\, \varepsilon^{-1}\, D_2^2\, (|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2), \quad \textit{for all } 1 \leq t < 2, \tag{3.9}$$

$$\int_{t-1}^{t} \|u(\tau)\|_{H^2}^2\, d\tau \,\leq\, \varepsilon^{-1}\, D_2^2\, (e^{-2\alpha t} |m|_\varepsilon^2 + |\ell|_\varepsilon^2), \quad \textit{for all } t \geq 2. \tag{3.10}$$

The next two corollaries of the Main Theorem are of special interest. First, we set $p = q = r = s = 0$ and obtain the following generalization of a result in [21]. The corollary specifies the largest size of the data, obtained by our theory, so that the global strong solutions exist. Note that the inequality (3.11) imposes a smallness condition on $\mathbb{K} = \mathbb{K}_0$ below.

Corollary 3.2. *Let $p = q = r = s = 0$ and $\mathbb{K} = \mathbb{K}_0 = (k_{0,0}, k_{1,0}, K_{0,0}, K_{1,0})$. Assume that*

$$|\mathbb{K}_0|^2 = k_{0,0}^2 + k_{1,0}^2 + K_{0,0}^2 + K_{1,0}^2 \le R_0^2, \tag{3.11}$$

Suppose $0 < \varepsilon \le \varepsilon_1$ and $(f, u_0) \in L^\infty(L^2) \times V^1$ satisfy

$$\begin{aligned} \|M_2\, u_0\|_{L^2}^2 \;&\le\; k_{0,0}^2, \quad \|u_0\|_{H^1}^2 \;\le\; k_{1,0}^2\, \varepsilon^{-1} \\ \|M_2\, \mathbb{P}\, f\|_\infty^2 \;&\le\; K_{0,0}^2, \quad \|(I - M_2)(\mathbb{P}\, f)\|_\infty^2 \;\le\; K_{1,0}^2\, \varepsilon^{-1}. \end{aligned} \tag{3.12}$$

Then the strong solution $u(t)$ exists for all $t \ge 0$. Moreover, for $t \ge 2$, one has

$$\|u(t)\|_{H^1}^2 \;\le\; \varepsilon^{-1}\, D_1^2\, \big(e^{-2\alpha t}(k_{0,0}^2 + k_{1,0}^2) + K_{0,0}^2 + K_{1,0}^2\big), \tag{3.13}$$

$$\int_{t-1}^{t} \|u(\tau)\|_{H^2}^2\, d\tau \;\le\; \varepsilon^{-1}\, D_2^2\, \big(e^{-2\alpha t}(k_{0,0}^2 + k_{1,0}^2) + K_{0,0}^2 + K_{1,0}^2\big). \tag{3.14}$$

Next we set $r = s = 1$ and $p = q = 0$. The goal is to obtain a better estimate for $\|u(t)\|_{H^1}^2$, for large t, which is independent of ε, see (3.17). This corollary will be used when we study the behavior of the global attractor for small $\varepsilon > 0$. The hypotheses used here insure that the expressions $M_2 \mathbb{P} f$ and $(I - M_2)(\mathbb{P}\, f)$ can be large, but not too large - in appropriate norms, as $\varepsilon \to 0$.

Corollary 3.3. *Let $r = s = 1$, $p = q = 0$ and $\mathbb{K} = \mathbb{K}_1 = (k_{0,0}, k_{1,0}, K_{0,1}, K_{1,1})$. Let $\mathbb{K}_0$ and R_0^2 be given by Corollary 3.2. Assume that $0 < \varepsilon \le \varepsilon_1$,*

$$k_{0,0}^2 + k_{1,0}^2 + \varepsilon(K_{0,1}^2 + K_{1,1}^2) \le R_0^2, \tag{3.15}$$

and $(f, u_0) \in L^\infty(L^2) \times V^1$ satisfy

$$\begin{aligned} \|M_2\, u_0\|_{L^2}^2 \;&\le\; k_{0,0}^2, \quad \|u_0\|_{H^1}^2 \;\le\; k_{1,0}^2\, \varepsilon^{-1} \\ \|M_2\, \mathbb{P}\, f\|_\infty^2 \;&\le\; K_{0,1}^2\, \varepsilon\ , \quad \|(I - M_2)(\mathbb{P}\, f)\|_\infty^2 \;\le\; K_{1,1}^2. \end{aligned} \tag{3.16}$$

Then the strong solution $u(t)$ exists for all $t \ge 0$, and the relations (3.13), (3.14) hold, with $\varepsilon(K_{0,1}^2 + K_{1,1}^2)$ replacing $(K_{0,0}^2 + K_{1,0}^2)$. In particular there is $T_1 > 1$ such that for $t \ge T_1$, one has

$$\|u(t)\|_{H^1}^2 \;\le\; 2\, D_1^2\, (K_{0,1}^2 + K_{1,1}^2), \tag{3.17}$$

$$\int_{t-1}^{t} \|u(\tau)\|_{H^2}^2\, d\tau \;\le\; 2\, D_2^2\, (K_{0,1}^2 + K_{1,1}^2). \tag{3.18}$$

Remark 3.4. It should be noted that, while the entries in $\mathbb{K}_0$ may be small - due to inequality (3.11) - the entries $K_{0,1}$ and $K_{1,1}$ in $\mathbb{K}_1$ may be (very) large for small $\varepsilon > 0$ - due to (3.15). Indeed, for arbitrarily large $K_{0,1}$ and $K_{1,1}$, there is $\varepsilon_2 \in (0, \varepsilon_1]$ such that (3.15) holds for all $\varepsilon \in (0, \varepsilon_2]$.

Remark 3.5. Another application of the Main Theorem occurs with $r = s = p = 1$. In this case, the inequality $\|M_2\, u_0\|^2_{L^2(\Omega^\varepsilon)} \le k^2_{0,0}$, which appears in (3.16), is replaced by $\|M_2\, u_0\|^2_{L^2(\Omega^\varepsilon)} \le k^2_{0,1}\varepsilon$. The rationale behind this assumption is that the term $M_2 u_0$ does not depend on x_3, see (4.67) and (4.59). Therefore, for small $\varepsilon > 0$, one has

$$\|M_2\, u_0\|^2_{L^2(\Omega^\varepsilon)} \;=\; C\,\varepsilon \|M_2\, u_0\|^2_{L^2(\mathbb{T}^2)} \;\le\; k^2_{0,1}\,\varepsilon.$$

If small $\varepsilon > 0$ is chosen so that $k^2_{0,1}\varepsilon \le k^2_{0,0}$, then one can invoke Corollary 3.3 with $\|M_2\, u_0\|^2_{L^2(\mathbb{T}^2)} \;=\; C\,\varepsilon^{-1}$. Thus, in this application, $\|M_2\, u_0\|^2_{L^2(\mathbb{T}^2)}$ may be chosen to be very large.

Next, we describe several aspects of the theory of global attractors of the weak and strong solutions of (1.1). First, we denote by CL^∞ the space $L^\infty(\mathbb{R}, L^2) \cap C(\mathbb{R}, L^2)$, endowed with the ucbs-topology, that is, the topology of "uniform convergence on bounded sets" of $\mathbb{R}$, see [35] and [40]. The norm of any function f in CL^∞ is defined by

$$\|f\|_\infty \stackrel{\mathrm{def}}{=\!=} \sup_{t\in\mathbb{R}} \|f(t)\|_{L^2}.$$

In the following Theorems 3.6 and 3.7, we assume that:

Forcing Function Assumption II: The forcing function f belongs to a set $\mathcal{H} = \mathcal{H}_\varepsilon$, where $\mathcal{H}_\varepsilon$ is a compact, time-translation invariant subset of CL^∞, and $f(t)$ is orthogonal to Z_1 in $L^2(\Omega^\varepsilon)^3$, i.e. $f(t) \in Z_1^\perp$, for all $t \in \mathbb{R}$.

One then obtains a family of equations:

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f; \quad \nabla\cdot u = 0, \qquad \text{where } f \in \mathcal{H}. \tag{3.19}$$

Reference is made below to the concept of the global attractor of the Leray-Hopf weak solutions of the Navier–Stokes equations, in the sense of [39], also see [9] and [41].

Denote by $\mathcal{M}$ the set of data $(f, u_0) \in \mathcal{H} \times V^1$ such that the strong solution $u(t)$ exists for all $t \ge 0$. The constant $d_1^2 \ge 1$, which is used below, arises in Lemma 6.1.

Theorem 3.6 (Attractor I Theorem)**.** *Let p, q, r, s, the vector $\mathbb{K}$ and ε be as in in the Main Theorem 3.1. Assume that*

$$\|M_2\,\mathbb{P}\, f\|^2_\infty \;\le\; K^2_{0,r}\,\varepsilon^r\;, \quad \|(I - M_2)(\mathbb{P}\, f)\|^2_\infty \;\le\; K^2_{1,s}\,\varepsilon^{-1+s}, \tag{3.20}$$

holds uniformly for $f \in \mathcal{H}$. Let

$$B_0^* \;=\; \{u \in V^1 \;:\; \|M_2 u\|^2_{L^2} \;\le\; k^2_{0,p}\varepsilon^p \quad \textit{and} \quad \|u\|^2_{H^1} \;\le\; k^2_{1,q}\,\varepsilon^{-1+q}\}, \tag{3.21}$$

$$B_1^* \;=\; \{u \in V^1 \;:\; \|u\|^2_{H^1} \;\le\; \varepsilon^{-1}\, D_1^2\, |\ell|^2_\varepsilon \quad \textit{and} \quad \|u\|^2_{L^2} \;\le\; d_1^2\, |\ell|^2_\varepsilon\}. \tag{3.22}$$

Then the set $\mathcal{H} \times B_0^*$ is contained in $\mathcal{M}$, the orbit $\gamma^+(\mathcal{H} \times B_0^*)$ is a bounded set in $\mathcal{H} \times V^1$, and the omega limit set $\mathcal{K}_\varepsilon = \omega(\mathcal{H} \times B_0^*)$ is a nonempty, compact, *invariant set in $\mathcal{M}$ that attracts $\mathcal{H} \times B_0^*$. Furthermore, the set $\mathcal{K}_\varepsilon$ is a subset of $\mathcal{H} \times (B_1^* \cap V^2)$.*

Assume in addition that $\mathbb{K}$ and ε satisfy

$$d_3^2\,|\ell|_\varepsilon^2 \,<\, \min(\,k_{0,p}^2\,\varepsilon^p,\; k_{1,q}^2\,\varepsilon^q\,), \tag{3.23}$$

where d_3^2 is a fixed positive number. Then $\mathfrak{A}_\varepsilon = \mathcal{K}_\varepsilon$ is an attractor of the strong solutions of (3.19) in $\mathcal{H} \times V^1$, and the basin of attraction $B(\mathfrak{A}_\varepsilon)$ contains the set $\mathcal{H} \times B_0^*$. Also, one has

$$\mathfrak{A}_\varepsilon \,\subset\, \mathcal{H} \times \Big\{ u \in V^1 \,:\, \|u\|_{H^1}^2 \,\le\, D_1^2\,(K_{0,r}^2\varepsilon^{r-1} + K_{1,s}^2\varepsilon^{s-1}) \Big\}. \tag{3.24}$$

Moreover, $\mathfrak{A}_\varepsilon$ is also the global attractor for the Leray-Hopf weak solutions and the globally defined strong solutions.

One might expect, as is the case in other theories, that for small $\varepsilon > 0$, the H^1-bound on the global attractor $\mathfrak{A}_\varepsilon$ may be independent of ε. Indeed, this is the case when the body forces have appropriate sizes ($r = s = 1$) as shown in the following theorem.

Theorem 3.7 (Attractor II Theorem)**.** *Let $r = s = 1$ and let positive numbers $K_{0,1}$ and $K_{1,1}$ be given. There is $\varepsilon_3 > 0$ such that if $\varepsilon \in (0, \varepsilon_3]$ and the property*

$$\|M_2\,\mathbb{P}\,f\|_\infty^2 \,\le\, K_{0,1}^2\,\varepsilon, \quad \|(I - M_2)(\mathbb{P}\,f)\|_\infty^2 \,\le\, K_{1,1}^2, \tag{3.25}$$

holds uniformly for $f \in \mathcal{H}$, then the global attractor $\mathfrak{A}_\varepsilon$ for globally defined strong solutions exists and satisfies

$$\mathfrak{A}_\varepsilon \,\subset\, \mathcal{H} \times \Big\{ u \in V^1 \,:\, \|u\|_{H^1}^2 \,\le\, D_1^2\,(K_{0,1}^2 + K_{1,1}^2) \Big\}. \tag{3.26}$$

Remark 3.8. A related theory of the global attractor for the nonautonomous Navier–Stokes equations appears in [35].

4 Fundamental Issues

In this section we present a number of basic lemmas which form the building blocks for the general theory of global existence and longtime dynamics of the Navier–Stokes equations on $\Omega_3 = \Omega^\varepsilon$, as is developed in Section 6.

4.1 Auxiliary Inequalities

We present here auxiliary inequalities for thin domains. Our objective is to derive the explicit dependencies on the parameter ε. The proofs of these lemmas are given in the Appendix.

Lemma 4.1 (Poincaré - Trace I)**.** *Let $r \in [1, \infty)$ and $\phi \in W^{1,r}(\Omega^\varepsilon)$. Then*

$$\|\phi\|_{L^r(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{r}}\|\phi\|_{L^r(\Gamma_j)} + C\varepsilon\|\partial_3\phi\|_{L^r(\Omega^\varepsilon)}, \qquad j = 0, 1, \tag{4.1}$$

$$\|\phi\|_{L^r(\Gamma)} \le C\varepsilon^{-\frac{1}{r}}\|\phi\|_{L^r(\Omega^\varepsilon)} + C\varepsilon^{1-\frac{1}{r}}\|\partial_3\phi\|_{L^r(\Omega^\varepsilon)}. \tag{4.2}$$

In particular, for $\phi \in H^1(\Omega^\varepsilon)$, we have

$$\|\phi\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}\|\phi\|_{L^2(\Gamma_j)} + C\varepsilon\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}, \qquad j = 0, 1, \tag{4.3}$$

$$\|\phi\|_{L^2(\Gamma)} \leq C\,\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)} + C\,\varepsilon^{\frac{1}{2}}\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}. \tag{4.4}$$

Lemma 4.2 (Poincaré - Trace II)**.** *Let $r \in [1, \infty)$ and $\phi \in W^{1,r}(\Omega^\varepsilon)$ satisfying*

$$\int_{h_0(x')}^{h_1(x')} \phi(x', y_3)\, dy_3 = 0, \qquad \textit{for } x' = (x_1, x_2) \in \mathbb{T}^2. \tag{4.5}$$

Then one has

$$\|\phi\|_{L^r(\Omega^\varepsilon)} \leq C\varepsilon\|\partial_3\phi\|_{L^r(\Omega^\varepsilon)}, \tag{4.6}$$

$$\|\phi\|_{L^r(\Gamma)} \leq C\varepsilon^{1-\frac{1}{r}}\|\partial_3\phi\|_{L^r(\Omega^\varepsilon)}. \tag{4.7}$$

In particular when $r = 2$, one obtains

$$\|\phi\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}, \tag{4.8}$$

$$\|\phi\|_{L^2(\Gamma)} \leq C\varepsilon^{\frac{1}{2}}\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}. \tag{4.9}$$

Lemma 4.3 (Ladyzhenskaya Inequality)**.** *Let $\varphi = \varphi(x_1, x_2)$ be independent of the third variable. Then one has*

$$\|\varphi\|_{L^4(\Omega^\varepsilon)} \leq C\varepsilon^{-\frac{1}{4}}\|\varphi\|^{1/2}_{L^2(\Omega^\varepsilon)}\|\varphi\|^{1/2}_{H^1(\Omega^\varepsilon)}.$$

Lemma 4.4 (GLNS[1] Inequality)**.** *For $\phi \in H^1(\Omega^\varepsilon)$, one has*

$$\|\phi\|_{L^6(\Omega^\varepsilon)} \leq C\varepsilon^{-\frac{1}{6}}\|\phi\|^{2/3}_{H^1(\Omega^\varepsilon)}\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}\Big\}^{1/3}. \tag{4.10}$$

If in addition, ϕ satisfies $\int_{h_0(x')}^{h_1(x')} \phi(x', y_3)dy_3 = 0$, for all $x' = (x_1, x_2) \in \mathbb{T}^2$, then one has

$$\|\phi\|_{L^6(\Omega^\varepsilon)} \leq C\|\phi\|_{H^1(\Omega^\varepsilon)}, \tag{4.11}$$

and in general

$$\|\phi\|_{L^r(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{3}{r}-\frac{1}{2}}\|\phi\|_{H^1(\Omega^\varepsilon)}, \qquad \textit{for } 2 \leq r \leq 6. \tag{4.12}$$

Lemma 4.5 (Agmon Inequality)**.** *Let $\phi \in H^2(\Omega^\varepsilon)$, then*

$$\begin{aligned}\|\phi\|_{L^\infty(\Omega^\varepsilon)} \leq C\varepsilon^{-\frac{1}{2}}&\|\phi\|^{1/4}_{L^2(\Omega^\varepsilon)}\|\phi\|^{1/2}_{H^2(\Omega^\varepsilon)}\\ &\Big\{\|\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon^2\|\partial_3\partial_3\phi\|_{L^2(\Omega^\varepsilon)}\Big\}^{1/4}.\end{aligned} \tag{4.13}$$

If, in addition, $\int_{h_0(x')}^{h_1(x')} \phi(x', y_3)dy_3 = 0$ for all $x' \in \mathbb{T}^2$, then

$$\|\phi\|_{L^\infty(\Omega^\varepsilon)} \leq C\|\partial_3\phi\|^{1/4}_{L^2(\Omega^\varepsilon)}\|\phi\|^{1/2}_{H^2(\Omega^\varepsilon)}(\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon\|\partial_3\partial_3\phi\|_{L^2(\Omega^\varepsilon)})^{1/4}. \tag{4.14}$$

[1] Garliagdo-Ladyzhenskaya-Nirenberg-Sobolev

4.2 Boundary Behavior

In this section, we interpret the Navier boundary conditions and show how they yield new estimates in the context of thin domains. In particular, we first examine the interactions between the two Navier boundary conditions, (1.12) and (1.13), and the normal and tangent vector fields along the boundary Γ. We also present the proof of a uniform Korn inequality, see Proposition 4.12.

In this paper we use notation $\partial/\partial\tau$ where $\tau \in \mathbb{R}^3$, $\tau \neq 0$, to denote the Gateaux derivative with respect to vector τ. For example, one has

$$\frac{\partial u}{\partial \tau} = \lim_{h\to 0} \frac{1}{h}(u(x+h\tau) - u(x)).$$

If $|\tau| = 1$, then this is the usual directional derivative.

4.2.1 General Properties

We present here a reformulation of the two Navier boundary conditions (1.11) in terms of directional derivatives on the boundary Γ. Let $\Omega_3 = \Omega^\varepsilon$ be given as above. It should be noted that in the following result, we do not require that either of the vector fields $\widehat{N}$ or $\widehat{\tau}$ be normalized, that is, $|\widehat{N}| = 1$ or $|\widehat{\tau}| = 1$.

Proposition 4.6. *Let $\mathcal{O}$ be an open subset of $\mathbb{T}^2 \times \mathbb{R}^1$ such that $\Gamma = \partial\Omega^\varepsilon \cap \mathcal{O}$ is nonempty. Let $\widehat{N}$ and $\widehat{\tau}$ be nonvanishing vector fields belonging to $C^1(\overline{\Omega^\varepsilon} \cap \mathcal{O}, \mathbb{R}^3)$ with the property that the restrictions $\widehat{N}\big|_\Gamma$ and $\widehat{\tau}\big|_\Gamma$ are respectively, normal to and tangent to Γ. Assume that u belongs to $C^1(\overline{\Omega^\varepsilon} \cap \mathcal{O}, \mathbb{R}^3)$ and satisfies the slip boundary condition* (1.12) *on Γ. Then one has*

$$\frac{\partial u}{\partial \widehat{\tau}} \cdot \widehat{N} + \frac{\partial \widehat{N}}{\partial \widehat{\tau}} \cdot u = 0, \qquad \text{on} \quad \Gamma. \tag{4.15}$$

If, in addition, u satisfies the second Navier boundary condition (1.13)*, then one has*

$$\frac{\partial u}{\partial \widehat{N}} \cdot \widehat{\tau} = u \cdot \frac{\partial \widehat{N}}{\partial \widehat{\tau}}, \qquad \text{on} \quad \Gamma. \tag{4.16}$$

Proof. By taking the derivative $\partial/\partial\widehat{\tau}$ of equation (1.12), we immediately obtain (4.15). Assume next that u satisfies (1.13) so that on Γ, one has $[(Du)\widehat{N}]\cdot\widehat{\tau} = 0$. One then obtains

$$0 = (\nabla u)\widehat{N} \cdot \widehat{\tau} + (\nabla u)^{\mathrm{t}} \widehat{N} \cdot \widehat{\tau} = \frac{\partial u}{\partial \widehat{N}} \cdot \widehat{\tau} + \widehat{N} \cdot (\nabla u)\widehat{\tau},$$

on Γ, which in turn yields

$$\frac{\partial u}{\partial \widehat{N}} \cdot \widehat{\tau} + \frac{\partial u}{\partial \widehat{\tau}} \cdot \widehat{N} = 0, \qquad \text{on} \quad \Gamma. \tag{4.17}$$

Therefore (4.16) follows from (4.15) and (4.17). □

When $u = (u_1, u_2, u_3)$ satisfies (1.12) on Γ, one has:

$$u_3 = \varepsilon \Big(u_1\, \partial_1 g_j + u_2\, \partial_2 g_j \Big) \qquad \text{on} \quad \Gamma_j, \qquad \text{for } j = 0, 1. \tag{4.18}$$

It follows that

$$|u_3| \le C\varepsilon(|u_1| + |u_2|), \qquad \text{on} \quad \Gamma. \tag{4.19}$$

One now obtains the following Poincaré-like inequality, even though u_3 does not satisfy the Dirichlet boundary condition on the top or bottom, nor does u_3 satisfy the zero vertical average condition (4.5).

