

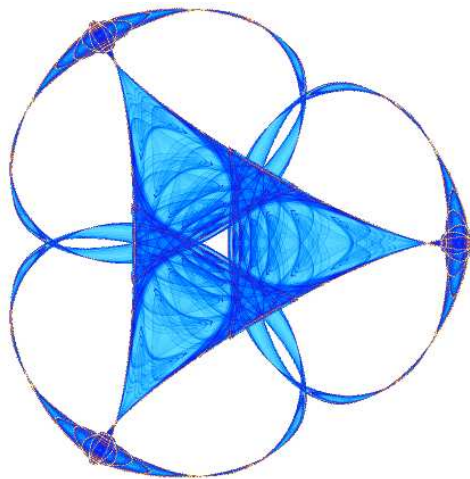
E-POINTS FOR DIAGONAL GAMES II

By

Ezio Marchi

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436

Phone: 612-624-6066 Fax: 612-626-7370

URL: <http://www.ima.umn.edu>

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Ezio Marchi^{*)**)}

Abstract

In this paper we study and compute E -points in an explicit way for a special general kind of $3k + 1$ and $3k + 2$ players.

Key words: E -points, computation, equilibrium, non-cooperative games.

^{*)} Founder and First Director, Instituto de Matemática Aplicada, San Luis, CONICET and Universidad Nacional de San Luis, San Luis, Argentina. Visiting the University of Barcelona, Barcelona, Spain.
e-mail: emarchi@sinectis.com.ar

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1. Introduction

We have begun to compute and study E -points for diagonal games in Marchi (2004) recently. We have done this for simple games. In this paper which is the continuation of the quoted one we compute and study E -points for a more complicated games.

With the notation of Marchi (2004) we have in general for the expected functions E_i for player $i \in N = \{1, \dots, n\}$ and $d(i) \subset N$ the set of friends of player $i \in N$. An E -point is a point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ such that

$$E_i(\bar{x}) \geq E_i(x_{d(i)}, \bar{x}_{N-d(i)}) \quad \forall i \quad \forall x_{d(i)}.$$

We remind the following result.

Proposition: \bar{x} is an E -point if and only if

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) = 0 \quad \forall \sigma_{d(i)} \in \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\lambda_i - E_i(\sigma_{d(i)}, \bar{x}_{N-d(i)}) \geq 0 \quad \forall \sigma_{d(i)} \notin \prod_{j \in d(i)} S(\bar{x}_j)$$

$$\sum_{\sigma_i \in \Sigma_i} \bar{x}_i(\sigma_i) = 1 \quad \forall i$$

$$\bar{x}_i(\sigma_i) \geq 0 \quad \forall i \quad \forall \sigma_i \in \Sigma_i.$$

where $S(\bar{x}_j)$ denotes the support of the mixed strategy \bar{x}_j .

From now on we consider n -person games in normal form where all the players have the same cardinality for their pure strategy set: $m = |\Sigma_i|$.

In the next paragraph we will study a 7-player diagonal game and a 10-player game with similar $d(i)$ structure. Next we generalize them in a general way. In the third section we study an 8-player game of the game type for the sets $d(i)$ and finally we generalize it to $3k + 2$ -player with similar structure function.

2. 7 and 10-players diagonal games

Here in this paragraph we will compute explicitly the E -points for two games, namely are with 7-players and the other are with 10-players.

The first one is given with the structure function $d(i)$ as $d(1) = \{1, 2, 6, 7\}$, $d(2) = \{2, 3, 7, 1\}$, $d(3) = \{1, 2, 3, 4\}$, $d(4) = \{2, 3, 4, 5\}$, $d(5) = \{3, 4, 5, 6\}$, $d(6) = \{4, 5, 6, 7\}$, $d(7) = \{5, 6, 7, 1\}$. Therefore due to the Proposition if the game is diagonal that is to say

$$A_i(\sigma_1 \dots \sigma_2) = a_i(\sigma_i) \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \quad \text{mod } 7 \quad a_i(\sigma_i) > 0$$

where

$$\delta(\sigma_{j_1}, \sigma_{j_2}, \sigma_{j_3}, \dots, \sigma_{j_k}) = \delta(\sigma_{j_1}, \sigma_{j_2}) \dots \delta(\sigma_{j_{k-1}}, \sigma_{j_k})$$

where the δ 's are Krorecker's deltas.