Lemma 4.7. *Let u be in $H^1(\Omega^\varepsilon)^3$ and satisfy the slip boundary condition* (1.12) *on Γ. Then one has*

$$\|u_3\|_{L^2} \le C\varepsilon \|u\|_{H^1}. \tag{4.20}$$

Proof. By using (4.3), (4.19) and (4.4), one obtains

$$\begin{aligned} \|u_3\|_{L^2} &\le C\varepsilon^{\frac12} \|u_3\|_{L^2(\Gamma)} + C\varepsilon \|\partial_3 u_3\|_{L^2} \\ &\le C\varepsilon^{\frac32} \|(u_1, u_2)\|_{L^2(\Gamma)} + C\varepsilon \|\partial_3 u_3\|_{L^2} \\ &\le C\varepsilon^{\frac32} \Big(\varepsilon^{-\frac12} \|(u_1, u_2)\|_{L^2(\Omega^\varepsilon)} + C\varepsilon^{\frac12} \|\partial_3 (u_1, u_2)\|_{L^2(\Omega^\varepsilon)} \Big) + C\varepsilon \|\partial_3 u_3\|_{L^2}, \end{aligned}$$

which implies (4.20). □

In our study the boundary conditions, we need extensions of the normal and tangential vectors from Γ_j, for $j = 0, 1$, to the whole domain Ω^ε. For the extension of the (outward) normal vector field we use:

$$N^j = N^j(x) = \widehat{N}^j / |\widehat{N}^j|, \quad \text{where } \widehat{N}^j = (-1)^j (\partial_1 h_j, \partial_2 h_j, -1), \tag{4.21}$$

for $j = 0, 1$, where $x \in \mathbb{T}^2 \times \mathbb{R}$. For the two tangential vector fields on Γ and their extensions, let

$$\tau^{i,j} = \widehat{\tau}^{i,j} / |\widehat{\tau}^{i,j}|, \quad \text{where } \widehat{\tau}^{i,j} = e^i + \partial_i h_j\, e^3, \qquad \text{for } i = 1, 2, \quad j = 0, 1, \tag{4.22}$$

where $\{e^1, e^2, e^3\}$ is the standard basis of $\mathbb{R}^3$. For each $j \in \{0, 1\}$, on the surface Γ_j, the two vectors $\tau^{1,j}$ and $\tau^{2,j}$ form a basis of the tangent space. Note that the vector fields in (4.21) and (4.22) on Ω^ε are independent of x_3. From the formulas above, we have, for $i = 1, 2$ and $j = 0, 1$, the following estimates of the Euclidean norms over Ω^ε:

$$|\nabla N^j|,\ |\nabla \tau^{i,j}|,\ |e^3 - N^1|,\ |e^3 + N^0|,\ |e^i - \tau^{i,j}| \le C\, \varepsilon, \tag{4.23}$$

It is convenient in some cases to use an orthonormal frame field

$$\{\sigma^1, \sigma^2, \sigma^3\} = \{\sigma^1, \sigma^2, N\} \tag{4.24}$$

with $\sigma^1 = \sigma^{1,j}$, $\sigma^2 = \sigma^{2,j}$ and $N = N^j$, on each surface Γ_j, for $j = 0, 1$, of Γ. The orthonormality refers to the relation $\sigma^k(x) \cdot \sigma^m(x) = \delta_{k,m}$ in $\mathbb{R}^3$, for all $x \in \Gamma$. There are many ways of constructing such a frame. Here is one. We begin with the pair $\widehat{\tau}^1 = \widehat{\tau}^{1,j}$ and $\widehat{\tau}^2 = \widehat{\tau}^{2,j}$ on Γ_j. Define

$$\widehat{\sigma}^1 = \widehat{\tau}^1, \quad \widehat{\sigma}^2 = \widehat{\tau}^2 + \gamma \widehat{\tau}^1, \quad \text{and set} \quad \sigma^k = \widehat{\sigma}^k / |\widehat{\sigma}^k| \quad \text{for } k = 1, 2,$$

where γ is chosen so that $\widehat{\sigma}^1 \cdot \widehat{\sigma}^2 = 0$. Explicitly, $\gamma = -(\widehat{\tau}^1 \cdot \widehat{\tau}^2)|\widehat{\tau}^1|^{-2} = O(\varepsilon^2)$.

Similar to Lemma 4.7, we obtain in the following weak Ponicaré inequalities for a couple of particular entries of ∇u, where u satisfies the Navier boundary conditions.

Lemma 4.8. *Assume that u belongs to $H^2(\Omega^\varepsilon)^3$ and satisfies the Navier boundary conditions* (1.11) *on Γ. Then one has:*

$$\|\partial_3 u_i\|_{L^2} \le C\varepsilon \|u\|_{H^2}, \qquad \textit{for } i = 1, 2. \tag{4.25}$$

Proof. We prove the inequality for the case $i = 1$. The argument for $i = 2$ is similar and is omitted. Using (4.3) with $j = 0$ (the case $j = 1$ also yields a similar result), we have

$$\|\partial_3 u_1\|_{L^2} \le C\varepsilon^{\frac{1}{2}} \|\partial_3 u_1\|_{L^2(\Gamma_0)} + C\varepsilon \|\partial_3 \partial_3 u_1\|_{L^2}. \tag{4.26}$$

In order to estimate $\|\partial_3 u_1\|_{L^2(\Gamma_0)}$, we write

$$\begin{aligned}
\partial_3 u_1 &= (\nabla u_1) \cdot (e^3 + N^0) - (\nabla u_1) \cdot N^0 \\
&= (\nabla u_1) \cdot (e^3 + N^0) - ((\nabla u)^{\mathrm{t}} e^1) \cdot N^0 \\
&= (\nabla u_1) \cdot (e^3 + N^0) - ((\nabla u)^{\mathrm{t}} (e^1 - \tau^{1,0}) \cdot N^0 - ((\nabla u)^{\mathrm{t}} \tau^{1,0} \cdot N^0 \\
&= (\nabla u_1) \cdot (e^3 + N^0) - ((\nabla u)^{\mathrm{t}} (e^1 - \tau^{1,0}) \cdot N^0 - ((\nabla u) N^0) \cdot \tau^{1,0}.
\end{aligned}$$

Let $\widehat{\tau} = \tau^{1,0}$ and $\widehat{N} = N^0$ in (4.16), we have $((\nabla u)N^0) \cdot \tau^{1,0} = ((\nabla N^0)\tau^{1,0}) \cdot u$ on Γ_0. Then thanks to (4.23), the Euclidean norm satisfies: $|\partial_3 u_1| \le C\varepsilon(|\nabla u| + |u|)$ on Γ. Together with (4.26) and (4.4), we obtain:

$$\|\partial_3 u_1\|_{L^2} \le C\,\varepsilon^{\frac{1}{2}} \cdot \varepsilon(\varepsilon^{-\frac{1}{2}} \|u\|_{H^1} + \varepsilon^{\frac{1}{2}} \|u\|_{H^2}) + C\varepsilon \|\nabla^2 u\|_{L^2} \le C\varepsilon \|u\|_{H^2},$$

hence we obtain (4.25) with $i = 1$. □

Remark 4.9. In the case $g_0 = 0$ as in [21], one has $\partial_3 u_1 = 0$ on Γ_0. Then (4.25), with $i = 1$ immediately follows from (4.26).

4.2.2 Uniform Korn Inequality

When $\varepsilon = 1$ the domain Ω^ε becomes

$$\Omega^1 = \{(y_1, y_2, y_3) \in \mathbb{T}^2 \times \mathbb{R} : g_0(y_1, y_2) < y_3 < g_1(y_1, y_2)\}. \tag{4.27}$$

This domain is also considered as a dilation of the domain Ω^ε by using the change of variables $y = (x_1, x_2, \varepsilon^{-1} x_3)$, for $x \in \Omega^\varepsilon$.

First, we recall the Korn inequality for the fixed domain Ω^1. The proof can be found in [4, 42, 47], for example. According to Remark 2.4, the null space of $D(\cdot)$ in $H^1(\Omega^\varepsilon)^3$ is

$$Z_2 = \{\, u = (a_1, a_2, a_3) \in \mathbb{R}^3 \,\} \subset L^2(\Omega^\varepsilon)^3. \tag{4.28}$$

Proposition 4.10 (Korn inequality for Ω^1)**.** *There is $c^* > 0$ such that if $u \in H^1(\Omega^1)^3$ and is orthogonal in $L^2(\Omega^1)$ to the space Z_2, then*

$$\|Du\|_{L^2(\Omega^1)} \le \|u\|_{H^1(\Omega^1)} \le c^* \|Du\|_{L^2(\Omega^1)}. \tag{4.29}$$

Corollary 4.11 (Korn inequality for individual Ω^ε)**.** *There is $C > 0$ such that if $u \in H^1(\Omega^\varepsilon)^3$ satisfies the slip condition* (1.12) *on Γ and is orthogonal in $L^2(\Omega^\varepsilon)$ to the space Z_2, then*

$$\|u\|^2_{H^1(\Omega^\varepsilon)} \le C\varepsilon^{-4} \|Du\|^2_{L^2(\Omega^\varepsilon)}, \qquad \text{for } 0 < \varepsilon \le 1. \tag{4.30}$$

We will prove this corollary below. Since the factor $C\varepsilon^{-4}$ in (4.30) is unbounded, when $\varepsilon \to 0$, our next objective is to prove a uniform Korn inequality on the domain $\Omega_3 = \Omega^\varepsilon$. The proof of the next result is given below, as well.

Proposition 4.12 (UKI: Uniform Korn Inequality)**.** *There are numbers $\varepsilon_0 \in (0, 1]$ and $c_1 > 0$ such that*

$$c_1 \|u\|^2_{H^1} \;\le\; E(u, u) \;=\; 2\, \|Du\|^2_{L^2} \;\le\; 2\|u\|^2_{H^1}, \qquad \text{for } 0 < \varepsilon \le \varepsilon_0, \tag{4.31}$$

and for all $u \in H^1(\Omega^\varepsilon)^3$ that satisfies the slip boundary condition (1.12) *on Γ and that is orthogonal in $L^2(\Omega^\varepsilon)$ to the space Z_1 (see* (2.19)*).*

We define a mapping $\Phi_\varepsilon : L^2(\Omega^1) \to L^2(\Omega^\varepsilon)$ by

$$\Phi_\varepsilon \phi \;=\; \varphi, \ \text{for } \phi \in L^2(\Omega^1), \ \text{with } \varphi(x) = \phi(y), \tag{4.32}$$

where $x \in \Omega^\varepsilon$ and $y = (x_1, x_2, \varepsilon^{-1} x_3) \in \Omega^1$. Thus Φ_ε is an isomorphism of $L^2(\Omega^1)$ onto $L^2(\Omega^\varepsilon)$. Note that the L^2-norms satisfy

$$\|\phi\|^2_{L^2(\Omega^1)} \;=\; \|\Phi_\varepsilon^{-1} \varphi\|^2_{L^2(\Omega^1)} \;=\; \varepsilon^{-1}\, \|\varphi\|^2_{L^2(\Omega^\varepsilon)}. \tag{4.33}$$

Of course, one has $\varphi \in H^1(\Omega^\varepsilon)$ if and only if $\phi \in H^1(\Omega^1)$, and

$$\widetilde{\partial}_1 \phi \;=\; \partial_1 \varphi, \quad \widetilde{\partial}_2 \phi \;=\; \partial_2 \varphi, \quad \widetilde{\partial}_3 \phi \;=\; \varepsilon \partial_3 \varphi, \tag{4.34}$$

where $\widetilde{\partial}_i$ denotes $\partial/\partial y_i$ for $i = 1, 2, 3$.

We will use below the notation $\widetilde{\nabla} = (\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3)$ and $\widetilde{\nabla}_2 = (\widetilde{\partial}_1, \widetilde{\partial}_2)$, and set

$$(\widetilde{D}\, v)_{i,j} = \frac{1}{2}\big(\widetilde{\partial}_i\, v_j + \widetilde{\partial}_j\, v_i\big), \quad (\widehat{D}\, v)_{i,j} = \frac{1}{2}\big(\varepsilon^{-\delta_{i,3}} \widetilde{\partial}_i\, v_j + \varepsilon^{-\delta_{j,3}} \widetilde{\partial}_j\, v_i\big), \tag{4.35}$$

for $i, j = 1, 2, 3$, where $\delta_{k,3}$ is the Kronecker symbol, for $k = i, j$.

By using (4.34) and (4.33), one obtains

$$\begin{aligned}\|\nabla\varphi\|^2_{L^2(\Omega^\varepsilon)} &= \|\nabla_2\,\varphi\|^2_{L^2(\Omega^\varepsilon)} + \|\partial_3\,\varphi\|^2_{L^2(\Omega^\varepsilon)} \\ &= \varepsilon\,(\|\widetilde{\nabla}_2\,\phi\|^2_{L^2(\Omega^1)} + \varepsilon^{-2}\,\|\widetilde{\partial}_3\,\phi\|^2_{L^2(\Omega^1)}).\end{aligned} \tag{4.36}$$

Hence it follows that

$$\varepsilon\,\|\widetilde{\nabla}\,\phi\|^2_{L^2(\Omega^1)} \le \|\nabla\varphi\|^2_{L^2(\Omega^\varepsilon)} \le \varepsilon^{-1}\,\|\widetilde{\nabla}\,\phi\|^2_{L^2(\Omega^1)}. \tag{4.37}$$

Proof of Corollary 4.11: Let $v = (v_1, v_2, v_3) = \Phi_\varepsilon^{-1}(u_1, u_2, \varepsilon u_3)$. That is,

$$v(y_1, y_2, y_3) = (u_1, u_2, \varepsilon u_3)(y_1, y_1, \varepsilon y_3), \quad y \in \Omega^1,$$

or equivalently,

$$u(x_1, x_2, x_3) = (v_1, v_2, \varepsilon^{-1} v_3)(x_1, x_2, \varepsilon^{-1} x_3), \quad x \in \Omega^\varepsilon.$$

One can verify that $v \perp Z_2$ in $L^2(\Omega^1)$, that $\partial_i u_j = \widetilde{\partial}_i v_j$, for $i, j = 1, 2$, and

$$\partial_3 u_3 = \varepsilon^{-2}\widetilde{\partial}_3 v_3, \quad \partial_3 u_j = \varepsilon^{-1}\widetilde{\partial}_3 v_j,\ \partial_j u_3 = \varepsilon^{-1}\widetilde{\partial}_j v_3, \quad \text{for } j = 1, 2.$$

Hence by using (4.33) one finds

$$\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} \le \varepsilon^{-3}\|\widetilde{\nabla} v\|^2_{L^2(\Omega^1)}, \tag{4.38}$$

$$\begin{aligned}\|\widetilde{D} v\|^2_{L^2(\Omega^1)} = \varepsilon^{-1}\Big\{ &\sum_{i,j=1,2} \frac{1}{4}\|\partial_i u_j + \partial_j u_i\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon^4 \|\partial_3 u_3\|^2_{L^2(\Omega^\varepsilon)} \\ &+ \varepsilon^2 \sum_{j=1,2} \frac{1}{4}\|\partial_3 u_j + \partial_j u_3\|^2_{L^2(\Omega^\varepsilon)}\Big\} \le \varepsilon^{-1}\|Du\|^2_{L^2(\Omega^\varepsilon)}.\end{aligned} \tag{4.39}$$

Combining with the Korn inequality (4.29) in Ω^1, one obtains

$$\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} \le \varepsilon^{-3}\|\widetilde{\nabla} v\|^2_{L^2(\Omega^1)} \le C\varepsilon^{-3}\|\widetilde{D} v\|^2_{L^2(\Omega^1)} \le C\varepsilon^{-4}\|Du\|^2_{L^2(\Omega^\varepsilon)}.$$

Similarly,

$$\|u\|^2_{L^2(\Omega^\varepsilon)} \le \varepsilon^{-1}\|v\|^2_{L^2(\Omega^1)} \le C\varepsilon^{-1}\|\widetilde{D} v\|^2_{L^2(\Omega^1)} \le C\varepsilon^{-2}\|Du\|^2_{L^2(\Omega^\varepsilon)}.$$

Therefore one obtains (4.30) □

The first step in proving the inequality (4.31) is the following estimate of a boundary integral:

Lemma 4.13. *Given $\beta > 0$, there is a positive constant $C(\beta)$, which does not depend on ε, such that one has*

$$\left|\int_\Gamma \{(\nabla u)u\} \cdot N d\sigma\right| \le \beta\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} + C(\beta)\|u\|^2_{L^2(\Omega^\varepsilon)}, \tag{4.40}$$

for any $\varepsilon \in (0,1]$ *and for all* $u \in H^1(\Omega^\varepsilon)^3$ *that satisfies the slip boundary condition* (1.12) *on* Γ. *Consequently,*

$$\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} \le 4\|Du\|^2_{L^2(\Omega^\varepsilon)} + C_2\|u\|^2_{L^2(\Omega^\varepsilon)}, \tag{4.41}$$

where C_2 *is a positive constant which does not depend on* ε.

Proof. We use here the normal vector N^j given by (4.21). Note that one has $\sqrt{1+|\nabla_2 h_j|^2}\, N^j = \widehat{N}^j$. The integral of the left term in inequality (4.40) is rewritten as

$$\int_\Gamma \{(\nabla u)u\}\cdot N\, d\sigma = \sum_{j=0,1}\int_{\mathbb{T}^2}\{(\nabla u)u\}\cdot \widehat{N}^j\, dx'.$$

Next we apply the relation (4.15) to the integrand on the right with $\widehat{\tau} = u$ and $\widehat{N} = \widehat{N}^j$ for $j = 0, 1$, to obtain

$$\begin{aligned}\int_\Gamma \{(\nabla u)u\}\cdot N\, d\sigma &= -\sum_{j=0,1}\int_{\mathbb{T}^2} u\cdot\{(\nabla \widehat{N}^j)u\}\, dx' \\ &= \int_{\mathbb{T}^2}\Big(u_1^2\partial_1^2 h_0 + u_2^2\partial_2^2 h_0 + 2u_1u_2\partial_1\partial_2 h_0\Big)\, dx' \\ &\quad - \int_{\mathbb{T}^2}\Big(u_1^2\partial_1^2 h_1 + u_2^2\partial_2^2 h_1 + 2u_1u_2\partial_1\partial_2 h_1\Big)\, dx'.\end{aligned} \tag{4.42}$$

Define the 2×2 symmetric matrix-valued function $\Psi^\varepsilon = \Psi = \Psi_{ij}(x', x_3)$, for $i, j = 1, 2$, by

$$\begin{aligned}\Psi_{ij}(x', x_3) &= \frac{1}{\varepsilon g}\Big((x_3 - h_0)\partial_i\partial_j h_1 + (h_1 - x_3)\partial_i\partial_j h_0\Big) \\ &= \partial_i\partial_j h_0 + \frac{x_3 - h_0}{g}\partial_i\partial_j g.\end{aligned}$$

Let $\overline{u} = (u_1, u_2)$. By means of a straightforward calculation, one finds that equation (4.42) can now be rewritten as:

$$\begin{aligned}\int_\Gamma \{(\nabla u)u\}\cdot N\, d\sigma &= \int_{\mathbb{T}^2}\overline{u}\cdot\Psi\overline{u}\Big|_{x_3=h_0} dx' - \int_{\mathbb{T}^2}\overline{u}\cdot\Psi\overline{u}\Big|_{x_3=h_1} dx' \\ &= -\int_{\Omega^\varepsilon}\frac{\partial}{\partial x_3}\,(\overline{u}\cdot\Psi\overline{u})\, dx.\end{aligned} \tag{4.43}$$

Notice that one has $|\Psi_{ij}|, |\partial_3\Psi_{ij}| \le C$. With the Hölder inequality and using $L^2 = L^2(\Omega^\varepsilon)$, one finds that

$$\left|\int_\Gamma \{(\nabla u)u\}\cdot N\, d\sigma\right| \le C\,\|u\|_{L^2(\Omega^\varepsilon)}\|\nabla u\|_{L^2(\Omega^\varepsilon)} + C\,\|u\|^2_{L^2(\Omega^\varepsilon)},$$

which implies (4.40), by using the Young inequality.

To prove (4.41), we note (as in [21]) that the definition of Du yields

$$2\|D\,u\|^2_{L^2(\Omega^\varepsilon)} \;=\; \|\nabla u\|^2_{L^2(\Omega^\varepsilon)} + \sum_{1\le i,j\le 3}\int_{\Omega^\varepsilon}(\partial_i\,u_j)(\partial_j u_i)\,dx.$$

By integrating the last term by-parts twice and using the slip boundary condition (1.12), one obtains

$$\sum_{1\le i,j\le 3}\int_{\Omega^\varepsilon}(\partial_i\,u_j)(\partial_j\,u_i)\,dx \;=\; \int_{\Omega^\varepsilon}(\nabla\cdot u)^2\,dx \;+\; \int_\Gamma\{(\nabla u)u\}\cdot N\,d\sigma.$$

Since $\int_{\Omega^\varepsilon}(\nabla\cdot u)^2\,dx\;\ge 0$, it then follows from inequalities (4.40) (with particular value $\beta = 1/2$) that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} &\le\; 2\|Du\|^2_{L^2(\Omega^\varepsilon)} + \Big|\int_\Gamma\{(\nabla u)u\}\cdot N\,d\sigma.\Big|\\ &\le\; 2\|Du\|^2_{L^2(\Omega^\varepsilon)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega^\varepsilon)} + C\,\|u\|^2_{L^2(\Omega^\varepsilon)}.\end{aligned}$$

Thus (4.41) follows. □

The main step used in the proof of UKI is the next lemma.

Lemma 4.14. *Given $\beta > 0$, there exist $\varepsilon_0 \in (0,1]$ and $C(\beta) > 0$ such that*

$$\|u\|^2_{L^2} \;\le\; \beta\,\|\nabla u\|^2_{L^2} + C(\beta)\,\|D\,u\|^2_{L^2}, \tag{4.44}$$

for all $\varepsilon \in (0,\varepsilon_0]$ and for all $u \in H^1(\Omega^\varepsilon)^3 \cap Z_1^\perp$ that satisfies the slip boundary condition (1.12) *on* Γ.

Proof. Fix $\beta > 0$. Suppose on the contrary, that there are sequences $\varepsilon_n \in (0,1]$ and $u_n \in H^1(\Omega^{\varepsilon_n})^3 \cap Z_1^\perp$, with $\varepsilon_n \to 0$, as $n\to\infty$, and $u\cdot N = 0$ on $\partial\Omega^{\varepsilon_n}$ such that

$$\|u_n\|^2_{L^2(\Omega^{\varepsilon_n})} > \beta\,\|\nabla u_n\|^2_{L^2(\Omega^{\varepsilon_n})} + n\,\|D\,u_n\|^2_{L^2(\Omega^{\varepsilon_n})}. \tag{4.45}$$

By multiplying equation (4.45) by ε_n^{-1} and using the relations (4.33) and (4.37), one obtains:

$$\|v_n\|^2_{L^2(\Omega^1)} > \beta\,\|\widetilde\nabla_2\,v_n\|^2_{L^2(\Omega^1)} + \beta\,\varepsilon_n^{-2}\|\widetilde\partial_3\,v_n\|^2_{L^2(\Omega^1)} + n\,\|\widehat D\,v_n\|^2_{L^2(\Omega^1)}, \tag{4.46}$$

where $v_n = \Phi^{-1}_{\varepsilon_n}u_n$.

Without loss of generality, we assume $\|v_n\|_{L^2(\Omega^1)} = 1$. Then (4.46) becomes:

$$1 > \beta\,\|\widetilde\nabla_2\,v_n\|^2_{L^2(\Omega^1)} + \beta\,\varepsilon_n^{-2}\|\widetilde\partial_3\,v_n\|^2_{L^2(\Omega^1)} + n\|\widehat D\,v_n\|^2_{L^2(\Omega^1)}. \tag{4.47}$$

It follows that the sequence $\|\tilde u_n\|_{H^1(\Omega^1)}$ is bounded and that

$$\|\widetilde\partial_3\,v_n\|_{L^2(\Omega^1)} \to 0 \quad\text{and}\quad \|\widehat D\,v_n\|_{L^2(\Omega^1)} \to 0, \qquad \text{as } n\to\infty.$$

Since

$$\frac{1}{n} \geq \|\widehat{D}\, v_n\|^2_{L^2(\Omega^1)} \geq \varepsilon_n^{-2}\|\widetilde{\partial}_3\, v_{n,3}\|^2_{L^2(\Omega^1)},$$

we see that

$$\varepsilon_n^{-2}\, \|\widetilde{\partial}_3\, v_{n,3}\|^2_{L^2(\Omega^1)} \to 0, \qquad \text{as } n \to \infty. \tag{4.48}$$

Consequently, there are subsequences, which we relabel as ε_n and v_n, and a vector field $v \in H^1(\Omega^1)^3$, such that $v_n \to v$ strongly in $L^2(\Omega^1)$ and weakly in $H^1(\Omega^1)$. Furthermore, one has $\|v\|_{L^2(\Omega^1)} = 1$ and $\widetilde{\partial}_3\, v = 0$.

Let $\widetilde{\Gamma} = \widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1$, where $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are the bottom and top boundaries of Ω^1. Then v_n satisfies

$$v_{n,3} = \varepsilon_n (v_{n,1}\, \partial_1\, g_j + v_{n,2}\, \partial_2\, g_j), \qquad \text{on } \widetilde{\Gamma}_j, \quad j = 0, 1, \tag{4.49}$$

From (4.49), one obtains

$$\|v_{n,3}\|_{L^2(\widetilde{\Gamma})} \leq C\, \varepsilon_n \big(\|v_{n,1}\|_{L^2(\widetilde{\Gamma})} + \|\widetilde{u}_{n,2}\|_{L^2(\widetilde{\Gamma})}\big).$$

Next from inequality (4.4), with $\varepsilon = 1$, one finds that

$$\|v_{n,3}\|_{L^2(\widetilde{\Gamma})} \leq C\, \varepsilon_n \sum_{i=1,2} \big(\|v_{n,i}\|_{L^2(\Omega^1)} + \|\widetilde{\partial}_3 v_{n,i}\|_{L^2(\Omega^1)}\big).$$

Hence $\|v_{n,3}\|_{L^2(\widetilde{\Gamma})} \to 0$, as $n \to \infty$, and subsequently $v_3 = 0$ on $\widetilde{\Gamma}$. Since $\widetilde{\partial}_3\, v = 0$, one has $v_3 = 0$ in Ω^1. Since $\|\widehat{D}_n\, v_n\|_{L^2(\Omega^1)} \to 0$, as $n \to \infty$, one has $\widetilde{\partial}_1\, v_2 + \widetilde{\partial}_2\, v_1 = \widetilde{\partial}_1\, v_1 = \widetilde{\partial}_2\, v_2 = 0$. Since one also has $v_3 = 0$ and $\widetilde{\partial}_3\, v = 0$, we obtain $\widetilde{\partial}_i v_j + \widetilde{\partial}_j v_i = 0$, for i, $j = 1, 2, 3$, which implies that $\widetilde{D}\, v = 0$. According to Remark 2.4, $v = a \in \mathbb{R}^3$. Thus $v = (a_1, a_2, 0)$ on Ω^1.

We will now show that $v \in Z_1^{\perp}$. Fix $\hat{a} = (\hat{a}_1, \hat{a}_2, 0) \in Z_1$, see Lemma 2.1. By assumption, one has $\langle u_n, \hat{a}\rangle_{L^2(\Omega^\varepsilon)} = 0$. This implies that $v_n = \Phi_{\varepsilon_n}^{-1} u_n$ satisfies $\langle v_n, \hat{a}\rangle_{L^2(\Omega^1)} = \varepsilon^{-1}\langle u_n, \hat{a}\rangle_{L^2(\Omega^\varepsilon)} = 0$. The last equation holds in the limit and one obtains $\langle v, \hat{a}\rangle_{L^2(\Omega^1)} = 0$. Thus one has $v \perp_{L^2(\Omega^1)} \hat{a}$, or $v \in Z_1^{\perp}$.

Next we show that $v \in Z_1$. By taking the difference of the two equations in (4.49), one obtains:

$$\begin{aligned} &\varepsilon_n^{-1} \Big(v_{n,3}(g_1(y')) - v_{n,3}(g_0(y'))\Big) \\ &\quad = \Big(v_{n,1}\, \partial_1\, g_1 + v_{n,2}\, \partial_2\, g_1\Big)\Big|_{y_3 = g_1(y')} - \Big(v_{n,1}\, \partial_1\, g_0 + v_{n,2}\, \partial_2\, g_0\Big)\Big|_{y_3 = g_0(y')}. \end{aligned}$$

We assume for the moment that the left hand side of the equation above tends to 0 in $L^2(\mathbb{T}^2)$ and $\lim_{n\to\infty} v_{n,i} = a_i$ in $L^2(\mathbb{T}^2)$, for $i = 1, 2$. It follows that the right hand side goes to $a_1 \partial g_1 + a_2 \partial g_2$ in $L^2(\mathbb{T}^2)$ as $n \to \infty$. and consequently, one has

$$\overline{a} \cdot \nabla_2\, g(y') = a_1 \partial_1 g(y') + a_2 \partial_2 g(y') = 0, \qquad \text{for all } y' \in \mathbb{T}^2,$$

where $\overline{a} = (a_1, a_2)$ This implies that $v \in Z_1$, see Lemma 2.1. Since $v \in Z_1 \cap Z_1^\perp$, one has $v = 0$, which contradicts the fact that the norm $\|v\|_{L^2(\Omega^1)}$ is 1.

In order to complete the proof of this lemma, we note on one hand that

$$\begin{aligned} \varepsilon_n^{-2} \int_{\mathbb{T}^2} |v_{n,3}(g_1(y')) - v_{n,3}(g_0(y'))|^2\, dy' &= \varepsilon_n^{-2} \int_{\mathbb{T}^2} \Big| \int_{g_0(y')}^{g_1(y')} \widetilde{\partial}_3 v_{n,3}(y_3)\, dy_3 \Big|^2 dy' \\ &\le C\, \varepsilon_n^{-2} \, \|\widetilde{\partial}_3\, v_{n,3}\|^2_{L^2(\Omega^1)} \end{aligned}$$

which goes to zero as $n \to \infty$ thanks to (4.48). On the other hand, using trace estimate (4.4) on Ω^1, we have

$$\|v_{n,i} - a_i\|_{L^2(\mathbb{T}^2)} \le C \|v_{n,i} - a_i\|_{L^2(\widetilde{\Gamma})} \le C(\|v_{n,i} - a_i\|_{L^2(\Omega^1)} + \|\widetilde{\partial}_3 v_{n,i}\|_{L^2(\Omega^1)}),$$

which goes to zero thanks to the fact that $v_n \to v = (a_1, a_2, 0)$ and $\widetilde{\partial}_3 v_n \to 0$ in $L^2(\Omega^1)$ as $n \to \infty$. □

Proof of Proposition 4.12: The right-inequality of (4.31) is trivial. We will combine Lemmas 4.13 and 4.14 to prove the remaining left-inequality. Fix β so that $C_2\,\beta = \frac{1}{2}$, where C_2 is the positive constant in Lemma 4.13. Let ε_0 be a positive number as in Lemma 4.14 for such value of β. Let $\varepsilon \in (0, \varepsilon_0]$. One combines (4.41) and (4.44) to yield

$$\begin{aligned} \|\nabla u\|^2_{L^2} &\le 4\|Du\|^2_{L^2} + C_2 \|u\|^2_{L^2} \le 4\|Du\|^2_{L^2} + C_2(\beta \|\nabla u\|^2_{L^2} + C(\beta)\|Du\|^2_{L^2}) \\ &\le 4\|Du\|^2_{L^2} + \frac{1}{2}\|\nabla u\|^2_{L^2} + C\|Du\|^2_{L^2}. \end{aligned}$$

Hence one has

$$\|\nabla u\|^2_{L^2} \le C\, \|Du\|^2_{L^2}. \tag{4.50}$$

Now that we have (4.50), applying Lemma 4.14 again gives

$$\|u\|^2_{L^2} \le \beta\, \|\nabla u\|^2_{L^2} + C(\beta)\, \|D\, u\|^2_{L^2} \le C \|D\, u\|^2_{L^2}.$$

Thus we obtain $\|u\|^2_{H^1} = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} \le C \|D\, u\|^2_{L^2}$. □

Hereafter, the positive numbers ε_0 and c_1 which appear in UKI (Proposition 4.12) are fixed.

Remark 4.15. The UKI (4.31), in general, does not hold for $u \in H^1(\Omega^\varepsilon)^3 \cap (Z_0^\perp \backslash Z_1^\perp)$ that satisfies the slip condition (1.12) on Γ. Indeed, consider $g_0(x_1, x_2) = \sin(2\pi x_1)$ and $g(x_1, x_2) = 1$. One has $\nabla_2 g_0(x_1, x_2) = (2\pi \cos(2\pi x_1), 0)$ and $\nabla_2 g(x_1, x_2) = (0, 0)$. Hence $Z_1 = \mathbb{R}^2 \times \{0\}$ and

$$Z_0 = \{(\bar{a}, 0) : \bar{a} \cdot \nabla_2 g_0 = 0\} = \{0\} \times \mathbb{R} \times \{0\}.$$

For $\varepsilon \in (0, 1]$, let $u^\varepsilon(x) = (1, 0, 2\pi\varepsilon \cos(2\pi x_1))$. Then one has $u^\varepsilon \in H^1(\Omega^\varepsilon)^3 \cap (Z_0^\perp \setminus Z_1^\perp)$, and u^ε satisfies the slip condition (1.12) on Γ, and even $\nabla \cdot u^\varepsilon = 0$ on Ω^ε. Simple calculations give $\|u^\varepsilon\|^2_{L^2} \ge \varepsilon$ and $\|Du\|^2_{L^2} = C\varepsilon^3$. Therefore $\|u\|^2_{H^1} \ge C_\varepsilon \|Du^\varepsilon\|^2_{L^2}$, where $C_\varepsilon = C\varepsilon^{-2} \to \infty$ as $\varepsilon \to 0$.

Remark 4.16. We now discuss the possibility of Domain Assumption (2.20). For our convenience, let $\widehat{Z}_i$, $i = 0, 1$, be the projection of Z_i to $\mathbb{R}^2$, i.e., $Z_i = \widehat{Z}_i \times \{0\}$. Then the Domain Assumption is simply $\widehat{Z}_0 = \widehat{Z}_1$. Let

$$\widehat{E}_i = \text{ span } \{\nabla_2 g_i(x'),\ x' \in \mathbb{T}^2\}, \quad i = 0, 1, \tag{4.51}$$

$$\widehat{E} = \text{ span } \{\nabla_2 g(x'),\ x' \in \mathbb{T}^2\}. \tag{4.52}$$

Then one has

$$\widehat{Z}_0 = \widehat{E}_0^\perp \cap \widehat{E}_1^\perp = (\widehat{E}_0 + \widehat{E}_1)^\perp, \quad \widehat{Z}_1 = \widehat{E}^\perp. \tag{4.53}$$

The Domain Assumption hence is equivalent to $\widehat{E} = \widehat{E}_0 + \widehat{E}_1$. Note that one always has $\widehat{E} \subset \widehat{E}_0 + \widehat{E}_1$. There are three cases (with some overlap) wherein the Domain Assumption holds.

Case (1): Either $\widehat{E}_0 = \{(0, 0)\}$ or $\widehat{E}_1 = \{(0, 0)\}$, i.e. $g_0 = const$ or $g_1 = const$.

Case (2): dim $\widehat{E} = 2$, i.e. $\widehat{E} = \mathbb{R}^2$.

Case (3): dim $\widehat{E} = 1$ and this is not Case (1). Thus $\widehat{E} = \widehat{E}_0 = \widehat{E}_1$ and is a one dimensional space.

Remark 4.17. More on the Uniform Korn Inequality on thin domains can be found in the article [26] by Lewicka and Mueller.

4.3 Properties of the Stokes Operator

We prove the "uniform equivalence" with respect to small ε between the norms $\|A^{\frac{1}{2}}u\|$ and $\|u\|_{H^1}$, and between the norms $\|Au\|$ and $\|u\|_{H^2}$.

Lemma 4.18. *For all $\varepsilon \in (0, \varepsilon_0]$ and $u \in \mathcal{D}(A^{\frac{1}{2}}) = V^1$, one has*

$$2\|u\|_{H^1}^2 \geq 2\|\nabla u\|_{L^2}^2 \geq 2\|Du\|_{L^2}^2 = E(u, u) = \|A^{\frac{1}{2}}u\|_{L^2}^2 \geq c_1 \|u\|_{H^1}^2, \tag{4.54}$$

In addition, one has

$$c_1 \|A^{\frac{1}{2}}u\|_{L^2}^2 \leq \|Au\|_{L^2}^2, \qquad \text{for all } u \in \mathcal{D}(A) = V^2. \tag{4.55}$$

Proof. Inequality (4.54) follows directly from inequalities (2.6) and (2.15), and the Korn inequality (4.31). Now, for $u \in \mathcal{D}(A)$ and $u \neq 0$, one has

$$\begin{aligned}\|A^{\frac{1}{2}}u\|_{L^2}^2 = \langle Au, u\rangle &\leq \|Au\|_{L^2}\|u\|_{L^2} \leq \|Au\|_{L^2}\|u\|_{H^1}\\ &\leq c_1^{-1/2}\|Au\|_{L^2}\|A^{\frac{1}{2}}u\|_{L^2},\end{aligned}$$

by the Korn inequality (4.31). Hence $c_1^{1/2}\|A^{\frac{1}{2}}u\|_{L^2} \leq \|Au\|_{L^2}$ and (4.55) follows. □

For the relation between Au and Δu, we recall an inequality from [15]:

Lemma 4.19 ([15]). *Let $u \in \mathcal{D}(A)$, then*

$$\|Au + \Delta u\|_{L^2} \leq C\varepsilon \|\nabla u\|_{L^2} + C\|u\|_{L^2} \leq C\|u\|_{H^1}, \qquad \text{for } 0 < \varepsilon \leq 1. \tag{4.56}$$

For the relation between $\|\Delta u\|_{L^2}$ and $\|u\|_{H^2}$, we have the following:

Lemma 4.20. *There is an ε_1, with $0 < \varepsilon_1 \leq \varepsilon_0$, such that for $0 < \varepsilon \leq \varepsilon_1$ one has*

$$\|u\|_{H^2}^2 \;\leq\; 4\,\|\Delta u\|_{L^2}^2 + C\,\|u\|_{H^1}^2, \quad \text{for all } u \in \mathcal{D}(A). \tag{4.57}$$

The proof of this technical lemma is given the Appendix. Finally one obtains

Proposition 4.21. *For $0 < \varepsilon \leq \varepsilon_1$, one has*

$$C^{-1}\,\|u\|_{H^2}^2 \;\leq\; \|Au\|_{L^2}^2 \leq 3\|u\|_{H^2}^2, \quad \text{for all } u \in \mathcal{D}(A). \tag{4.58}$$

Proof. On one hand, one has

$$\|Au\|_{L^2}^2 = \|\mathbb{P}\,(-\Delta u)\|_{L^2}^2 \leq \|\Delta u\|_{L^2} \leq 3\|u\|_{H^2}^2, \qquad \text{for } u \in \mathcal{D}(A) = V^2.$$

On the other hand, by combining Lemmas 4.19 and 4.20, the Korn inequality (4.31) and inequalities (4.54) and (4.55), one has

$$\begin{aligned}
\|u\|_{H^2}^2 = \|u\|_{H^1}^2 + \|\nabla^2 u\|_{L^2}^2 &\leq \|u\|_{H^1}^2 + C(\|\Delta u\|_{L^2}^2 + \|u\|_{H^1}^2) \\
&\leq C\|u\|_{H^1}^2 + C(\|Au\|_{L^2}^2 + \|Au + \Delta u\|_{L^2}^2) \\
&\leq C\|u\|_{H^1}^2 + C\|Au\|_{L^2}^2 \leq C\|Du\|_{L^2}^2 + C\|Au\|_{L^2}^2 \\
&\leq C\|A^{\frac{1}{2}}u\|_{L^2}^2 + C\|Au\|_{L^2}^2 \leq C\|Au\|_{L^2}^2,
\end{aligned}$$

which implies (4.58). □

4.4 Averaging Operators

In this section we examine three related averaging operators M_0, M_2 and M_3, each involves an averaging in the "thin" direction, that is, along the x_3-axis.

4.4.1 The operator $\mathbf{M_0}$

The operator $M_0 : L^2(\Omega^\varepsilon) \to L^2(\Omega^\varepsilon)$ is defined by setting

$$\varphi(x_1, x_2) \;=\; (M_0\,\phi)(x) \;=\; \frac{1}{\varepsilon g(x_1, x_2)} \int_{h_0(x_1,x_2)}^{h_1(x_1,x_2)} \phi(x_1, x_2, s)\, ds, \tag{4.59}$$

where $\phi \in L^2(\Omega^\varepsilon)$ and $x = (x_1, x_2, x_3) \in \Omega^\varepsilon$. One can verify that M_0 is an orthogonal projection on $L^2(\Omega^\varepsilon)$. Hence

$$\|M_0\,\phi\|_{L^2}^2 + \|\phi - M_0\,\phi\|_{L^2}^2 = \|\phi\|_{L^2}^2, \qquad \text{for all } \phi \in L^2(\Omega^\varepsilon). \tag{4.60}$$

For our convenience in the subsequent computations, we define the 2D vector field $\psi = (\psi_1, \psi_2)$ on $\mathbb{T}^2 \times \mathbb{R}$ by

$$\begin{aligned}\psi = (\psi_1, \psi_2) &= \nabla_2 h_0 + \frac{x_3 - h_0}{g} \nabla_2 g = \nabla_2 h_1 - \frac{h_1 - x_3}{g} \nabla_2 g \\ &= \frac{1}{\varepsilon g} \{(x_3 - h_0) \nabla_2 h_1 + (h_1 - x_3) \nabla_2 h_0\},\end{aligned} \tag{4.61}$$

where ∇_2 denotes the 2D gradient operator. Explicitly, each component of ψ is $\psi_i(x_1, x_2, x_3) = \frac{1}{g}\Big\{(x_3 - h_0)\, \partial_i g_1(x_1, x_2) + (h_1 - x_3)\, \partial_i g_0(x_1, x_2)\Big\}$, for $i = 1, 2$. As a result, one readily obtains the estimate

$$\|\psi\|_{L^\infty(\Omega^\varepsilon)} \le C\, \varepsilon, \tag{4.62}$$

where C depends only on g_0 and g_1. Note that

$$\psi\big|_{\Gamma_j} = \nabla_2\, h_j, \quad \text{for } j = 0, 1, \quad \text{and} \quad \partial_3 \psi = \frac{1}{g} \nabla_2\, g. \tag{4.63}$$

Since $g(x') \ge c_0 > 0$, for $x' \in \mathbb{T}^2$, see (1.17), it follows that when g_0 and g_1 are C^{m+1} functions, one has

$$\|\partial_3 \psi\|_{L^\infty} \le C, \quad \|\nabla_2 \psi\|_{L^\infty} \le C\varepsilon, \quad \|\nabla^m \psi\|_{L^\infty} \le C(m), \quad m \ge 2, \tag{4.64}$$

where C and $C(m)$ are positive and do not depend on ε.

Using the function ψ, one easily describes the partial derivatives of $M_0\phi$ as follows:

Lemma 4.22. *Let $\phi \in H^1(\Omega^\varepsilon)$ and let ψ satisfy* (4.61)*. Then one has:*

$$\partial_i\, M_0\, \phi = M_0(\partial_i\, \phi) + M_0(\psi_i\, \partial_3 \phi), \qquad \textit{for } i = 1, 2. \tag{4.65}$$

Proof. For $i = 1, 2$, we have

$$\partial_i\, M_0 \phi = -\frac{\partial_i g}{g} M_0 \phi + M_0(\partial_i\, \phi) + \frac{1}{\varepsilon g} \Big\{ \phi\big|_{x_3 = h_1} \partial_i h_1 - \phi\big|_{x_3 = h_0} \partial_i h_0 \Big\}.$$

Next note that

$$\begin{aligned}\phi\big|_{x_3 = h_1} \partial_i\, h_1 - \phi\big|_{x_3 = h_0} \partial_i h_0 &= \int_{h_0}^{h_1} \partial_3(\phi \psi_i)\, dx_3 \\ &= \int_{h_0}^{h_1} \Big(\psi_i \partial_3\, \phi + \phi \frac{\partial_i g}{g}\Big)\, dx_3 = \varepsilon g \Big\{ M_0(\psi_i\, \partial_3 \phi) + \frac{\partial_i g}{g} M_0\, \phi \Big\},\end{aligned}$$

which in turn implies (4.65). □

Applying Lemma 4.22 recursively, we obtain the following:

Lemma 4.23. *Let $m \in \mathbb{N}$. Assume that g_0 and g_1 are C^m functions and $\phi \in H^m(\Omega^\varepsilon)$. Then one has:*

$$\|M_0\,\phi\|_{H^m} \le C(m)\|\phi\|_{H^m}, \tag{4.66}$$

where positive number $C(m)$ is independent of ε.