We remind that a completely mixed strategy is a vector $x = (x_1, \dots, x_n)$ such that $\forall i \forall \sigma_i \in \sum_i; x_i(\sigma_i) > 0$.

Then is we want to compute the completely mixed E-points for our 7-person game for the Proposition we have to solve

$$\begin{aligned} \lambda_1 - a_1(\sigma) x_3(\sigma) x_4(\sigma) x_5(\sigma) &= 0 & \forall \sigma \\ \lambda_2 - a_2(\sigma) x_4(\sigma) x_5(\sigma) x_6(\sigma) &= 0 & \forall \sigma \\ \lambda_3 - a_3(\sigma) x_5(\sigma) x_6(\sigma) x_7(\sigma) &= 0 & \forall \sigma \\ \lambda_4 - a_4(\sigma) x_6(\sigma) x_7(\sigma) x_1(\sigma) &= 0 & \forall \sigma \\ \lambda_5 - a_5(\sigma) x_2(\sigma) x_1(\sigma) x_2(\sigma) &= 0 & \forall \sigma \\ \lambda_6 - a_6(\sigma) x_1(\sigma) x_2(\sigma) x_3(\sigma) &= 0 & \forall \sigma \\ \lambda_7 - a_7(\sigma) x_2(\sigma) x_3(\sigma) x_4(\sigma) &= 0 & \forall \sigma \end{aligned} \quad (1)$$

For simplicity reasons until we need it we drop the explicit notation of the independent variable that it is to say we write $a_i = a_i(\sigma)$ and $x_i(\sigma) = x_i$. From (1) calling $\mu_i = \mu_i(\sigma) = \lambda_i / a_i$ we obtain

$$\begin{aligned} x_6 &= \frac{\mu_2}{\mu_1} x_3, \quad x_7 = \frac{\mu_3}{\mu_2} x_4, \quad x_1 = \frac{\mu_4}{\mu_3} x_5, \quad x_2 = \frac{\mu_5}{\mu_4} x_6, \quad x_3 = \frac{\mu_6}{\mu_5} x_7 \\ x_4 &= \frac{\mu_6}{\mu_6} x_1, \quad x_5 = \frac{\mu_1}{\mu_7} x_2. \end{aligned}$$

By replacing terms we have from these last equalities

$$x_2 = \frac{\mu_3 \mu_7}{\mu_1 \mu_4} x_1, \quad x_7 = \frac{\mu_3 \mu_7}{\mu_2 \mu_6} x_1.$$

Replacing them in the fifth equation of (1) we get

$$\mu_5 - x_1^3 \frac{\mu_3^2 \mu_7^2}{\mu_2 \mu_6 \mu_4 \mu_1} = 0 \quad (2)$$

In a similar way it is possible to obtain the order equations, or by symmetry arguments, which happen to be

$$\begin{aligned} \mu_6 - x_2^3 \frac{\mu_1^2 \mu_4^2}{\mu_3 \mu_7 \mu_5 \mu_2} &= 0 \\ \mu_7 - x_3^3 \frac{\mu_5^2 \mu_2^2}{\mu_1 \mu_3 \mu_4 \mu_6} &= 0 \end{aligned} \quad (3)$$

From (2) one obtains

$$x_1^3(\sigma) = \frac{\lambda_1 \lambda_2 \lambda_4 \lambda_5 \lambda_6}{\lambda_3^2 \lambda_7^2} \frac{a_3^2(\sigma) a_7^2(\sigma)}{a_1(\sigma) a_2(\sigma) a_4(\sigma) a_5(\sigma) a_6(\sigma)}$$

Using the condition $\sum_{\sigma} x_1(\sigma) = 1$ one gets

$$b_1 = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_3^2(\sigma) a_7^2(\sigma)}{a_1(\sigma) a_2(\sigma) a_4(\sigma) a_5(\sigma) a_6(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_1 \lambda_2 \lambda_4 \lambda_5 \lambda_6}{\lambda_3^2 \lambda_7^2}$$