Proof. Thanks to (4.60), we have $\|M_0\,\phi\|_{L^2} \le \|\phi\|_{L^2}$. For $m = 1$, it follows from (4.62) and (4.65) that

$$\|\nabla M_0\,\phi\|_{L^2} \le \|\nabla_2\,\phi\|_{L^2} + C\varepsilon\|\partial_3\,\phi\|_{L^2} \le C\|\phi\|_{H^1},$$

and hence (4.66). For $m \ge 2$, inequality (4.66) is proved by induction; we omit the details. □

4.4.2 The operators $\mathbf{M_2}$ and $\mathbf{M_3}$

The other two averaging operators of interest, M_2 and M_3, map vector fields into vector fields. For $u \in L^2(\Omega^\varepsilon)^3$, we define the vector fields

$$v = (v_1, v_2, v_3), \quad \overline{v} = (v_1, v_2), \quad \text{and} \quad \widehat{v} = (v_1, v_2, 0),$$

and the operators

$$M_2 u = \widehat{v} \quad \text{and} \quad M_3 u = v, \tag{4.67}$$

with $v_1 = M_0(u_1)$, $v_2 = M_0(u_2)$, and $v_3 = \overline{v}\cdot\psi$, where ψ satisfies (4.61). Since $M_2^2 = M_2$ and $M_3^2 = M_3$, both M_2 and M_3 are projections in $L^2(\Omega^\varepsilon)^3$. Note that M_2 is an orthogonal projection.

Lemma 4.24. *Let $u \in L^2(\Omega^\varepsilon)^3$. Assume that $\nabla\cdot u = 0$ in Ω^ε and that $u\cdot N = 0$ on Γ. Then the vector field $v = M_3 u$ defined by (4.67) satisfies both $\nabla\cdot v = 0$ in Ω^ε and $v\cdot N = 0$ on Γ. In addition the 2D vector field $\overline{v}$ satisfies the g-divergence property:*

$$\nabla_2\,\cdot(g\,\overline{v}) \;=\; 0, \quad \text{on} \quad \mathbb{T}^2. \tag{4.68}$$

Proof. Thanks to (4.63), one finds for $j = 0, 1$, that $v_3\big|_{\Gamma_j} = \overline{v}\cdot\nabla_2 h_j$, hence $v\cdot N = 0$ on Γ. We will show that

$$\nabla_2\cdot\overline{v} \;=\; -\overline{v}\cdot\Big(\frac{1}{g}\nabla_2\,g\Big). \tag{4.69}$$

Since $\frac{1}{g}\nabla_2\,\cdot(g\,\overline{v}) \;=\; \nabla_2\cdot\overline{v} + \overline{v}\cdot\Big(\frac{1}{g}\nabla_2\,g\Big)$, the equation (4.68) is a consequence of (4.69). Also $\nabla\cdot v = 0$ follows (4.69) and the fact that

$$\partial_3 v_3 = \overline{v}\cdot\Big(\frac{1}{g}\nabla_2\,g\Big). \tag{4.70}$$

To prove (4.69), we note that for $i = 1, 2$, one has

$$\partial_i M_0(u_i) = \partial_i\big(\frac{1}{\varepsilon g}\int_{h_0}^{h_1} u_i\, ds\big) = M_0(\partial_i u_i) - \frac{\partial_i g}{g} M_0(u_i) + [BC]_i,$$

where

$$[BC]_i = \frac{1}{\varepsilon g}\big[u_i\big|_{x_3=h_1}(\partial_i h_1) - u_i\big|_{x_3=h_0}(\partial_i h_0)\big].$$

Hence

$$\nabla_2 \cdot \overline{v} = -\overline{v}\cdot\frac{1}{g}\nabla_2 g + M_0(\partial_1 u_1 + \partial_2 u_2) + [BC]_1 + [BC]_2.$$

Since $\nabla \cdot u = 0$ in Ω^ε, one has

$$\begin{aligned} M_0(\partial_1 u_1 + \partial_2 u_2) &= -M_0(\partial_3 u_3) = -\frac{1}{\varepsilon g}\int_{h_0}^{h_1} \partial_3 u_3\, ds \\ &= -\frac{1}{\varepsilon g}\big[u_3\big|_{x_3=h_1} - u_3\big|_{x_3=h_0}\big] = [BC]_3. \end{aligned}$$

Since $u \cdot N = 0$ on Γ, one has $[BC]_3 + [BC]_1 + [BC]_2 = 0$, which completes to proof of the lemma. □

Note that if u is in $H^1(\Omega^\varepsilon)^3$, then the Euclidean norms on $\mathbb{R}^3$ satisfy:

$$|v_3| \le C\,\varepsilon|\overline{v}|, \quad |\partial_3 v_3| \le C\,|\overline{v}|, \quad |\nabla_2 v_3| \le C\,\varepsilon|\nabla_2\overline{v}|, \tag{4.71}$$

thanks to (4.62), (4.70) and (4.64). Combining this with Lemma 4.23, we have

Lemma 4.25. *Let $m = 0, 1, 2, \dots$, suppose g_0 and g_1 are C^{m+1} functions. Let $u \in H^m(\Omega^\varepsilon)^3$, $v = Mu$ and $w = u - v$. One has*

$$\|v\|_{H^m}\,,\ \|w\|_{H^m} \le C(m)\|u\|_{H^m}, \tag{4.72}$$

where $C(m)$ are independent of ε.

4.4.3 Basic inequalities

We now derive some auxiliary inequalities for $v = M_3 u$ and $\widehat{v} = M_2 u$ with $w = u - v$ and $\widehat{w} = u - \widehat{v}$. We assume here that $u \in H^1(\Omega^\varepsilon)^3 \cap H_1(\Omega^\varepsilon)$, see (2.3). Thus u satisfies the slip boundary condition (1.12) on Γ and $\nabla \cdot u = 0$ in Ω^ε. We will also refer to the 2D vector fields $\overline{u} = (u_1, u_2)$, $\overline{v} = (v_1, v_2)$, and $\overline{w} = (w_1, w_2)$. Note that $w = (w_1, w_2, u_3 - v_3)$ and $\widehat{w} = (w_1, w_2, u_3)$.

Lemma 4.26 (Ladyzhenskaya inequalities)**.** *One has:*

$$\|v\|_{L^4} \le C\,\varepsilon^{-\frac14}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}, \tag{4.73}$$

If in addition, $u \in H^2(\Omega^\varepsilon)^3$, then

$$\|\nabla v\|_{L^4} \le C\,\varepsilon^{-\frac14}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}. \tag{4.74}$$

Proof. From Lemmas 4.3 and 4.25, one has

$$\|\overline{v}\|_{L^4} \le C\,\varepsilon^{-\frac{1}{4}}\|\overline{v}\|_{L^2}^{1/2}\|\overline{v}\|_{H_1}^{1/2} \le C\,\varepsilon^{-\frac{1}{4}}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2}. \tag{4.75}$$

By (4.71), one has

$$\|v_3\|_{L^4} \le C\,\varepsilon\|\overline{v}\|_{L^4} \le C\,\varepsilon^{\frac{3}{4}}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2}, \tag{4.76}$$

which implies (4.73). Similarly, when $u \in H^2(\Omega^\varepsilon)^3$, then Lemma 4.3 implies

$$\|\nabla_2\overline{v}\|_{L^4} \le C\,\varepsilon^{-\frac{1}{4}}\|u\|_{H^1}^{1/2}\|u\|_{H_2}^{1/2},$$
$$\|\nabla_2 v_3\|_{L^4} \le C\,\varepsilon\|\nabla_2\overline{v}\|_{L^4} \le C\,\varepsilon^{\frac{3}{4}}\|u\|_{H^1}^{1/2}\|u\|_{H_2}^{1/2}.$$

From (4.71), one obtains

$$\|\partial_3 v_3\|_{L^4} \le C\,\|\overline{v}\|_{L^4} \le C\,\varepsilon^{-\frac{1}{4}}\|u\|_{L^2}^{1/2}\|u\|_{H_1}^{1/2},$$

which implies (4.74). □

Lemma 4.27 (Poincaré Inequality)**.** *One has*

$$\|w\|_{L^2} \le C\,\varepsilon\|\nabla w\|_{L^2} \le C\,\varepsilon\|u\|_{H^1} \quad \textit{and} \quad \|\widehat{w}\|_{L^2} \le C\,\varepsilon\|u\|_{H^1}. \tag{4.77}$$

Proof. By (4.8), we obtain

$$\|\overline{w}\|_{L^2} \le C\,\varepsilon\|\partial_3\overline{w}\|_{L^2} \le C\,\varepsilon\|\nabla w\|_{L^2} \le C\,\varepsilon\|u\|_{H^1}. \tag{4.78}$$

For w_3, we first use (4.3) with $j = 0$ to obtain

$$\int_{\Omega^\varepsilon}|w_3|^2\,dx \le C\,\varepsilon\int_{\mathbb{T}^2}|w_3(x', h_0(x'))|^2\,dx' + C\,\varepsilon^2\int_{\Omega^\varepsilon}|\partial_3 w_3|^2\,dx.$$

Since $u\cdot N = 0$ on Γ, we have $u_3 = \overline{u}\cdot\nabla_2\,h_j$ on Γ_j, for $j = 0, 1$, and

$$w_3 = u_3 - \overline{v}\cdot\nabla_2 h_j = (\overline{u} - \overline{v})\cdot\nabla_2 h_j, \qquad \text{on} \quad \Gamma_j, \quad \text{for } j = 0, 1. \tag{4.79}$$

Now $|\nabla_2 h_j| \le C\,\varepsilon$ on Γ_j and by (4.9), it follows that

$$\int_{\mathbb{T}^2}|w_3(x', h_j(x'))|^2 dx' \le C\varepsilon\int_{\Omega^\varepsilon}|\partial_3\overline{w}|^2 dx.$$

Taking $j = 0$, we thus obtain

$$\|w_3\|_{L^2}^2 \le C\varepsilon^2\|\partial_3\overline{w}\|_{L^2}^2 + C\varepsilon^2\|\partial_3 w_3\|_{L^2}^2 \le C\varepsilon^2\|\nabla w\|_{L^2}^2, \tag{4.80}$$

which implies (4.77) for w. Since $\widehat{w}_3 = u_3$, the inequality for $\widehat{w}$, follows from (4.78) and (4.20). □

If in addition, $u \in W^{1,r}(\Omega^\varepsilon)$, for $r \in [1,\infty)$, then equation (4.79) implies that $|w| \le C|\overline{w}|$ on Γ, and therefore (4.7) yields:

$$\|w\|_{L^r(\Gamma)} \le C\,\varepsilon^{(r-1)/r}\|\partial_3 w\|_{L^r(\Omega^\varepsilon)}. \tag{4.81}$$

For ∇w, we obtain the following weaker form of Poincaré inequality which serves us well for our current studies. Except for the use of the Poincaré-Trace Lemma 4.1 the proof is the same as that in [21].

Lemma 4.28 (Poincaré Inequality)**.** *Assume in addition that $u \in \mathcal{D}(A)$. Then one has:*

$$\|\nabla w\|_{L^2} \le C\,\varepsilon\|u\|_{H^2}. \tag{4.82}$$

Proof. The first step is to estimate $\|\partial_i w_j\|_{L^2}$, for $i,j = 1,2$. One has $\partial_i w_j = \partial_i u_j - \partial_i v_j$. Using Lemma 4.22 with $\phi = u_j$, one obtains $\partial_i v_j = M_0(\partial_i u_j) + M_0(\psi_i \partial_3 u_j)$. Therefore, one has

$$\partial_i w_j = \phi - M_0(\psi_i\partial_3 u_j), \quad \text{where } \phi = (I - M_0)\partial_i u_j.$$

Since M_0 is an orthogonal projection, one has

$$\|\partial_i w_j\|^2_{L^2} = \|\phi\|^2_{L^2} + \|M_0(\psi_i\partial_3 u_j)\|^2_{L^2} \le \|\phi\|^2_{L^2} + \|\psi_i\partial_3 u_j\|^2_{L^2}.$$

Now (4.8) implies that $\|\phi\|^2_{L^2} \le C\varepsilon^2\|\partial_3\phi\|^2_{L^2}$. Since

$$\partial_3\phi = \partial_3(\partial_i u_j - M_0(\partial_i u_j)) = \partial_3\partial_i u_j,$$

one obtains (see (4.62), (4.64), and (4.60))

$$\|\partial_i w_j\|^2_{L^2} \le C\,\varepsilon^2(\|\partial_3\partial_i u_j\|^2_{L^2} + \|\partial_3 u_j\|^2_{L^2}) \le C\varepsilon^2\|u\|^2_{H^2}.$$

Since $\nabla\cdot w = 0$ in Ω^ε, one has $\partial_3 w_3 = -\partial_1 w_1 - \partial_2 w_2$. Therefore, one has $\|\partial_3 w_3\|_{L^2} \le C\varepsilon\|u\|_{H^2}$ as well. For $\partial_i w_3$ with $i = 1,2$, we note that (4.3) implies that

$$\|\partial_i w_3\|_{L^2(\Omega^\varepsilon)} \le C\,\varepsilon^{\frac12}\|\partial_i w_3\|_{L^2(\Gamma_j)} + C\,\varepsilon\|\partial_3\partial_i w_3\|_{L^2(\Omega^\varepsilon)}, \quad \text{for } j = 0,1. \tag{4.83}$$

For the trace term on Γ_j, one has $w_k = w_k(x', h_j(x'))$, for $k = 1,2,3$, where $x' = (x_1,x_2) \in \mathbb{T}^2$. Since $w\cdot N = 0$ on Γ, one has $w_3 = \overline{w}\cdot\nabla_2 h_j$ on Γ_j, for $j = 0,1$. Hence one has

$$\partial_i w_3 + \partial_3 w_3\,\partial_i h_j = \partial_i\overline{w}\cdot\nabla_2 h_j + (\partial_3\overline{w}\cdot\nabla_2 h_j)(\partial_i h_j) + \overline{w}\cdot\nabla_2\partial_i h_j.$$

By using the trace inequality (4.4) on Γ and the standard L^∞-bounds on derivatives of h_j, one readily obtains:

$$\begin{aligned}
\|\partial_3 w_3\,\partial_i h_j\|_{L^2(\Gamma_j)} &\le C\,\varepsilon\Big\{\frac{1}{\sqrt{\varepsilon}}\|\partial_3 w_3\|_{L^2} + \sqrt{\varepsilon}\|\partial_3\partial_3 w_3\|_{L^2}\Big\},\\
\|\partial_i\overline{w}\cdot\nabla_2 h_j\|_{L^2(\Gamma_j)} &\le C\,\varepsilon\Big\{\frac{1}{\sqrt{\varepsilon}}\|\partial_i\overline{w}\|_{L^2} + \sqrt{\varepsilon}\|\partial_3\partial_i\overline{w}\|_{L^2}\Big\},\\
\|(\partial_3\overline{w}\cdot\nabla_2 h_j)(\partial_i h_j)\|_{L^2(\Gamma_j)} &\le C\,\varepsilon^2\Big\{\frac{1}{\sqrt{\varepsilon}}\|\partial_3\overline{w}\|_{L^2} + \sqrt{\varepsilon}\|\partial_3\partial_3\overline{w}\|_{L^2}\Big\},\\
\|\overline{w}\cdot\nabla_2\partial_i h_j\|_{L^2(\Gamma_j)} &\le C\,\varepsilon\Big\{\frac{1}{\sqrt{\varepsilon}}\|\overline{w}\|_{L^2} + \sqrt{\varepsilon}\|\partial_3\overline{w}\|_{L^2}\Big\},
\end{aligned}$$

for $j = 0, 1$. Taking $j = 0$ and summing up, one obtains

$$\|\partial_i w_3\|_{L^2} \le C\{\varepsilon(\|\overline{w}\|_{L^2} + \|\partial_i \overline{w}\|_{L^2} + \|\partial_3 w_3\|_{L^2} + \|\partial_3\partial_i w_3\|_{L^2}) + \varepsilon^2(\|\partial_3\overline{w}\|_{L^2} + \|\partial_3\partial_3 w_3\|_{L^2} + \|\partial_3\partial_i\overline{w}\|_{L^2}) + \varepsilon^3\|\partial_3\partial_3\overline{w}\|_{L^2}\}. \tag{4.84}$$

Hence

$$\|\partial_i w_3\|_{L^2} \le C\varepsilon\|u\|_{H^2}.$$

Finally, by Lemma 4.8, $\|\partial_3 w_i\|_{L^2} = \|\partial_3 u_i\|_{L^2} \le C\varepsilon\|u\|_{H^2}$, for $i = 1, 2$. The proof is complete. □

Lemma 4.29 (LGNPS[2] Inequality)**.** *One has*

$$\|w\|_{L^6} \le C\|w\|_{H^1}, \tag{4.85}$$

and, more generally, for $q \in [2, 6]$,

$$\|w\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}\|w\|_{H^1}. \tag{4.86}$$

Assume in addition that $u \in \mathcal{D}(A)$, *then one has*

$$\|\nabla w\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{1}{2}}\|u\|_{H^2} \quad \textit{where} \quad q \in [2, 6]. \tag{4.87}$$

Proof. Combining (4.77) and (4.10), we obtain (4.85). Using interpolating inequalities, we have (4.86). Inequality (4.87) is derived similarly thanks to (4.82). □

Lemma 4.30 (Agmon inequality)**.** *Assume* $u \in \mathcal{D}(A)$. *Then*

$$\|w\|_{L^\infty} \le C\varepsilon^{\frac{1}{2}}\|u\|_{H^2}. \tag{4.88}$$

Proof. Combining (4.14), (4.77) and (4.82)

$$\begin{aligned}\|w\|_{L^\infty} &\le C\varepsilon^{-\frac{1}{2}}\Big\{\varepsilon\|\partial_3 w\|\Big\}^{1/4}\|w\|_{H^2}^{1/2}\Big\{\varepsilon\|\partial_3 w\| + \varepsilon\|\partial_3 w\| + \varepsilon^2\|\partial_3\partial_3 w\|\Big\}^{1/4}\\ &\le C\|\partial_3 w\|_{L^2}^{1/4}\|w\|_{H^2}^{1/2}(\|\partial_3 w\|_{L^2} + \varepsilon\|\partial_3\partial_3 w\|_{L^2})^{1/4}\\ &\le C\varepsilon^{\frac{1}{4}}\|\partial_3 w\|_{L^2}^{1/4}\|u\|_{H^2}^{3/4} \le C\varepsilon^{\frac{1}{2}}\|u\|_{H^2}.\end{aligned}$$

We obtain (4.88). □

5 Nonlinear Term

The main goal of this section is to obtain the following estimate for the trilinear term in the study of the Navier–Stokes equations.

[2] Ladyzhenskaya-Garliagdo-Nirenberg-Poincaré-Sobolev

Proposition 5.1. *For any $\varepsilon \in (0,1]$, $u \in \mathcal{D}(A)$ and $\beta > 0$, one has*

$$|\langle (u\cdot\nabla)u, Au\rangle| \le \beta\,\|u\|_{H^2}^2 + C\varepsilon^{\frac12}\|u\|_{H^1}\|u\|_{H^2}^2 + C_\beta\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2, \tag{5.1}$$

where $C > 0$ is independent of β and ε; and $C_\beta > 0$ is independent of ε.

Hereafter we fix $\varepsilon_1 > 0$ such that $\varepsilon_1 \le \varepsilon_0$, Lemma 4.20 and Proposition 4.21, hold. As a consequence of Proposition 5.1, Lemma 4.18 and Proposition 4.21, one obtains

Corollary 5.2. *For any $\varepsilon \in (0,\varepsilon_1]$, $u \in \mathcal{D}(A)$ and $\beta \in (0,\infty)$, one has*

$$\begin{aligned} |\langle (u\cdot\nabla)u, Au\rangle| \le \beta\,\|Au\|_{L^2}^2 &+ C\varepsilon^{\frac12}\|A^{\frac12}u\|_{L^2}\,\|Au\|_{L^2}^2 \\ &+ C_\beta\varepsilon^{-1}\|u\|_{L^2}^2\,\|A^{\frac12}u\|_{L^2}^2, \end{aligned} \tag{5.2}$$

where $C > 0$ is independent of β and ε; and $C_\beta > 0$ is independent of ε.

One first has the following simple estimates:

Lemma 5.3. *Let $u \in \mathcal{D}(A)$ and $v = M_3u$, $w = u - M_3u$. Then*

$$\begin{aligned} \|\, v|\nabla v|\,\|_{L^2} &\le C\varepsilon^{-\frac12}\|u\|_{L^2}^{1/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2}, \\ \|\, v|\nabla w|\,\|_{L^2} &\le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{H^2}. \end{aligned}$$

Proof. Use Hölder's inequality and the estimates of $\|v\|_{L^4}$, $\|\nabla v\|_{L^4}$ and $\|\nabla w\|_{L^4}$ from Lemma 4.26 and Inequality (4.86). □

Proof of Proposition 5.1: Let $u \in \mathcal{D}(A)$ and $u = v + w$, where $v = M_3u$. We write the trilinear term as

$$\langle (u\cdot\nabla)u, Au\rangle = I_0 + I_1 - \langle (v\cdot\nabla)u, \Delta u\rangle, \tag{5.3}$$

where $I_0 = \langle (w\cdot\nabla)u, Au\rangle$ and $I_1 = \langle (v\cdot\nabla)u, Au + \Delta u\rangle$.

Estimate of I_0. From (4.88), Proposition 4.12 and Proposition 4.21

$$|I_0| \le \|w\|_{L^\infty}\|u\|_{H^1}\|u\|_{H^2} \le C\varepsilon^{\frac12}\|u\|_{H^1}\|u\|_{H^2}^2. \tag{5.4}$$

Estimate of I_1. By Lemmas 4.19 and 5.3:

$$\begin{aligned} |I_1| &\le C\|v\cdot\nabla u\|_{L^2}(\varepsilon\|u\|_{H^1} + \|u\|_{L^2}) \\ &\le C(\|v\cdot\nabla v\|_{L^2} + \|v\cdot\nabla w\|_{L^2})(\varepsilon\|u\|_{H^1} + \|u\|_{L^2}) \\ &\le C\varepsilon^{\frac12}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^2\|u\|_{H^2}^{1/2} + C\varepsilon\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{3/2}\|u\|_{H^2} \\ &\quad + C\varepsilon^{-\frac12}\|u\|_{L^2}^{3/2}\|u\|_{H^1}\|u\|_{H^2}^{1/2} + C\|u\|_{L^2}^{3/2}\|u\|_{H^1}^{1/2}\|u\|_{H^2} \\ &= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}. \end{aligned}$$

By using the interpolation inequality $\|u\|_{H^1} \le C\,\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}$ for $I_{1,1}$ and $I_{1,3}$, with the Young inequality and $\|u\|_{H^1} \le \|u\|_{H^2}$, one obtains

$$\begin{aligned} I_{1,1} &\le C\,\varepsilon^{\frac{1}{2}}\|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^2} \le \alpha\|u\|_{H^2}^2 + C_\alpha\,\varepsilon\|u\|_{L^2}^2\|u\|_{H^1}^2, \\ I_{1,2} &\le C\,\varepsilon\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^2, \\ I_{1,3} &\le C\,\varepsilon^{-\frac{1}{2}}\|u\|_{L^2}^2\|u\|_{H^2} \le \alpha\|u\|_{H^2}^2 + C_\alpha\,\varepsilon^{-1}\|u\|_{L^2}^4, \\ I_{1,4} &\le \alpha\|u\|_{H^2}^2 + C_\alpha\,\|u\|_{L^2}^3\|u\|_{H^1}. \end{aligned}$$

Summing up the four inequalities yields

$$|I_1| \le 3\alpha\|u\|_{H^2}^2 + C\,\varepsilon^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}^2 + C_\alpha\,\varepsilon^{-1}\|u\|_{L^2}^2\|u\|_{H^1}^2. \tag{5.5}$$

For the next step we examine the term $-\langle (v\cdot\nabla)u, \Delta u\rangle$. By integrating by parts, we obtain

$$-\langle (v\cdot\nabla)u, \Delta u\rangle = I_2 + I_3, \tag{5.6}$$

where

$$I_2 = \int_{\Omega^\varepsilon}\sum_{k,i,j=1}^{3}\partial_k v_i\partial_i u_j\partial_k u_j\,dx \qquad \text{and} \qquad I_3 = \int_\Gamma (v\cdot\nabla u)\cdot\frac{\partial u}{\partial N}\,d\sigma.$$

Estimate of I_2. Here we follow the argument used in [21]. By using $u = v + w$, one has

$$\begin{aligned} I_2 &= \int_{\Omega^\varepsilon}\sum_{k,i,j=1}^{3}\partial_k v_i(\partial_i v_j + \partial_i w_j)(\partial_k v_j + \partial_k w_j)\,dx \\ &= I_{2,1} + \sum_{i,j,k=1}^{3}\int_{\Omega^\varepsilon}\partial_k v_i\partial_i v_j\partial_k v_j\,dx, \end{aligned}$$

where

$$I_{2,1} = \sum_{k,i,j=1}^{3}\int_{\Omega^\varepsilon}(\partial_k v_i\partial_i v_j\partial_k w_j + \partial_k v_i\partial_i w_j\partial_k v_j + \partial_k v_i\partial_i w_j\partial_k w_j)\,dx.$$

Note that

$$\begin{aligned} |I_{2,1}| &\le C\int_{\Omega^\varepsilon}|\nabla v|^2|\nabla w| + |\nabla v|\,|\nabla w|^2 dx \\ &\le C\|\nabla v\|_{L^4}^2\|\nabla w\|_{L^2} + C\|\nabla w\|_{L^4}^2\|\nabla v\|_{L^2} \end{aligned}$$

Next by using Lemma 4.26 with (4.77) and (4.87) with $q = 4$, one obtains

$$\begin{aligned} |I_{2,1}| &\le C\varepsilon^{-\frac{1}{2}}\|v\|_{H^1}\|v\|_{H^2}(\varepsilon\|u\|_{H^2}) + C(\varepsilon^{\frac{1}{4}}\|u\|_{H^2})^2\|v\|_{H^1} \\ &\le C\varepsilon^{\frac{1}{2}}\|u\|_{H^1}\|u\|_{H^2}^2. \end{aligned} \tag{5.7}$$

Continuing, we define $I_{2,2}$ by the equation

$$\sum_{i,j,k=1}^{3} \int_{\Omega^\varepsilon} \partial_k v_i \partial_i v_j \partial_k v_j \, dx = \sum_{i,j,k=1}^{2} \int_{\Omega^\varepsilon} \partial_k v_i \partial_i v_j \partial_k v_j \, dx + I_{2,2}.$$

With (4.68), elementary calculations give

$$\begin{aligned}
&\Big| \sum_{i,j,k=1}^{2} \int_{\Omega^\varepsilon} \partial_k v_i \, \partial_i v_j \, \partial_k v_j \, dx \Big| \\
&\qquad = \Big| \int_{\Omega^\varepsilon} \mathrm{div}\bar{v} \{ \partial_1 v_2 \, \partial_2 v_1 + (\partial_1 v_2)^2 + (\partial_2 v_1)^2 \} + (\partial_1 v_1)^3 + (\partial_2 v_2)^3 dx \Big| \\
&\qquad = \Big| \int_{\Omega^\varepsilon} \mathrm{div}\bar{v} \{ \partial_1 v_2 \, \partial_2 v_1 + (\partial_1 v_2)^2 + (\partial_2 v_1)^2 + (\partial_1 v_1)^2 \\
&\qquad\qquad + (\partial_2 v_2)^2 - \partial_1 v_1 \, \partial_2 v_2 \} \, dx \Big| \\
&\qquad \le C \int_{\Omega^\varepsilon} |\bar{v}| |\nabla \bar{v}|^2 dx \le C \|\bar{v}\|_{L^2} \|\nabla \bar{v}\|_{L^4}^2 .
\end{aligned}$$

Applying Lemma 4.26 to $\|\nabla \bar{v}\|_{L^4}^2$ yields

$$\left| \sum_{i,j,k=1}^{2} \int_{\Omega^\varepsilon} \partial_k v_i \, \partial_i v_j \, \partial_k v_j \, dx \right| \le C\varepsilon^{-\frac{1}{2}} \|u\|_{L^2} \|u\|_{H^1} \|u\|_{H^2}. \tag{5.8}$$

In the expression for $I_{2,2}$, each of the terms: $\partial_k v_i$, $\partial_i v_j$, or $\partial_k v_j$ is either $\partial_3 v_m$, or $\partial_m v_3$, for some $m = 1, 2, 3$. Since $|\partial_3 v_3| \le C|\bar{v}|$ and $|\partial_m v_3| \le C\varepsilon(|\bar{v}| + |\nabla_2 \bar{v}|)$ for $m = 1, 2$, one has

$$\begin{aligned}
|I_{2,2}| &\le C \int_{\Omega^\varepsilon} (|\bar{v}| + \varepsilon |\nabla_2 \bar{v}|) \, |\nabla v|^2 \, dx \\
&\le C \|v\|_{L^2} \|\nabla v\|_{L^4}^2 + C\varepsilon \|\nabla v\|_{L^4}^2 \|\nabla v\|_{L^2}.
\end{aligned}$$

Applying Lemma 4.26 gives

$$|I_{2,2}| \le C\varepsilon^{-\frac{1}{2}} \|u\|_{L^2} \|u\|_{H^1} \|u\|_{H^2} + C\varepsilon^{\frac{1}{2}} \|u\|_{H^1}^2 \|u\|_{H^2}. \tag{5.9}$$

Combining (5.7), (5.8) and (5.9) gives

$$\begin{aligned}
|I_2| &\le C\varepsilon^{\frac{1}{2}} \|u\|_{H^1} \|u\|_{H^2}^2 + C\varepsilon^{-\frac{1}{2}} \|u\|_{L^2} \|u\|_{H^1} \|u\|_{H^2} + C\varepsilon^{\frac{1}{2}} \|u\|_{H^1}^2 \|u\|_{H^2} \\
&\le \alpha \|u\|_{H^2}^2 + C_\alpha \varepsilon^{-1} \|u\|_{L^2}^2 \|u\|_{H^1}^2 + C\varepsilon^{\frac{1}{2}} \|u\|_{H^1} \|u\|_{H^2}^2 .
\end{aligned} \tag{5.10}$$

Estimate of I_3. The boundary term is

$$I_3 = \int_\Gamma b \cdot \frac{\partial u}{\partial N} \, d\sigma, \qquad \text{where } b = (v \cdot \nabla) u.$$

Next we express b and v in terms of the orthonormal frame field $\{\sigma^1, \sigma^2, N\}$ on Γ, see (4.24), using $v \cdot N = 0$ on Γ:

$$b = b^{(1)}\sigma^1 + b^{(2)}\sigma^2 + b^{(3)}N \quad \text{and} \quad v = v^{(1)}\sigma^1 + v^{(2)}\sigma^2.$$

Since $b^{(k)} = b \cdot \sigma^k$, one has $|b^{(k)}| \le |b|$, for $k = 1, 2, 3$. Using the boundary relation (4.16) and the estimates in (4.23), one finds, for $k = 1, 2$, that

$$\left| b^{(k)}\sigma^k \cdot \frac{\partial u}{\partial N} \right| = \left| b^{(k)}\, u \cdot \frac{\partial N}{\partial \sigma^k} \right| \le C\,|b|\,|u|\,\varepsilon \le C\,\varepsilon |v|\,|\nabla u|\,|u|.$$

Also, thanks to (4.15) one has

$$\begin{aligned} b^{(3)} &= (v \cdot \nabla u) \cdot N = [(\nabla u)v] \cdot N = -u \cdot [(\nabla N)v] \\ &= -u \cdot [(\nabla N)(v^{(1)}\sigma^1 + v^{(2)}\sigma^2)] = -u \cdot \left(v^{(1)} \frac{\partial N}{\partial \sigma^1} + v^{(2)} \frac{\partial N}{\partial \sigma^2} \right). \end{aligned}$$

Thus

$$\left| b^{(3)} N \cdot \frac{\partial u}{\partial N} \right| \le |b^{(3)}|\,|\nabla u| \le C\,|u|\,|v| \,\left(\left| \frac{\partial N}{\partial \sigma^1} \right| + \left| \frac{\partial N}{\partial \sigma^2} \right| \right)\, |\nabla u| \le C\,\varepsilon |u||v||\nabla u|.$$

Therefore

$$|I_3| \le C\varepsilon \int_\Gamma |v|\,|u|\,|\nabla u|\,d\sigma \le C\varepsilon \|v\|_{L^4(\Gamma)} \|u\|_{L^4(\Gamma)} \|\nabla u\|_{L^2(\Gamma)}.$$

By using $u = v + w$, we have

$$|I_3| \le C\varepsilon (\|v\|^2_{L^4(\Gamma)} + \|v\|_{L^4(\Gamma)} \|w\|_{L^4(\Gamma)})(\|\nabla v\|_{L^2(\Gamma)} + \|\nabla w\|_{L^2(\Gamma)}).$$

It follows from (4.71) and (4.73) that

$$\|v\|_{L^4(\Gamma)} \le C\|\bar{v}\|_{L^4(\Gamma)} \le C\varepsilon^{-\frac{1}{4}} \|\bar{v}\|_{L^4(\Omega^\varepsilon)} \le C\varepsilon^{-\frac{1}{2}} \|u|^{1/2}_{L^2} \|u|^{1/2}_{H^1}.$$

Applying (4.81) with $r = 2$ and $r = 4$, one has

$$\|w\|_{L^2(\Gamma)} \le C\varepsilon^{\frac{1}{2}} \|\nabla w\|_{L^2(\Omega^\varepsilon)}, \quad \|w\|_{L^4(\Gamma)} \le C\varepsilon^{\frac{3}{4}} \|\nabla w\|_{L^4(\Omega^\varepsilon)}.$$

Using (4.81) with $r = 4$, then the interpolation inequality and (4.87), with $q = 2$ and $q = 6$, yields

$$\|w\|_{L^4(\Gamma)} \le C\,\varepsilon^{\frac{3}{4}} \|\partial_3 w\|_{L^4(\Omega^\varepsilon)} \le C\,\varepsilon^{\frac{3}{4}} \|\partial_3 w\|^{1/4}_{L^2} \|\partial_3 w\|^{3/4}_{L^6} \le C\,\varepsilon \|u\|_{H^2}.$$

It follows that

$$\begin{aligned} |I_3| &\le C\,\varepsilon \left\{ \varepsilon^{-1} \|u\|_{L^2} \|u\|_{H^1} + (\varepsilon^{-\frac{1}{2}} \|u\|^{1/2}_{L^2} \|u\|^{1/2}_{H^1})(\varepsilon \|u\|_{H^2}) \right\} \\ &\quad \times \left\{ \varepsilon^{-\frac{1}{2}} \|u\|_{H^1} + \varepsilon^{\frac{1}{2}} \|u\|_{H^2} \right\} \\ &\le C\,\varepsilon \left\{ \varepsilon^{-1} \|u\|_{L^2} \|u\|_{H^1} + \varepsilon^{\frac{1}{2}} \|u\|^{1/2}_{L^2} \|u\|^{1/2}_{H^1} \|u\|_{H^2} \right\} \\ &\quad \times \left\{ \varepsilon^{-\frac{1}{2}} \|u\|_{H^1} + \varepsilon^{\frac{1}{2}} \|u\|_{H^2} \right\} \\ &\le C\,\varepsilon^{-\frac{1}{2}} \|u\|_{L^2} \|u\|^2_{H^1} + C\varepsilon \|u\|^{1/2}_{L^2} \|u\|^{3/2}_{H^1} \|u\|_{H^2} \\ &\quad + C\,\varepsilon^{\frac{1}{2}} \|u\|_{L^2} \|u\|_{H^1} \|u\|_{H^2} + C\varepsilon^2 \|u\|^{1/2}_{L^2} \|u\|^{1/2}_{H^1} \|u\|^2_{H^2} \\ &= I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4}. \end{aligned}$$

We have for $I_{3,1}$:

$$I_{3,1} \leq C\varepsilon^{-\frac{1}{2}} \|u\|_{L^2} \|u\|_{H^1} \|u\|_{H^2} \leq \alpha \|u\|_{H^2} + C_\alpha \varepsilon^{-1} \|u\|_{L^2}^2 \|u\|_{H^1}^2.$$

The other terms are easily bounded by

$$I_{3,2} + I_{3,3} + I_{3,4} \leq C\varepsilon^{\frac{1}{2}} \|u\|_{H^1} \|u\|_{H^2}^2.$$

It follows that

$$|I_3| \leq \alpha \|u\|_{H^2} + C_\alpha \varepsilon^{-1} \|u\|_{L^2}^2 \|u\|_{H^1}^2 + C\varepsilon^{\frac{1}{2}} \|u\|_{H^1} \|u\|_{H^2}^2. \tag{5.11}$$

Combining the equations (5.3), (5.6), with the estimates (5.4), (5.5), (5.10) and (5.11), we obtain (5.1). The proof is complete. □

6 Global Solutions and Longtime Dynamics

We now turn to the proofs of the Main Results described in Section 3. However, we first recall the definitions and the resulting properties of Leray-Hopf weak solutions and maximally defined strong solutions for the Navier–Stokes equations (1.1). For more information, see [2, 5, 10, 11, 40, 41, 43, 48]. We will refer to the spaces H, $V^1 = \mathcal{D}(A^{\frac{1}{2}})$, $V^2 = \mathcal{D}(A)$, and V^{-1} here, see (2.9)-(2.11). The norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ used in this section are in the space $L^2(\Omega^\varepsilon)$. As usual, we assume that f satisfies the Forcing Function Assumption.

A function $v = v(t) : [0, \infty) \to H$, is a Leray-Hopf **weak solution** of the Navier–Stokes equations (1.1) with $v(0) = v_0 \in H$, if the following properties hold:

- One has $v \in L^\infty[0, \infty; H) \cap L^2_{\text{loc}}[0, \infty; V^1)$.
- One has $dv/dt \in L^q_{\text{loc}}[0, \infty; V^{-1})$, for some q with $1 \leq q < \infty$.
- There is a subset $\mathcal{E}$ in $[0, \infty)$, with the complement $[0, \infty) \setminus \mathcal{E}$ having zero Lebesgue measure, such that $0 \in \mathcal{E}$; and for $t_0 \in \mathcal{E}$, all $w \in V^1$ and $t \geq t_0$, the function v satisfies

$$\begin{aligned} \langle v(t) - v(t_0), w \rangle + \int_{t_0}^t E(v(s), w)\, ds + \int_{t_0}^t \langle B(v(s), v(s)), w \rangle\, ds \\ = \int_{t_0}^t \langle f(s), w \rangle\, ds. \end{aligned} \tag{6.1}$$

  Moreover, $v(t)$ is strongly continuous (in $L^2(\Omega^\varepsilon)$) from the right at every $t_0 \in \mathcal{E}$.

- For all $t_0 \in \mathcal{E}$ and all $t \geq t_0$, the function v satisfies

$$\|v(t)\|^2 + 2\int_{t_0}^t E(v(s), v(s))\, ds \leq \|v(t_0)\|^2 + 2\int_{t_0}^t \langle f(s), v(s) \rangle\, ds. \tag{6.2}$$

It follows from these properties that a weak solution v also satisfies:

- $v \in C[0,\infty; H_w)$, where H_w denotes the Hilbert space H with the weak topology.
- $dv/dt \in L^{\frac{4}{3}}_{\mathrm{loc}}[0,\infty; V^{-1})$, and $v \in C^{0,\theta}_{\mathrm{loc}}[0,\infty; V^{-1})$, for some $\theta > 0$.

A function $v = v(t)$ is a **strong solution** of the Navier–Stokes equations (1.1) on the interval $I = [t_0, T_1)$, where $0 \le t_0 < T_1 \le \infty$, provided that the following hold:

- one has $t_0 \in \mathcal{E}$ and $v(t_0) \in V^1$;
- the function v is the restriction of a Leray-Hopf weak solution to the interval I;
- the solution v satisfies

$$v \in C(I; V^1) \cap L^2_{\mathrm{loc}}(I; V^2). \tag{6.3}$$

A strong solution $v(t)$ of (1.1), on an interval $[t_0, T_0)$, is said to be **maximally defined** if either $T_0 = \infty$, or $v(t)$ has no proper extension, as a strong solution, to an interval $[t_0, T_1)$, where $T_1 > T_0$. When v is a maximally defined strong solution on $[t_0, T_0)$ and $T_0 < \infty$, then one has:

$$\lim_{t \to T_0^-} \|A^{\frac{1}{2}} v(t)\|^2 = \infty. \tag{6.4}$$

As noted in [41], the following properties of a maximally defined strong solution v on $[t_0, T_0)$ follow from this definition:

(i) one has

$$dv/dt \in L^2_{\mathrm{loc}}[t_0, T_0; H); \tag{6.5}$$

(ii) v is a mild solution in the spaces H and V^1, i.e., the Variation of Constants Formula:

$$v(t) = e^{-A(t-t_0)} v(t_0) + \int_{t_0}^{t} e^{-A(t-s)} [f(s) - B(v(s), v(s))]\, ds, \tag{6.6}$$

is valid in these spaces, for $t_0 \le t < T_0$; and there exist $\theta_0 > 0$, $\theta_1 > 0$ such that

(iii) v satisfies $v \in C^{0,\theta_0}_{\mathrm{loc}}[t_0, T_0; H) \cap C^{0,\theta_1}_{\mathrm{loc}}(t_0, T_0; V^1)$.

6.1 Proof of the Main Theorem

We will use here the notation of Sections 2 and 3. We begin with the following lemma which gives energy estimates for the strong solutions of the Navier–Stokes equations that satisfy the relations in (3.2). Note that the expressions $|m|_\varepsilon^2$, $|\widehat{m}|_\varepsilon^2$, $|\ell|_\varepsilon^2$ and α appear in (3.3), (3.4), and (3.5).

Lemma 6.1. *Assume that* $0 < \varepsilon \le \varepsilon_0$*, the data* $(f, u_0) \in L^\infty(L^2) \times V^1$ *satisfy* (3.2)*, and let* $u(t)$ *denote the corresponding maximally defined strong solution of* (1.1)*. Then there are constants* $d_1^2 \ge 1$ *and* $d_2^2 \ge 1$ *which do not depend on* ε*, such that:*

$$\|u(t)\|_{L^2}^2 \le d_1^2(e^{-2\alpha t}|m|_\varepsilon^2 + |\ell|_\varepsilon^2), \qquad \textit{for } t \ge 0, \tag{6.7}$$

$$\int_{t-1}^t \|A^{\frac{1}{2}} u(\tau)\|_{L^2}^2 \, d\tau \le d_2^2(e^{-2\alpha t}|m|_\varepsilon^2 + |\ell|_\varepsilon^2), \qquad \textit{for } t \ge 1. \tag{6.8}$$

Proof. Because it is needed later, we begin the proof by seeking an estimate for a Leray-Hopf weak solution $u = u(t)$, with $u(0) = u_0 \in H$. It follows from (6.2) that

$$2\int_0^t \|A^{\frac{1}{2}} u(\tau)\|^2 \, d\tau \le \|u_0\|^2 + 2\int_0^t \langle f(\tau), u(\tau)\rangle \, d\tau. \tag{6.9}$$

Next we set $\widehat{v} = M_2 u$ and $\widehat{w} = u - \widehat{v} = (I - M_2)u$. Since $u(t) \in V^1$, almost everywhere, and $\mathbb{P}$ and M_2 are orthogonal projections on $L^2(\Omega^\varepsilon)$, one has

$$\langle f, u\rangle = \langle (I - M_2)\mathbb{P} f, \widehat{w}\rangle + \langle M_2 \mathbb{P} f, \widehat{v}\rangle.$$

By using (4.77) and $\|\widehat{v}\|_{L^2} \le \|u\|_{H^1}$, one obtains

$$\begin{aligned} |\langle f, u\rangle| &\le C\,(\varepsilon\|u\|_{H^1}\|(I - M_2)\mathbb{P} f\|_\infty + \|u\|_{H^1}\|M_2\mathbb{P} f\|_\infty) \\ &\le \frac{c_1}{2}\|u\|_{H^1}^2 + C\,(\varepsilon^2\|(I - M_2)\mathbb{P} f\|_\infty^2 + \|M_2\mathbb{P} f\|_\infty^2), \end{aligned}$$

and relations (4.54), (3.2)–(3.4) imply

$$|\langle f, u\rangle| \le \frac{1}{2}\|A^{\frac{1}{2}} u\|^2 + C\,(K_{1,s}^2 \varepsilon^{s+1} + K_{0,r}^2 \varepsilon^r) \le \frac{1}{2}\|A^{\frac{1}{2}} u\|^2 + C\,|\ell|_\varepsilon^2. \tag{6.10}$$

By combining this with (6.9) one obtains the following inequality for the Leray-Hopf weak solutions:

$$\int_0^t \|A^{\frac{1}{2}} u(\tau)\|^2 \, d\tau \le \|u_0\|^2 + t\, C_3\, |\ell|_\varepsilon^2, \qquad \text{for } t \ge 0, \tag{6.11}$$

where the positive number C_3 is independent of ε.

In the case of the strong solution $u = u(t)$, where $u_0 \in V^1$, one takes the scalar product of equations (3.19) with u, and uses (6.5) to obtain $\langle u, \partial_t u\rangle = \frac{1}{2}\frac{d}{dt}\|u\|^2$ (see [41], Lemma 37.9, or [43]) and

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|A^{\frac{1}{2}} u\|^2 \le |\langle f, u\rangle|.$$

It then follows from (6.10) that

$$\frac{d}{dt}\|u\|^2 + \|A^{\frac{1}{2}}u\|^2 \leq C\,|\ell|_\varepsilon^2. \tag{6.12}$$

By using $\|A^{\frac{1}{2}}u\|^2 \geq c_1\|u\|_{H^1}^2 \geq 2\alpha\|u\|_{L^2}^2$ and the Gronwall inequality, one obtains

$$\|u(t)\|^2 \leq e^{-2\alpha t}\|u_0\|^2 + C\,|\ell|_\varepsilon^2, \qquad \text{for } t \geq 0. \tag{6.13}$$

Next we use (4.77) and (3.2) to estimate

$$\|u_0\|_{L^2}^2 = \|M_2 u_0\|_{L^2}^2 + \|u_0 - M_2 u_0\|_{L^2}^2 \leq k_{0,p}^2\,\varepsilon^p + C\,\varepsilon^2\|u_0\|_{H^1}^2 \leq C\,|m|_\varepsilon^2. \tag{6.14}$$

The existence of the constant $d_1^2 \geq 1$, as well as the validity of inequality (6.7), follow immediately from (6.13) and (6.14). For $t \geq 1$, by integrating the inequality (6.12) from $t-1$ to t, and using (6.7), one obtains

$$\int_{t-1}^t \|A^{\frac{1}{2}}u(s)\|^2\,ds \leq \|u(t-1)\|^2 + C\,|\ell|_\varepsilon^2 \leq d_1^2(e^{-2\alpha(t-1)}|m|_\varepsilon^2 + |\ell|_\varepsilon^2) + C|\ell|_\varepsilon^2.$$

Hence the inequality (6.8) follows. □

A key feature in our proof of the Global Existence Theorem is the use of the Uniform Gronwall Inequality, see [8] and [41], Appendix D.

Lemma 6.2 (Uniform Gronwall Inequality)**.** *Let y, g, and h be functions in $L^1([0,T),\mathbb{R})$, where $0 < T \leq \infty$. Assume that y is nonnegative and absolutely continuous on $[0,T)$ and that*

$$\frac{d}{dt}y(t) \leq g(t)\,y(t) + h(t), \quad \textit{almost everywhere on } (0,T). \tag{6.15}$$

Then for $0 < t < T$ and $\tau = \max(0, t-1)$, one has

$$y(t) \leq \Big(\frac{1}{t-\tau}\int_\tau^t y(s)\,ds + \int_\tau^t h(s)\,ds\Big)\,exp\Big(\int_\tau^t g(s)\,ds\Big). \tag{6.16}$$

Next let $0 < \varepsilon \leq \varepsilon_1$. One takes the scalar product of equation (1.1) with Au noting that $\langle Au, \partial_t u\rangle = \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2$ (since (6.3) holds), and use Corollary 5.2, with $\beta = \frac{1}{4}$, and the Young inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2 + \|Au\|^2 &\leq |\langle (u\cdot\nabla)u, Au\rangle| + |\langle f, Au\rangle| \\ &\leq \frac{1}{2}\|Au\|^2 + C\,\varepsilon^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|\,\|Au\|^2 + C\left(\|\mathbb{P}\,f\|_\infty^2 + \varepsilon^{-1}\|u\|^2\|A^{\frac{1}{2}}u\|^2\right).\end{aligned}$$

This implies that there are positive constants C_0 and C_1, which do not depend on ε, such that

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|^2 + \|Au\|^2 \leq C_0\,\varepsilon^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|\,\|Au\|^2 + h, \tag{6.17}$$

where $h = C_1 \left(\|\mathbb{P} f\|_\infty^2 + \varepsilon^{-1} \|u\|^2 \|A^{\frac{1}{2}} u\|^2 \right)$. Note from (3.2) and (3.4) that

$$\begin{aligned} \|\mathbb{P} f\|_\infty^2 &\leq \|M_2 \mathbb{P} f\|_\infty^2 + \|(I - M_2)(\mathbb{P} f)\|_\infty^2 \\ &\leq K_{0,r}^2 \varepsilon^r + K_{1,s}^2 \varepsilon^{-1+s} \leq \varepsilon^{-1} |\ell|_\varepsilon^2. \end{aligned} \tag{6.18}$$

Denote $R_\varepsilon^2 = d_1^2 (|m|_\varepsilon^2 + |\ell|_\varepsilon^2)$. One has from (6.7) that

$$\|u(t)\|^2 \leq R_\varepsilon^2, \quad \text{for all } t \geq 0. \tag{6.19}$$

Combining this with (6.8) and (6.18), one obtains

$$\int_0^t h(s)\, ds \leq C_2\, \varepsilon^{-1} \Big(|\ell|_\varepsilon^2 + R_\varepsilon^2 (|m|_\varepsilon^2 + |\ell|_\varepsilon^2) \Big), \quad \text{for } 0 < t < 1, \tag{6.20}$$

$$\int_{t-1}^t h(s)\, ds \leq C_2\, \varepsilon^{-1} \Big(|\ell|_\varepsilon^2 + R_\varepsilon^2 (e^{-2\alpha t} |m|_\varepsilon^2 + |\ell|_\varepsilon^2) \Big), \quad \text{for } t \geq 1, \tag{6.21}$$

where $C_2 \geq \max(d_1^2, d_2^2) \geq 1$ is a constant which does not depend on ε. We define

$$d_0^2 = 2(1 + C_2). \tag{6.22}$$

Also we fix $R_2 > 0$ so that

$$R_2^2 \leq \min\Big(\frac{2\, d_0^2}{d_1^2}, \frac{1}{4\, C_0^2} \Big). \tag{6.23}$$

Main Assumption: We assume that K-vector $\mathbb{K}$ satisfies

$$\boxed{\widehat{R}_\varepsilon^2 \stackrel{\text{def}}{=\!=} 2 d_0^2 \left(|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2 \right) \leq R_2^2.} \tag{6.24}$$

We note that $R_0^2 = R_2^2\, (2 d_0^2)^{-1}$ is the constant mentioned in the statement of the Main Theorem. Since $2 d_0^2 R_\varepsilon^2 \leq d_1^2 \widehat{R}_\varepsilon^2$ it follows from (6.23) that

$$R_\varepsilon^2 \leq \frac{d_1^2}{2\, d_0^2}\, \widehat{R}_\varepsilon^2 \leq \frac{d_1^2}{2\, d_0^2}\, R_2^2 \leq 1. \tag{6.25}$$

Lemma 6.3. *Assume that* $0 < \varepsilon \leq \varepsilon_1$ *and the Main Assumption* (6.24) *holds. Let* (f, u_0) *satisfy* (3.2). *Then the corresponding strong solution* $u(t)$ *of* (1.1) *satisfies*

$$\|A^{\frac{1}{2}} u(t)\|^2 \leq \begin{cases} \varepsilon^{-1} d_0^2\, (|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2), & \textit{for } 0 \leq t < 1, \\ \varepsilon^{-1} d_0^2\, (e^{-2\alpha t} |m|_\varepsilon^2 + |\ell|_\varepsilon^2), & \textit{for all } t \geq 1. \end{cases} \tag{6.26}$$

Proof. Without loss of generality, we assume $(f, u_0) \neq (0,0)$. Hence $\widehat{R}_\varepsilon^2 > 0$. Let $u = u(t)$ denote the maximally defined strong of (1.1), with $u(0) = u_0$, and let $[0,T)$ denote the interval of existence of u. Note that (4.54) and (6.22) imply that

$$\|A^{\frac{1}{2}} u_0\|^2 \le 2\|u_0\|_{H^1}^2 \le 2k_{1,q}^2 \varepsilon^{-1+q} \le 2\varepsilon^{-1}|\widehat{m}|_\varepsilon^2 \le \varepsilon^{-1} d_0^2 |\widehat{m}|_\varepsilon^2 < \varepsilon^{-1}\widehat{R}_\varepsilon^2. \tag{6.27}$$

We claim that

$$\|A^{\frac{1}{2}} u(t)\|^2 < \varepsilon^{-1}\widehat{R}_\varepsilon^2, \qquad \text{for all } t \in [0,T). \tag{6.28}$$

Indeed, if (6.28) fails, then there is a T_0, with $0 < T_0 < T$, such that

$$\begin{aligned} \|A^{\frac{1}{2}} u(t)\|^2 &< \varepsilon^{-1}\widehat{R}_\varepsilon^2, \qquad \text{for } 0 \le t < T_0, \\ \|A^{\frac{1}{2}} u(T_0)\|^2 &= \varepsilon^{-1}\widehat{R}_\varepsilon^2. \end{aligned} \tag{6.29}$$

It follows from (6.23), (6.24) and (6.29) that

$$C_0\, \varepsilon^{\frac{1}{2}} \|A^{\frac{1}{2}} u(t)\| \le C_0\, R_2 \le \frac{1}{2}, \qquad \text{for } 0 \le t < T_0. \tag{6.30}$$

Consequently, the inequalities (6.17) and (6.30) imply that

$$\frac{d}{dt}\|A^{\frac{1}{2}} u(t)\|^2 + \frac{1}{2}\|Au(t)\|^2 \le h(t), \qquad \text{for } 0 < t < T_0. \tag{6.31}$$

From (4.55) and (3.5), one has $\|Au\|^2 \ge c_1\|A^{\frac{1}{2}}u\|^2 = 4\alpha\|A^{\frac{1}{2}}u\|^2$. Hence inequality (6.31) implies that

$$\frac{d}{dt}\|A^{\frac{1}{2}} u(t)\|^2 + 2\alpha\, \|A^{\frac{1}{2}} u(t)\|^2 \le h(t), \qquad \text{for } 0 < t < T_0. \tag{6.32}$$

Assume that $T_0 \le 1$. For $t \in (0, T_0]$, by using the Gronwall inequality, (6.27) and (6.20), as well as (6.25), one finds that

$$\begin{aligned} \|A^{\frac{1}{2}} u(t)\|^2 &\le e^{-2\alpha t}\|A^{\frac{1}{2}} u_0\|^2 + \int_0^t h(s)\, ds \\ &\le \varepsilon^{-1}\Big(2e^{-2\alpha t}|\widehat{m}|_\varepsilon^2 + C_2|\ell|_\varepsilon^2 + C_2(|m|_\varepsilon^2 + |\ell|_\varepsilon^2)\Big) \\ &\le \varepsilon^{-1} d_0^2\,(|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2) = \frac{1}{2}\varepsilon^{-1}\,\widehat{R}_\varepsilon^2. \end{aligned} \tag{6.33}$$

Since $\widehat{R}_\varepsilon^2 > 0$, relation (6.29) then implies that $T_0 > 1$.

When $T_0 > 1$, one sets $g(t) = -2\alpha$, $y(t) = \|A^{\frac{1}{2}} u(t)\|^2$ and uses the Uniform Gronwall Inequality for the differential inequality (6.32), the estimates (6.8), (6.21), and (6.25) to obtain

$$\begin{aligned} \|A^{\frac{1}{2}} u(t)\|^2 &\le \int_{t-1}^t \|A^{\frac{1}{2}} u(s)\|^2\, ds + \int_{t-1}^t h(s)\, ds \\ &\le d_2^2(e^{-2\alpha t}|m|_\varepsilon^2 + |\ell|_\varepsilon^2) + C_2\, \varepsilon^{-1}\Big(|\ell|_\varepsilon^2 + (e^{-2\alpha t}|m|_\varepsilon^2 + |\ell|_\varepsilon^2)\Big) \\ &\le \varepsilon^{-1} d_0^2\,(e^{-2\alpha t}|m|_\varepsilon^2 + |\ell|_\varepsilon^2), \end{aligned} \tag{6.34}$$

for $1 \le t \le T_0$. Since $|m|_\varepsilon^2 \le |\widehat{m}|_\varepsilon^2$, it follows from (6.23) and (6.24) that

$$\|A^{\frac{1}{2}} u(T_0)\|^2 \le \varepsilon^{-1} d_0^2 \, (|\widehat{m}|_\varepsilon^2 + |\ell|_\varepsilon^2) = \frac{1}{2} \varepsilon^{-1} \, \widehat{R}_\varepsilon^2,$$

which contradicts the relation (6.29). Therefore (6.28) holds true. This in turn implies that $T = \infty$, see (6.4). Subsequently, the estimates in (6.26) follow from (6.33) for $t < 1$ and (6.34) for $t \ge 1$. The proof is complete. □

Proof of Main Theorem: The inequalities (3.7) and (3.8) follow from (6.26) and (4.54) with $D_1^2 = c_1^{-1} \, d_0^2$. The estimates (3.9) and (3.10) are obtained by integrating the differential inequality (6.31) and using the inequalities (6.26), (6.20), (6.21), (6.25) and (4.54). □

Corollaries 3.2 and 3.3 are now direct consequences of Theorem 3.1.

6.2 Longtime Dynamics

We include here a brief introduction to the theory of infinite dimensional dynamical systems, as this theory is used in the context of the Navier–Stokes equations. While most of the material presented here is based on [41], other references, such as [3, 12, 31, 32, 44], are very useful

Let $\mathcal{H} = \mathcal{H}_\varepsilon$ denote a subset of

$$CL^\infty \;=\; L^\infty(\mathbb{R}, L^2) \cap C(\mathbb{R}, L^2) \;=\; L^\infty(\mathbb{R}, L^2(\Omega^\varepsilon)) \cap C(\mathbb{R}, L^2(\Omega^\varepsilon)).$$

We say that $\mathcal{H}$ satisfies the **Hölder property** if there are z, $s \in (0, 1]$ such that for every $f \in \mathcal{H}$, there is a positive constant $L = L(f)$ such that

$$\|f(t_1) - f(t_2)\|_{L^2} \;\le\; L\,|t_1 - t_2|^z, \quad \text{for all } t_1,\, t_2 \in \mathbb{R} \text{ with } |t_1 - t_2| < s. \tag{6.35}$$

For $f \in CL^\infty$, we define the time-translation f_τ by $f_\tau(t) = f(\tau + t)$, for all τ, $t \in \mathbb{R}$, and we define the norm $\|f\|_\infty$ as

$$\|f\|_\infty \stackrel{\text{def}}{=\!=} \sup_{t \in \mathbb{R}} \|f(t)\|_{L^2}. \tag{6.36}$$

We will be using the metrizable topology of uniform convergence on bounded sets (the **ucbs-topology**) on CL^∞. Thus one has $f^n \to g$ (in this topology) if and only if, for every bounded set $I \subset \mathbb{R}$, one has

$$\sup_{t \in I} \|f^n(t) - g(t)\|_{L^2} \;\to\; 0, \quad \text{as } n \to \infty.$$

We note that, with this topology, the mapping $(f, \tau) \to \sigma(f, \tau) = f_\tau$ defines a (two-sided) flow on CL^∞, see [41]. We say that $\mathcal{H}$ is a (time translation) **invariant** subset if one has $f_\tau \in \mathcal{H}$ whenever $f \in \mathcal{H}$ and $\tau \in \mathbb{R}$, that is to say, $\mathcal{H}$ is an invariant set for σ.

Next we consider the family of Navier–Stokes equations given by (3.19), where $\mathcal{H}$ is an invariant set in CL^∞ with the Hölder property. We assume further that $\mathcal{H}$ is a compact set in the ucbs-topology. For example, if in addition, one has

$$\mathcal{H} \subset L^\infty(\mathbb{R}, H^r(\Omega^\varepsilon)) \cap C(\mathbb{R}, H^r(\Omega^\varepsilon)),$$

for some r with $r > 0$, and

$$\sup_{f\in\mathcal{H}} \sup_{t\in\mathbb{R}} \|f(t)\|_{H^r(\Omega^\varepsilon)} < \infty,$$

then - because of the compact imbedding $H^r(\Omega_\varepsilon) \hookrightarrow L^2(\Omega_\varepsilon)$ - it follows from the Ascoli-Arzelá Theorem that $\mathcal{H}$ is a compact set in the ucbs-topology. For example, $\mathcal{H}$ may be a quasi periodic minimal set, see [40].

6.2.1 Skew Product Structures

With $f \in \mathcal{H}$, the basic existence and uniqueness theorem for strong solutions applies to each of the equations in (3.19), see Theorem 64.4 in [41]. In particular, for each $v_0 \in V^1$ and each $f \in \mathcal{H}$, we let $s(f,t)v_0$ represent the maximally defined strong solution of (3.19) that satisfies $s(f,0)v_0 = v_0$. Set $T_0 = T_0(f, v_0)$, where $T_0 \in (0,\infty]$ and $[0,T_0)$ denotes the interval of definition of $s(f,t)v_0$. The strong solutions $v = v(t) = s(f,t)v_0$ have additional properties. In particular, they are Lipschitz continuous functions of the data (f, v_0), uniformly on bounded intervals, see Theorem 64.8 in [41]. As noted above, the topology on $\mathcal{H} \times V^1$ is the product topology with the ucbs-topology on $\mathcal{H}$ and the strong topology - that is, the H^1-norm topology - on V^1.

The **semiflow** generated by the maximally defined strong solutions of (3.19) is denoted by $\pi(t) = \pi^\varepsilon(t)$, where

$$\pi(\tau)(f, v_0) \stackrel{\text{def}}{=} (f_\tau, s(f,\tau)v_0), \qquad \text{for } \tau \in [0, T_0(f, v_0)). \tag{6.37}$$

Next define Σ^+ by:

$$\Sigma^+ = \Big\{(f, v_0, \tau) \in \mathcal{H} \times V^1 \times [0,\infty) : \tau \in [0, T_0(f, v_0))\Big\}, \tag{6.38}$$

and recall that

$$\mathcal{M} = \Big\{(f, v_0) \in \mathcal{H} \times V^1 : T_0(f, v_0) = \infty\Big\}. \tag{6.39}$$

The proof of the following result is standard, see for example, Section 6.5.3 in [41] or [40].

Lemma 6.4. *The following hold:*

(i) The semiflow mapping

$$\pi : (f, v_0, \tau) \longrightarrow \pi(\tau)(f, v_0) = (f_\tau, s(f,\tau)v_0)$$

is a continuous mapping of Σ^+ into $\mathcal{H} \times V^1$ with $\pi(0)(f, v_0) = (f, v_0)$.

(ii) *The set* Σ^+ *is an open set in* $\mathcal{H} \times V^1 \times [0, \infty)$.

(iii) *Whenever one has* $\tau \in [0, T_0(f, v_0))$ *and* $\sigma \in [0, T_0(f_\tau, s(f, \tau)v_0))$ *then one has* $\tau + \sigma \in [0, T_0(f, v_0))$ *and*

$$s(f, \tau + \sigma)v_0 = s(f_\tau, \sigma)\, s(f, \tau)v_0. \tag{6.40}$$

In other words, whenever the right-side of equation (6.40) *is defined, then the left side of* (6.40) *is defined* and *equality holds.*

(iv) *One has* $\pi(t)\,\mathcal{M} \subset \mathcal{M}$, *for all* $t \geq 0$, *i.e.* $\mathcal{M}$ *is a positively invariant set in* $\mathcal{H} \times V^1$.

This brings us to two very important concepts. A continuous mapping $\phi : (-\infty, T_1) \to V^1$ is said to be a **negative continuation** of the strong solution $s(f, t)v_0$ provided that ϕ satisfies: (1) $\phi(0) = v_0$, (2) $T_1 = T_0(f, v_0) > 0$, and (3) for all $\tau \in (-\infty, T_1)$, ϕ satisfies

$$s(f_\tau, t)\phi(\tau) = \phi(\tau + t), \qquad \text{for all } t \in [0, T_1 - \tau). \tag{6.41}$$

A **global solution** through the point $(f, v_0) \in \mathcal{M}$ is a continuous mapping $\phi : \mathbb{R} \to V^1$ such that: (1) $\phi(0) = v_0$ and (2) ϕ satisfies

$$s(f_\tau, t)\phi(\tau) = \phi(\tau + t), \qquad \text{for all } \tau \in \mathbb{R} \text{ and all } t \in [0, \infty).$$

It is important to note that, when a negative continuation ϕ of a solution $s(f, t)v_0$ exists, it need not be unique. This lack of uniqueness is a major complication that arises in infinite dimensional dynamical systems. Nevertheless, it is convenient to adopt a notational convention here: For $\tau \leq 0$, we set $s(f, \tau)v_0 := \phi(\tau)$. In this way, (6.41) reads

$$s(f_\tau, t)\, s(f, \tau)v_0 = s(f, \tau + t)v_0, \qquad \text{for all } t \in [0, T_1 - \tau).$$

For any global solution ϕ, $s(f, \tau)\phi(0) = s(f, \tau)v_0 = \phi(\tau)$ is defined for all $\tau \in \mathbb{R}$, and it satisfies:

$$s(f_\tau, t)\, s(f, \tau)v_0 = s(f, \tau + t)v_0, \qquad \text{for all } \tau \in \mathbb{R} \text{ and all } t \geq 0. \tag{6.42}$$

Remark 6.5. One uses the Hölder property for $\mathcal{H}$ to show that, under appropriate conditions, a mild solution for (3.19) is a strong solution, as well, see [41], Theorem 42.9.

6.2.2 Hull and Omega Limit Set

Let K be any subset of $\mathcal{M}$. We define the (positive) **orbit** of K and the (positive) **hull** of K as

$$\gamma^+(K) \overset{\text{def}}{=\!=} \bigcup_{t \geq 0} \pi(t)K \quad \text{and} \quad H^+(K) = \overline{\gamma^+(K)},$$

that is, $H^+(K)$ is the closure of the orbit $\gamma^+(K)$ in $\mathcal{H} \times V^1$. The **omega limit set** of K is

$$\omega(K) \stackrel{\text{def}}{=\!=} \bigcap_{\tau \geq 0} H^+(\pi(\tau)\, K).$$

An example, which arises below, is $K = \mathcal{H} \times \widehat{B}_0$, where $\widehat{B}_0$ is a nonempty, bounded, open set in V^1. In this example, $\gamma^+(K)$ is a bounded set in $\mathcal{M}$, that is to say, there is a constant $b \geq 0$ such that for all $(f, u) \in \gamma^+(K)$ one has $\|u\|_{H^1} \leq b$. In this example, $\omega(K)$ is a nonempty, compact, invariant set for π, that is, (6.43) holds, with $\mathcal{K} = \omega(K)$.

6.2.3 Invariant Sets

Assume for the time being that, there is a single datum $(g, v) \in \mathcal{H} \times V^1$, with the property that the strong solution $s(g, t)v$ satisfies

$$\sup_{t \geq 0} \|A^{\frac{1}{2}} s(g, t)v\|^2 \; = \; \rho^2 \; < \; \infty.$$

It follows then that for every $r \geq \rho$, the set

$$K = K_r = \left\{(f, v_0) \in \mathcal{H} \times V^1 \; : \; \sup_{t \geq 1} \|A^{\frac{1}{2}} s(f, t)v_0\|^2 \leq r^2\right\}$$

is a nonempty, compact positively invariant set for the semiflow $\pi = \pi(t)$ and $K_r \subset \mathcal{M}$. Since $s(f, t)$ is a compact operator for $t > 0$, the set K_r has compact closure. Furthermore, the omega limit set $\mathcal{K} = \omega(K_r)$ is a nonempty, compact, invariant set, that is to say,

$$\pi(t)\, \mathcal{K} = \mathcal{K}, \qquad \text{for all } t \geq 0. \tag{6.43}$$

The identity (6.43) implies that for every $(f, v_0) \in \mathcal{K}$, there is a global solution $v(t) = s(f, t)v_0$; v is defined for <u>all</u> $t \in \mathbb{R}$; and $(f_\tau, v(\tau)) \in \mathcal{K}$, for all $\tau \in \mathbb{R}$. Since $\mathcal{K} \subset \mathcal{M}$, it follows that for all $(f, v_0) \in \mathcal{K}$, the global solution $s(f, t)v_0$ satisfies (6.42).

We need the following result. The proof of this lemma in the case that $\mathcal{H}$ is a quasi periodic minimal set appears in [40]. This proof uses the mild solution formulation (6.6). Since the argument for the more general case considered here is essentially the same, we will not include the details.

Lemma 6.6. *Let $\mathcal{H}$ be a compact, invariant set in CL^∞ with the Hölder property. Let $\mathcal{K}$ be a bounded set in $\mathcal{H} \times V^1$, and assume that $\mathcal{K}$ is an invariant set for the semiflow $\pi(t)$. Let $\overline{\mathcal{K}}$ denote the closure of $\mathcal{K}$ in $\mathcal{H} \times V^1$. Then $\overline{\mathcal{K}}$ is a bounded, invariant set, and one has $\overline{\mathcal{K}} \subset \mathcal{H} \times V^2$. Moreover, for every $(f, v_0) \in \overline{\mathcal{K}}$, the global strong solution $s(f, t)v_0$ is both a mild solution and a classical solution of equation (3.19) in $V^{2r} = \mathcal{D}(A^r)$, for every r with $0 \leq r < 1$, for all $t \in \mathbb{R}$. Furthermore, $\overline{\mathcal{K}}$ is a compact, invariant set in $\mathcal{H} \times V^{2r}$, for each r with $0 \leq r < 1$. In addition, one has*

$$s(f, \cdot)v_0 \in C^{0,1-r}_{loc}(\mathbb{R}; V^{2r}) \cap C(\mathbb{R}; V^2). \tag{6.44}$$

6.2.4 Attractors

A set $\mathfrak{A}$ in $\mathcal{M}$ is said to be an **attractor** for the semiflow π on $\mathcal{M}$ provided that

- $\mathfrak{A}$ is a compact, invariant set in $\mathcal{M}$, and
- there is a bounded neighborhood U of $\mathfrak{A}$ in $\mathcal{M}$, such that $\mathfrak{A}$ attracts U.

To say that $\mathfrak{A}$ **attracts** U, we mean that

$$d(\pi(t)\,U, \mathfrak{A}) \;\to\; 0, \qquad \text{as } t \to \infty,$$

where $d(B, \mathfrak{A})$ is defined, for every bounded set B in $\mathcal{M}$, as

$$d(B, \mathfrak{A}) = \inf\{\varepsilon > 0 \,:\, B \subset N_\varepsilon(\mathfrak{A})\},$$

and $N_\varepsilon(\mathfrak{A})$ is the ε-neighborhood of $\mathfrak{A}$ in the $(\mathcal{H} \times V^1)$-topology. See [41], Chapter 2, for additional information. A short history of the concept of the "attractor", with references, is in [41], pages 53–59.) The concept we use here is widely accepted because, when $\mathfrak{A}$ is an attractor - in the sense used above - then $\mathfrak{A}$ has two very important stability properties:

- $\mathfrak{A}$ is Lyapunov stable, and
- $\mathfrak{A}$ is asymptotically stable,

see [41], Theorem 23.10. Because of the stability properties, an attractor is "robust" under small changes in the parameters of the model, see [41], Section 2.3.6. While the attractor $\mathfrak{A}$ is robust, this does not mean that the corresponding solution $s(f,t)v_0$ on $\mathfrak{A}$ is either boring, or non-chaotic, or even non-turbulent.

For any attractor $\mathfrak{A}$ in $\mathcal{M}$, we define $B(\mathfrak{A})$, the **basin** of $\mathfrak{A}$, as the collection of all $(f, v_0) \in \mathcal{M}$ such that $d(\pi(t)(f, v_0), \mathfrak{A}) \to 0$, as $t \to \infty$. We say that an attractor $\mathfrak{A}$ is a **global attractor** for $\pi(t)$ when $B(\mathfrak{A}) = \mathcal{M}$, see [41].

The theory of global attractors for the Navier–Stokes equations is still being developed. In 2D, both the weak solutions and the strong solutions of the Navier–Stokes equations on suitable bounded domains have global attractors, see [3, 5, 8, 25, 36, 37, 41, 44], for example. In 3D, the weak solutions of the Navier–Stokes equations has a global attractor, see [39, 7]. In the case of the strong solutions, the existence of global attractors is known for thin-domains, as is noted in Section 1 and is shown below.

How does one find the global attractor for the weak solutions of the 3D problem? Since the weak solutions are not known to be uniquely determined by the data (f, v_0), when $v_0 \in H$, one is led to the technique of treating each weak solution $\varphi(t) = S(t)v_0$ as a point in a suitable function space, see [38] and [39]. In the case of the Navier–Stokes equations, the appropriate function space is the Fréchet space $L^2_{\text{loc}}[0,\infty; H)$. This induces a semiflow $\pi_w(\tau)$ on $\mathcal{H} \times L^2_{\text{loc}}[0,\infty; H)$, where

$$\pi_w(\tau)(f, \varphi) = (f_\tau, \varphi_\tau), \qquad \text{for } \tau \geq 0,$$

and $\varphi_\tau(t) = \varphi(\tau+t)$. The global attractor $\mathfrak{A}_w$ is a nonempty, compact, invariant subset of $\mathcal{H} \times L^2_{\rm loc}[0,\infty; H)$, see [39]. Let $(f,\varphi) \in \mathfrak{A}_w$, where φ is a global weak solution of (3.19) that satisfies

$$\varphi \in L^\infty(\mathbb{R}, H) \cap L^2_{\rm loc}(\mathbb{R}, V^1). \tag{6.45}$$

It is important to note that if $(f,\varphi) \in \mathfrak{A}_w$, then φ_τ is a global solution of the shifted equation in (3.19), where f is replaced by f_τ. By using the methodology of Foias and Temam [9], one can readily show that the weak solution satisfies the continuity property: $\varphi \in C(\mathbb{R}, H_w)$, where H_w denotes the Hilbert space H with the weak topology.

6.3 Proofs of Attractor Theorems

In the proofs of the two Attractor Theorems, we will use the notation and assumptions which are stated in Section 2 and in Subsection 6.1. Recall that we require that the Main Assumption (6.24) is to hold uniformly for $f \in \mathcal{H}$.

Proof of Theorem 3.6: Let $\mathbb{K}$, ε_1, R_0^2, D_1^2 and D_2^2 be given by Theorem 3.1.

It follows from Theorem 3.1 that $\mathcal{H} \times B_0^*$ is a subset of $\mathcal{M}$, the orbit $\gamma^+(\mathcal{H} \times B_0^*)$ is bounded in $\mathcal{H} \times V^1$, and consequently, the omega limit set $\mathcal{K}_\varepsilon = \omega(\mathcal{H} \times B_0^*)$ is a nonempty, compact, invariant set for the semiflow π on $\mathcal{M}$. Also, one has $\mathcal{K}_\varepsilon \subset \mathcal{H} \times B_1^*$, by (3.8) and (6.7). By virtue of Lemma 6.6, $\mathcal{K}_\varepsilon \subset \mathcal{H} \times V^2$.

Now, we assume that (3.23) is satisfied. We define $d_3^2 = \max(D_1^2, d_1^2, C_3 c_1^{-1})$, where C_3 is the positive constant in (6.11). Due to (3.23), there is $\delta > 0$ such that

$$\delta + d_3^2\,|\ell|_\varepsilon^2 \,<\, k_{0,p}^2 \varepsilon^p \text{ and } \delta + d_3^2\,|\ell|_\varepsilon^2 \,<\, k_{1,q}^2 \varepsilon^q. \tag{6.46}$$

Let

$$B_\delta \,=\, \{u \in V^1 \,:\, \|u\|_{H^1}^2 \,<\, \delta + D_1^2\,\varepsilon^{-1}\,|\ell|_\varepsilon^2 \quad \text{and} \quad \|u\|_{L^2}^2 \,<\, \delta + d_1^2\,|\ell|_\varepsilon^2\}. \tag{6.47}$$

Let $u_0 \in V^1$ satisfy (3.2). Then there is a $T_1 > 0$ such that the strong solution $u(t) = s(f,t)u_0$ belongs to B_δ, for all $t \geq T_1$. Indeed, let $T_1 \geq 1$ such that $d_3^2\, e^{-2\alpha T_1} |m|_\varepsilon^2\, \varepsilon^{-1} < \delta$. Then for $t \geq T_1$, it follows from relations (3.8) and (6.46) that

$$\|u(t)\|_{H^1}^2 \,\leq\, (D_1^2\, e^{-2\alpha T_1}\,|m|_\varepsilon^2 + D_1^2\,|\ell|_\varepsilon^2)\,\varepsilon^{-1} \,<\, (\delta + d_3^2\,|\ell|_\varepsilon^2)\,\varepsilon^{-1} \,<\, k_{1,q}^2\,\varepsilon^{-1+q}.$$

Likewise, (6.7) and (6.46) imply for $t \geq T_1$ that

$$\|M_2 u(t)\|^2 \,\leq\, \|u(t)\|^2 \,\leq\, d_1^2 e^{-2\alpha T_1} |m|_\varepsilon^2 + d_1^2\,|\ell|_\varepsilon^2 < \delta + d_3^2\,|\ell|_\varepsilon^2 \,<\, k_{0,p}^2\,\varepsilon^p.$$

Now fix τ so that $\tau \geq T_1 \geq 1$. Then

$$u(\tau + t) = s(f, \tau + t)u_0 = s(f_\tau, t)u(\tau)$$

is in B_δ, for all $t \geq 0$. Note that the set B_δ is an open set in V^1 and

$$\mathcal{H} \times B_1^* \subset \mathcal{H} \times B_\delta \subset \mathcal{H} \times B_0^*.$$

Since $\mathcal{K}_\varepsilon$ attracts the open neighborhood $\mathcal{H} \times B_\delta$, it follows that $\mathfrak{A}_\varepsilon \stackrel{\text{def}}{=\!=} \mathcal{K}_\varepsilon$ is an attractor for π in V^1, and the basin of attraction $B(\mathfrak{A}_\varepsilon)$ contains $\mathcal{H} \times B_0^*$, and (3.24) holds since $\mathfrak{A}_\varepsilon = \mathcal{K}_\varepsilon \subset \mathcal{H} \times B_1^*$. See [12, 41, 44], for more details.

Next let $u = u(t)$ be any Leray-Hopf weak solution with $u(0) = u_0 \in H$. It follows from (4.54) and (6.11), that there is $C_3 > 0$ such that

$$\frac{1}{t}\int_0^t \|u(s)\|_{H^1}^2\,ds \le \frac{1}{c_1 t}\|u_0\|^2 + \frac{C_3}{c_1}|\ell|_\varepsilon^2, \qquad \text{for all } t > 0. \tag{6.48}$$

Let $\rho > 0$ be arbitrary and $\|u_0\|_{L^2}^2 \le \rho^2$. Set $T_2 = \rho^2\,(c_1\delta)^{-1}$. Then one has

$$\frac{1}{T_2}\int_0^{T_2} \|u(s)\|_{H^1}^2\,ds \le \delta + d_3^2|\ell|_\varepsilon^2 < \min(k_{1,q}^2\,\varepsilon^{-1+q},\, k_{0,p}^2\,\varepsilon^p).$$

Hence, the set

$$\{t \in [0, T_2] \,:\, \|u(t)\|_{H^1}^2 \le \min(k_{1,q}^2\,\varepsilon^{-1+q},\, k_{0,p}^2\,\varepsilon^p)\}$$

has positive measure. Thus there is a $\tau \in [0, T_2]$, where

$$\|u(\tau)\|_{H^1}^2 \le \min(k_{1,q}^2\,\varepsilon^{-1+q},\, k_{0,p}^2\,\varepsilon^p) \quad \text{and} \quad (f_\tau, u(\tau)) \in \mathcal{H} \times B_0^*.$$

For every such τ, the Main Theorem 3.1 implies that $v(t) = S(f_\tau, t)u(\tau)$ is a strong solution of (3.19), for all $t \ge 0$, and it satisfies (6.26). By using these relations, the proof that $\mathfrak{A}_\varepsilon$ is also the global attractor of the weak solutions, is straightforward, see [21], Theorem 7.1. We omit the details. □

Proof of Theorem 3.7: Let $r = s = 1$. We invoke Theorem 3.6 with $p = q = 0$. Let $\mathbb{K} = \mathbb{K}_1 = (k_{0,0}, k_{1,0}, K_{0,1}, K_{1,1})$ with $k_{0,0}^2 + k_{1,0}^2 < R_0^2$. Then there is $\varepsilon_3 \in (0, \varepsilon_1]$ so that

$$0 < (K_{0,1}^2 + K_{1,1}^2)\varepsilon < \min\Big(\frac{k_{0,0}^2}{d_3^2}, \frac{k_{1,0}^2}{d_3^2}, R_0^2 - k_{0,0}^2 - k_{1,0}^2\Big). \tag{6.49}$$

is satisfied for all $\varepsilon \in (0, \varepsilon_3]$. Note that (6.49) implies (3.15) and (3.23), and (3.25) is (3.20). Hence the existence of the attractor $\mathfrak{A}_\varepsilon$ and the estimate (3.26) follow from Theorem 3.6. □

6.4 Reduced Problem

The global existence Theorem 3.1 and the existence of the global attractors $\mathfrak{A}_\varepsilon$, for small $\varepsilon > 0$, is only the first chapter in the theory of the longtime dynamics of the Navier–Stokes equations over thin 3D domains. In the next chapter, we examine an important feature of the attractor, viz. the robustness, or upper semicontinuity, of the attractor at $\varepsilon = 0$, see for example Theorem 23.14 in [41]. This study, which includes a description of the Reduced Problem at $\varepsilon = 0$, will be presented in a sequel to the current article, see [17].

Some aspects of the Reduced Problem, including the role of the related 2D g-Navier-Stokes equations, appear in [21]. In the forthcoming article, [17], we describe the limiting behavior in H^1, rather than the L^2 theory which is treated in [21]. In order to treat the Reduced Problem at $\varepsilon = 0$ properly, it is appropriate - if not essential - that one reformulates the dynamics of the problem in the dilated domain Ω^1, rather than in the thin domain Ω^ε. This reformulation is included in [17].

6.5 On the Uniform Gronwall Inequality

In the setting of the Navier–Stokes equations, the Uniform Gronwall Inequality has been used to study the the solutions of the v-equation (not the u-equation) that arise in the analysis of Theorem 1.1, see [33, 34, 35]. In this case, it is shown that

$$\frac{d}{dt}\|A^{\frac{1}{2}}v(t)\|^2 \le C\,\|A^{\frac{1}{2}}v(t)\|^k + h(t), \tag{6.50}$$

where $k = 3$, a nonlinear differential inequality. Furthermore, the inequality (6.50), with $k = 4$ is also a common feature arising in the study of the 2D Navier–Stokes equations, with either periodic or Dirichlet boundary conditions, see [41] for example. The reader should compare this with (6.31), where one has $k = 2$, a linear differential inequality.

As far as we know, the first application of the Uniform Gronwall Inequality for the study of the global regularity and ultimate boundedness of solutions of the (full) u-equations of the Navier–Stokes equations on a thin 3D domain occurs in this article. It is interesting that the simplest form of the differential inequality, where $k = 2$, arises in this 3D problem with the Navier boundary conditions. We do not know of any other such example, be it in 2D or 3D, of Navier-Stokes equations, with either periodic or Dirichlet boundary conditions, with this feature.

7 Appendix

In this Appendix we attend to the "loose ends". In particular, we present here the proofs of Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, and 4.20. Note that the proofs of Lemmas 4.1–4.5 are meant for the general h_0 with $\|h_0\|_{W^{1,\infty}(\mathbb{T}^2)} \le C(h_0)$. We use the notation $Q_3 = (0,1)^3$ below.

Proof of Lemma 4.1: For $x' = (x_1, x_2) \in \mathbb{T}^2$, and $h_0(x') \le x_3, y_3 \le h_1(x')$, one has

$$\phi(x', x_3) = \phi(x', y_3) + \int_{y_3}^{x_3} \partial_3\phi(x', z_3)\, dz_3 \tag{7.1}$$

For $j = 0, 1$, let $y_3 = h_j(x')$ in (7.1),

$$\phi(x', x_3) = \phi(x', h_j(x')) + \int_{h_0(x')}^{x_3} \partial_3\phi(x', z_3)\, dz_3,$$

hence

$$|\phi(x',x_3)|^r \le C|\phi(x',h_j(x'))|^r + C(\varepsilon g(x'))^{r-1}\int_{h_0(x')}^{x_3} |\partial_3\phi(x',z_3)|^r\, dz_3. \tag{7.2}$$

Integrating (7.2) in variable $x = (x', x_3)$ over Ω^ε yields

$$\begin{aligned}\int_{\Omega^\varepsilon} |\phi|^r dx &\le C\int_{\Omega^\varepsilon} |\phi(x',h_j(x'))|^r dx' + \varepsilon^{r-1}C\int_{\Omega^\varepsilon}\int_{h_0(x')}^{x_3} |\partial_3\phi(x',z_3)|^r dz_3\, dx \\ &\le \varepsilon\, C\int_{\mathbb{T}^2} |\phi(x',h_j(x'))|^r dx' + \varepsilon^r C\int_{\Omega^\varepsilon} |\partial_3\phi(x',z_3)|^r dz_3\, dx'.\end{aligned}$$

Hence $\|\phi\|^r_{L^r(\Omega^\varepsilon)} \le C\varepsilon\|\phi\|_{L^r(\Gamma_j)} + C\varepsilon^r\|\partial_3\phi\|_{L^r(\Omega^\varepsilon)}$ and (4.1) follows.

Now, let

$$\eta_0(x',x_3) = \frac{h_1(x') - x_3}{\varepsilon g(x')}, \quad \eta_1(x',x_3) = \frac{x_3 - h_0(x')}{\varepsilon g(x')}.$$

For $j = 0, 1$, one has

$$\phi(x',h_j(x')) = \eta_j(x',h_j(x'))\phi(x',h_j(x')) = -\int_{h_0(x')}^{h_1(x')} \partial_3(\eta_j(x',y_3)\phi(x',y_3))\, dy_3,$$

which easily yields

$$\begin{aligned}|\phi(x',h_0(x'))| &\le C\int_{h_0(x')}^{h_1(x')} \varepsilon^{-1}|\phi(x',y_3)| + |\partial_3\phi(x',y_3)|\, dy_3, \\ |\phi(x',h_j(x'))|^r &\le C\int_{h_0(x')}^{h_1(x')} \varepsilon^{-1}|\phi(x',y_3)|^r + \varepsilon^{r-1}|\partial_3\phi(x',y_3)|^r\, dy_3.\end{aligned}$$

Integrating over $\mathbb{T}^2$ gives

$$\int_{\mathbb{T}^2} |\phi(x',h_j(x'))|^r dx' \le \frac{C}{\varepsilon}\int_{\Omega^\varepsilon} |\phi|^r dx + C\varepsilon^{r-1}\int_{\Omega^\varepsilon} |\partial_3\phi|^r dx. \tag{7.3}$$

Since

$$\|\phi\|^r_{L^r(\Gamma)} = \sum_{j=0,1}\int_{\mathbb{T}^2} |\phi(x',h_j(x'))|^r\sqrt{1+|\nabla_2 h_j|^2}dx' \le C\int_{\mathbb{T}^2} |\phi(x',h_j(x'))|^r dx',$$

the inequality (4.2) follows from (7.3). □

Proof of Lemma 4.2: Integrating (7.1) in y_3 from $h_0(x')$ to $h_1(x')$ and using (4.5) yields

$$\varepsilon g(x')\phi(x',x_3) = 0 + \int_{h_0(x')}^{h_1(x')}\int_{y_3}^{x_3} \partial_3\phi(x',z_3)\, dz_3\, dy_3.$$

Hence

$$\varepsilon g(x')|\phi(x',x_3)| \le \varepsilon g(x') \int_{h_0((x')}^{h_1(x')} |\partial_3\phi(x',z_3)|\,dz_3,$$

$$|\phi(x',x_3)| \le \int_{h_0(x')}^{h_1(x')} |\partial_3\phi(x',z_3)|\,dz_3,$$

and

$$|\phi(x',x_3)|^r \le (\varepsilon g(x'))^{r-1} \int_{h_0(x')}^{h_1(x')} |\partial_3\phi(x',z_3)|^r dz_3. \tag{7.4}$$

Integrating over Ω^ε,

$$\begin{aligned}\int_{\Omega^\varepsilon} |\phi|^r dx &\le \int_{\mathbb{T}^2}\int_{h_0(x')}^{h_1(x')}\int_{h_0(x')}^{h_1(x')} (\varepsilon g(x'))^{r-1}|\partial_3\phi(x',z_3)|^r dz_3\,dx_3\,dx' \\ &\le C\varepsilon^r \int_{\mathbb{T}^2}\int_{h_0(x')}^{h_1(x')} |\partial_3\phi(x',z_3)|^r dz_3\,dx' \\ &= C\varepsilon^r \int_{\Omega^\varepsilon} |\partial_3\phi|^r\,dx.\end{aligned}$$

Thus (4.6) follows. For $j=0,1$, letting $x_3 = h_j(x')$ in (7.4) and integrating over $\mathbb{T}^2$, one obtains

$$\int_{\mathbb{T}^2} |\phi(x',h_j(x'))|^r dx' \le C\varepsilon^{r-1}\int_{\Omega^\varepsilon} |\partial_3\phi|^r\,dx.$$

and hence (4.7) follows. □

Proof of Lemma 4.3: This is a direct consequence of 2D Ladyzhenskaya's inequality (see [24]):

$$\|\varphi\|_{L^4(\mathbb{T}^2)} \le C\|\varphi\|^{1/2}_{L^2(\mathbb{T}^2)}\|\varphi\|^{1/2}_{H^1(\mathbb{T}^2)},$$

for any $\varphi \in H^1(\mathbb{T}^2)$. □

Proof of Lemma 4.4: Note that (4.11) follows from (4.10) and (4.8); and (4.12) is obtained by using interpolating inequalities and the estimates (4.8), (4.11). We now prove (4.10). Recall the anisotropic Sobolev inequality from [13, 14, 45] for domain Q_3:

$$\|\varphi\|_{L^6(Q_3)} \le c_0 \prod_{i=1}^{3}\Big\{\|\varphi\|_{L^2(Q_3)} + \|\partial_{y_i}\varphi\|_{L^2(Q_3)}\Big\}^{1/3}. \tag{7.5}$$

Let $\phi\in H^1(\Omega^\varepsilon)$. Let $\varphi(y)=\phi(x)$ where

$$x_1 = y_1,\quad x_2=y_2,\quad x_3 = h_0 + (\varepsilon g)\,y_3 \quad\text{or}\quad y_3 = \frac{x_3-h_0}{\varepsilon g}. \tag{7.6}$$

Then $\varphi \in H^1(Q_3)$. Note for $i = 1, 2$, that

$$\partial_{y_i}\varphi = \partial_i\phi + \partial_3\phi(\partial_i h_0 + y_3\varepsilon\partial_i g), \quad \partial_{y_3}\varphi = \varepsilon g\partial_3\phi.$$

One has

$$\begin{aligned}
\|\varphi\|_{L^2(Q_3)} &\le C\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)}, \\
\|\partial_{y_i}\varphi\|_{L^2(Q_3)} &\le C\varepsilon^{-\frac{1}{2}}\|\nabla\phi\|_{L^2(\Omega^\varepsilon)}, \quad i = 1, 2, \\
\|\partial_{y_3}\varphi\|_{L^2(Q_3)} &\le C\varepsilon^{-\frac{1}{2}}\varepsilon\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}.
\end{aligned}$$

From (7.5), one obtains

$$\varepsilon^{-\frac{1}{6}}\|\phi\|_{L^6(\Omega^\varepsilon)} \le C\Big\{\ \varepsilon^{-\frac{1}{2}}\|\phi\|_{H^1(\Omega^\varepsilon)}\Big\}^{2/3}\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)}\Big\}^{1/3}.$$

Thus (4.10) follows. □

Proof of Lemma 4.5: First, one notes that (4.14) follows directly from (4.13) and (4.8). Hence it suffices to prove (4.13). We recall the anisotropic Agmon inequality from [45]. There is an absolute constant C such that for $\varphi \in H^2(Q_3)$, one has

(7.7)

$$\|\varphi\|_{L^\infty(Q_3)} \le C\|\varphi\|^{1/4}_{L^2(Q_3)}\prod_{i=1}^{3}\big(\|\partial_{y_i}\partial_{y_i}\varphi\|_{L^2(Q_3)} + \|\partial_{y_i}\varphi\|_{L^2(Q_3)} + \|\varphi\|_{L^2(Q_3)}\big)^{1/4}.$$

Let $\phi \in H^2(\Omega^\varepsilon)$. Using the same change of variables as in (7.6), one obtains

$$\begin{aligned}
&\partial_{y_3}\partial_{y_3}\varphi = \varepsilon^2 g^2\partial_3\partial_3\phi, \\
&\partial_{y_i}\partial_{y_i}\varphi = \partial_i\partial_i\phi + \partial_3\partial_i\phi(\partial_i h_0 + y_3\varepsilon\partial_i g) + \partial_3\phi(\partial_i\partial_i h_0 + y_3\varepsilon\partial_i\partial_i g) \\
&\qquad + \Big\{\partial_i\partial_3\phi + \partial_3\partial_3\phi(\partial_i h_0 + y_3\varepsilon\partial_i g)\Big\}(\partial_i h_0 + y_3\varepsilon\partial_i g), \quad i = 1, 2.
\end{aligned}$$

It follows that

$$\|\partial_{y_3}\partial_{y_3}\varphi\|_{L^2(Q_3)} \le C\varepsilon^{-\frac{1}{2}}\varepsilon^2\|\partial_3\partial_3\phi\|_{L^2(\Omega^\varepsilon)},\ \|\partial_{y_i}\partial_{y_i}\varphi\|_{L^2(Q_3)} \le C\varepsilon^{-\frac{1}{2}}\|\phi\|_{H^2(\Omega^\varepsilon)},$$

for $i = 1, 2$. Then (7.7) yields

$$\begin{aligned}
\|\phi\|_{L^\infty(\Omega^\varepsilon)} \le C&\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)}\Big\}^{1/4}\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{H^2(\Omega^\varepsilon)}\Big\}^{1/4}\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{H^2(\Omega^\varepsilon)}\Big\}^{1/4} \\
&\times\Big\{\varepsilon^{-\frac{1}{2}}\|\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon^{-\frac{1}{2}}\varepsilon\|\partial_3\phi\|_{L^2(\Omega^\varepsilon)} + \varepsilon^{-\frac{1}{2}}\varepsilon^2\|\partial_3\partial_3\phi\|_{L^2(\Omega^\varepsilon)}\Big\}^{1/4}.
\end{aligned}$$

Hence (4.13) follows. □

Proof of Lemma 4.20: Lemma 4.20 can be proved by modifying the proof of Proposition 3.7 in [21]. However we present below another proof which is a special case of the result in [16] when the friction coefficients on both of top and bottom boundaries are zero.

First, integration by parts yields

$$\int_{\Omega^\varepsilon} |\nabla^2 u|\, dx = \int_{\Omega^\varepsilon} |\Delta u|^2\, dx + \int_\Gamma \Big(\frac{1}{2}\frac{\partial |\nabla u|^2}{\partial N} - \frac{\partial u}{\partial N}\cdot \Delta u\Big)\, d\sigma.$$

Let I_0 denote the integral on the boundary. For each $j \in \{0,1\}$, let $N = N^j$ defined in (4.21), $\tau_1 = \tau^{1,j}$ defined in (4.22), and $\tau_2 = N \times \tau_1$. Then on each j, the corresponding set $\{\tau_1, \tau_2, N\}$ is an orthonormal frame on Γ_j. One also has,

$$|\nabla \tau_1|,\ |\nabla \tau_2|,\ |\nabla N|,\ |\nabla^2 \tau_1|,\ |\nabla^2 \tau_2|,\ |\nabla^2 N| \le C\varepsilon \text{ in } \mathbb{T}^2 \times \mathbb{R}. \tag{7.8}$$

Using the above orthonormal frame, one has

$$\begin{aligned}\frac{1}{2}\frac{\partial |\nabla u|^2}{\partial N} &= \frac{1}{2}\frac{\partial}{\partial N}\left|\frac{\partial u}{\partial N}\right|^2 + \frac{1}{2}\frac{\partial}{\partial N}\left|\frac{\partial u}{\partial \tau_1}\right|^2 + \frac{1}{2}\frac{\partial}{\partial N}\left|\frac{\partial u}{\partial \tau_2}\right|^2\\ &= \frac{\partial^2 u}{\partial N \partial N}\cdot \frac{\partial u}{\partial N} + \frac{\partial^2 u}{\partial N \partial \tau_1}\cdot \frac{\partial u}{\partial \tau_1} + \frac{\partial^2 u}{\partial N \partial \tau_2}\cdot \frac{\partial u}{\partial \tau_2}\\ &= J_3 + J_1 + J_2.\end{aligned}$$

Consider J_3. Let $Y_1(x), Y_2(x), Y_3(x)$ be orthonormal and ϕ be smooth. Then

$$\frac{\partial \phi}{\partial Y_j} = (\nabla \phi)\cdot Y_j = \sum_{l=1}^3 \partial_l \phi Y_{j,l},$$

$$\sum_{j=1}^3 \frac{\partial^2 \phi}{\partial Y_j \partial Y_j} = \sum_{j,k,l=1}^3 \partial_k(\partial_l \phi Y_{j,l}) Y_{j,k} = \sum_{j,k,l=1}^3 \partial_k \partial_l \phi Y_{j,l} Y_{j,k} + \partial_l \phi \partial_k Y_{j,l} Y_{j,k}$$

$$= \sum_{j,k,l=1}^3 \partial_k \partial_l \phi Y_{j,l} Y_{j,k} + \sum_{j,k,l=1}^3 \partial_l \phi \partial_k (Y_{j,l} Y_{j,k}) - \sum_{j,k,l=1}^3 \partial_l \phi Y_{j,l} \partial_k Y_{j,k}.$$

Since $\sum_{j=1}^3 Y_{j,l} Y_{j,k} = \delta_{lk}$ we obtain

$$\sum_{j=1}^3 \frac{\partial^2 \phi}{\partial Y_j \partial Y_j} = \Delta \phi + 0 - \sum_j \frac{\partial \phi}{\partial Y_j}(\nabla \cdot Y_j).$$

Let $Y_1 = \tau_1, Y_2 = \tau_2, Y_3 = N$, we derive

$$\frac{\partial^2 u}{\partial N \partial N} = \Delta u - \Big(\frac{\partial^2 u}{\partial \tau_1 \partial \tau_1} + \frac{\partial^2 u}{\partial \tau_2 \partial \tau_2}\Big) - \Big\{\frac{\partial u}{\partial N}(\nabla \cdot N) + \frac{\partial u}{\partial \tau_1}(\nabla \cdot \tau_1) + \frac{\partial u}{\partial \tau_2}(\nabla \cdot \tau_2)\Big\}.$$

It follows that

$$J_3 = \Delta u \cdot \frac{\partial u}{\partial N} - \Big(\frac{\partial^2 u}{\partial \tau_1 \partial \tau_1} + \frac{\partial^2 u}{\partial \tau_2 \partial \tau_2}\Big)\cdot \frac{\partial u}{\partial N} + J_3',$$

with $|J_3'| \le C\varepsilon |\nabla u|^2$.

Now consider J_1 and J_2. Let τ, τ' be tangential unit vectors to the boundary. Taking the directional derivative $\partial/\partial\tau'$ of the identity

$$\frac{\partial u}{\partial N} \cdot \tau = \frac{\partial N}{\partial \tau} \cdot u$$

on the boundary gives

$$\frac{\partial^2 u}{\partial \tau' \partial N} \cdot \tau + \frac{\partial u}{\partial N} \cdot \frac{\partial N}{\partial \tau'} = \frac{\partial u}{\partial \tau'} \cdot \frac{\partial N}{\partial \tau} + u \cdot \frac{\partial^2 N}{\partial \tau' \partial \tau}. \tag{7.9}$$

For $\tau, \tau' \in \{\tau_1, \tau_2\}$, it follows from (7.9) and (7.8) that

$$\left| \frac{\partial^2 u}{\partial \tau' \partial N} \cdot \tau \right| \le C\varepsilon(|\nabla u| + |u|). \tag{7.10}$$

One has

$$\frac{\partial^2 u}{\partial N \partial \tau} = \nabla u(\frac{\partial \tau}{\partial N}) + \{(\nabla^2 u) \cdot N\}\tau,$$

where $(\nabla^2 u) \cdot N$ is the matrix $\left(\sum_{k=1}^3 \partial_k \partial_j u_i \, N_k\right)_{i,j=1,2,3}$. Note that

$$\{(\nabla^2 u) \cdot N\}\tau = \sum_{k,j=1}^{3} \partial_k \partial_j u_i \, N_k \tau_j = \{(\nabla^2 u) \cdot \tau\}N,$$

$$\frac{\partial^2 u}{\partial N \partial \tau} - \frac{\partial^2 u}{\partial \tau \partial N} = \nabla u(\frac{\partial \tau}{\partial N} - \frac{\partial N}{\partial \tau}).$$

We obtain for $i = 1, 2$,

$$J_i = \frac{\partial^2 u}{\partial \tau_i \partial N} \cdot \frac{\partial u}{\partial \tau_i} + \nabla u(\frac{\partial \tau_i}{\partial N} - \frac{\partial N}{\partial \tau_i}) \cdot \frac{\partial u}{\partial \tau_i} = \frac{\partial^2 u}{\partial \tau_i \partial N} \cdot \frac{\partial u}{\partial \tau_i} + J_i',$$

with $|J_i'| \le C\varepsilon|\nabla u|^2$. Combining the above identities, we derive

$$\begin{aligned} I_0 &= \int_\Gamma \frac{\partial^2 u}{\partial \tau_1 \partial N} \cdot \frac{\partial u}{\partial \tau_1} + \int_\Gamma \frac{\partial^2 u}{\partial \tau_2 \partial N} \cdot \frac{\partial u}{\partial \tau_2} d\sigma \\ &\quad - \int_\Gamma (\frac{\partial^2 u}{\partial \tau_1 \partial \tau_1} + \frac{\partial^2 u}{\partial \tau_2 \partial \tau_2}) \cdot \frac{\partial u}{\partial N} d\sigma + J \\ &= I_1 + I_2 + I_4 + J, \end{aligned}$$

where $J = \int_\Gamma (J_1' + J_2' + J_3')\, d\sigma$ satisfies

$$|J| \le C\varepsilon \int_\Gamma |\nabla u|^2 d\sigma \le C\|\nabla u\|_{L^2}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2.$$

To estimate I_1, I_2 and I_4 we need the integration by parts on Γ which is established in Lemma 7.1 below. Using Lemma 7.1, we have $I_4 = I_1 + I_2 + I_4'$ with

$$|I_4'| \le C\varepsilon \int_\Gamma \left| \frac{\partial u}{\partial \tau_1} \cdot \frac{\partial u}{\partial N} \right| + \left| \frac{\partial u}{\partial \tau_2} \cdot \frac{\partial u}{\partial N} \right| d\sigma \le C\varepsilon \int_\Gamma |\nabla u|^2 \, d\sigma.$$

We need to estimate I_1 and I_2. First, considering I_1, one has

$$\frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\frac{\partial u}{\partial\tau_1} = (\frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\tau_1)(\frac{\partial u}{\partial\tau_1}\cdot\tau_1) + (\frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\tau_2)(\frac{\partial u}{\partial\tau_1}\cdot\tau_2)$$
$$+(\frac{\partial^2 u}{\partial\tau_1\partial N}\cdot N)(\frac{\partial u}{\partial\tau_1}\cdot N) = J_4 - \frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\{N(u\cdot\frac{\partial N}{\partial\tau_1})\}.$$

It follows from (7.10) that $|J_4| \le C(\varepsilon|\nabla u|^2 + \varepsilon|\nabla u|)$. Using Lemma 7.1 again,

$$-\int_\Gamma \frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\{N(u\cdot\frac{\partial N}{\partial\tau_1})\}\,d\sigma = \int_\Gamma \frac{\partial u}{\partial N}\cdot\frac{\partial}{\partial\tau_1}\{N(u\cdot\frac{\partial N}{\partial\tau_1})\}\,d\sigma + J_4',$$

where

$$|J_4'| \le C\varepsilon\int_\Gamma\left|\frac{\partial u}{\partial N}\cdot\{N(u\cdot\frac{\partial N}{\partial\tau_1})\}\right|d\sigma \le C\varepsilon^2\int_\Gamma |\nabla u|\,|u|\,d\sigma$$

Also,

$$\int_\Gamma\left|\frac{\partial u}{\partial N}\cdot\frac{\partial}{\partial\tau_1}\{N(u\cdot\frac{\partial N}{\partial\tau_1})\}\right|\,d\sigma \le C\varepsilon\int_\Gamma |\nabla u|\Big(|\nabla u| + |u|\Big)\,d\sigma.$$

Therefore,

$$|I_1| = \left|\int_\Gamma \frac{\partial^2 u}{\partial\tau_1\partial N}\cdot\frac{\partial u}{\partial\tau_1}\,d\sigma\right| \le C\varepsilon\int_\Gamma(|\nabla u|^2 + |\nabla u|\,|u|)\,d\sigma.$$

Similarly, one obtains

$$|I_2| \le C\varepsilon\int_\Gamma(|\nabla u|^2 + |\nabla u|\,|u|)\,d\sigma.$$

Summing up

$$|I_0| \le C\varepsilon\int_\Gamma(|\nabla u|^2 + |u|^2)\,d\sigma.$$

Combining with the trace theorem (see (4.4)), we have

$$\|\nabla^2 u\|_{L^2}^2 \le \|\Delta u\|_{L^2}^2 + |I_0| \le \|\Delta u\|_{L^2}^2 + C_1\|u\|_{H^1}^2 + C_2\varepsilon^2\|\nabla^2 u\|_{L^2}^2.$$

For small ε such that $C_2\varepsilon^2 \le 3/4$, one has $\|\nabla^2 u\|_{L^2}^2 \le 4\|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2$, and therefore

$$\|u\|_{H^2}^2 \le 4\|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2.$$

This completes the proof of Lemma 4.20. □

What remains to be proved is the following integration by parts on the boundary.

Lemma 7.1. *For two smooth vector fields u, v on Γ and a tangential vector field $a(x)$ to Γ, we have*

$$\int_\Gamma \frac{\partial u(x)}{\partial a(x)} \cdot v(x)\, d\sigma = -\int_\Gamma \frac{\partial v(x)}{\partial a(x)} \cdot u(x)\, d\sigma - \sum_{j=0,1} \int_{\mathbb{T}^2} u(x)\cdot v(x) \Big\{ \frac{d}{dx_1}(a_1 H_j) + \frac{d}{dx_2}(a_2 H_j) \Big\}\, dx', \tag{7.11}$$

where $H_j = \sqrt{1+|\nabla_2 h_j|^2}$, for $j = 0, 1$. Consequently, for $i = 1, 2$, we have

$$\Big| \int_\Gamma \frac{\partial u(x)}{\partial \tau_i(x)} \cdot v(x)\, d\sigma + \int_\Gamma \frac{\partial v(x)}{\partial \tau_i(x)} \cdot u(x)\, d\sigma \Big| \le C\varepsilon \int_\Gamma |u\cdot v|\, d\sigma. \tag{7.12}$$

Proof. We first have

$$\int_\Gamma \frac{\partial u(x)}{\partial a(x)} \cdot v(x)\, d\sigma = -\int_\Gamma \frac{\partial v(x)}{\partial a(x)} \cdot u(x)\, d\sigma + \int_\Gamma \frac{\partial}{\partial a(x)} (u(x)\cdot v(x))\, d\sigma.$$

Denote $F = u(x)\cdot v(x)$. One has

$$\begin{aligned}\int_\Gamma \frac{\partial F(x)}{\partial a(x)}\, d\sigma &= \sum_{j=0,1} \int_{\mathbb{T}^2} (\partial_1 F a_1 + \partial_2 F a_2 + \partial_3 F a_3)(x', h_j(x'))\, H_j(x')\, dx' \\ &= \sum_{j=0,1} \int_{\mathbb{T}^2} \Big\{ \Big(\frac{d}{dx_1} F(x', h_j(x')) - \partial_3 F \partial_1 h_j \Big) a_1 \\ &\quad + \Big(\frac{d}{dx_2} F(x', h_j(x')) - \partial_3 F \partial_2 h_j \Big) a_2 + \partial_3 F a_3 \Big\}\, H_j(x')\, dx'.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}\int_\Gamma \frac{\partial F(x)}{\partial a(x)} d\sigma = &- \sum_{j=0,1} \int_{\mathbb{T}^2} F \Big\{ \frac{d}{dx_1}(a_1 H_j) + \frac{d}{dx_2}(a_2 H_j) \Big\}\, dx' \\ &+ \sum_{j=0,1} \int_{\mathbb{T}^2} \partial_3 F (a_3 - a_1 \partial_1 h_j - a_2 \partial_2 h_j)\, H_j\, dx'.\end{aligned}$$

Since $a\cdot N = 0$ yields $a_3 - a_1\partial_1 h_j - a_2 \partial_2 h_j = 0$, we have

$$\int_\Gamma \frac{\partial F(x)}{\partial a(x)}\, d\sigma = - \sum_{j=0,1} \int_{\mathbb{T}^2} F \Big\{ \frac{d}{dx_1}(a_1 H_j) + \frac{d}{dx_2}(a_2 H_j) \Big\}\, dx', \tag{7.13}$$

thus (7.11) follows.

For $a = \tau_i$, $i = 1, 2$ one has from (7.8) that $|\frac{d}{dx_1}(a_1 H_j)|, |\frac{d}{dx_2}(a_2 H_j)| \le C\varepsilon$, hence (7.13) yields

$$\Big| \int_\Gamma \frac{\partial F(x)}{\partial \tau_i(x)} d\sigma \Big| \le C\varepsilon \int_\Gamma |F|\, d\sigma,$$

and finally we obtain (7.12). □

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