Similarly for the other two equations in (3)

$$x_2^3(\sigma) = \frac{\lambda_2 \lambda_3 \lambda_5 \lambda_6 \lambda_7}{\lambda_1^2 \lambda_4^2} \frac{a_1^2(\sigma) a_4^2(\sigma)}{a_2(\sigma) a_3(\sigma) a_5(\sigma) a_6(\sigma) a_7(\sigma)}$$

$$x_3^3(\sigma) = \frac{\lambda_1 \lambda_3 \lambda_4 \lambda_6 \lambda_7}{\lambda_2^2 \lambda_5^2} \frac{a_2^2(\sigma) a_5^2(\sigma)}{a_1(\sigma) a_3(\sigma) a_4(\sigma) a_6(\sigma) a_7(\sigma)}$$

and then

$$b_2 = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_1^2(\sigma) a_4^2(\sigma)}{a_2(\sigma) a_3(\sigma) a_5(\sigma) a_6(\sigma) a_7(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_2 \lambda_3 \lambda_5 \lambda_6 \lambda_7}{\lambda_1^2 \lambda_4^2}$$

$$b_3 = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_2^2(\sigma)a_5^2(\sigma)}{a_1(\sigma)a_3(\sigma)a_4(\sigma)a_6(\sigma)a_7(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_1\lambda_3\lambda_4\lambda_6\lambda_7}{\lambda_2^2\lambda_5^2}$$

Now multiplying these three last numbers we get

$$\lambda_6^3 = b_1 b_2 b_3$$

Thus the value

$$\lambda_6 = (b_1 b_2 b_3)^{1/3}$$

is determined. By analogous arguments it is possible to derive

$$\lambda_7 = (b_2 b_3 b_4)^{1/3}, \lambda_1 = (b_3 b_4 b_5)^{1/3}, \lambda_2 = (b_4 b_5 b_6)^{1/3}, \lambda_3 = (b_5 b_6 b_7)^{1/3},$$

$$\lambda_4 = (b_6 b_7 b_1)^{1/3}, \lambda_5 = (b_7 b_1 b_2)^{1/3}$$

where the value b_i are obtained by symmetry from the previous b_1 , b_2 and b_3 . Thus the problem of getting an E -point completely mixed for this game is solved. It is clear that it is the unique E -point completely mixed that this game possesses.

Now we are going to study a 10-person game. Consider the game with structure function $d(i) = \{i, i+1, i+5, i+6, \dots, 10, 1, 2, \dots, i-1\} \bmod 10$ and payoff function

$$A_i(\sigma_1, \dots, \sigma_{10}) = a_i(\sigma_i) \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \quad a_i(\sigma_i) > 0$$

Then from equation (1), the E -point has to satisfy

$$\mu_i - x_{i+2} x_{i+3} x_{i+4} = 0 \quad \bmod 10 \quad (4)$$

From here we get by dividing two consecutive

$$x_{i+5} = \frac{\mu_{i+1}}{\mu_i} x_{i+2} \quad \bmod 10$$

and by replacing the respective x 's we get

$$x_6 = \frac{\mu_2}{\mu_1} x_3, \quad x_7 = \frac{\mu_3}{\mu_2} x_4, \quad x_8 = \frac{\mu_4}{\mu_3} x_5, \quad x_9 = \frac{\mu_3 \mu_{10}}{\mu_4 \mu_1} x_8$$

$$x_{10} = \frac{\mu_3 \mu_{10} \mu_7}{\mu_5 \mu_2 \mu_9} x_8, \quad x_1 = \frac{\mu_7}{\mu_6} x_8, \quad x_3 = \frac{\mu_9}{\mu_8} x_{10}, \quad x_4 = \frac{\mu_{10} \mu_7}{\mu_6 \mu_9} x_8, \quad x_5 = \frac{\mu_1}{\mu_{10}} x_2$$

which we wish to have them in term of x_8 . Replacing in (4) with $i = 6$ we obtain

$$x_8^3 = \frac{\mu_6 \mu_9 \mu_2 \mu_5 \mu_8 \mu_1 \mu_4}{\mu_3^2 \mu_7^2 \mu_{10}^2} \quad (5)$$

Doing the same thing with respect to x_9 and x_{10} and respectively using equation (4) with $i = 7$, and 8 we get

$$x_9^3 = \frac{\mu_7 \mu_{10} \mu_3 \mu_6 \mu_9 \mu_2 \mu_5}{\mu_3^2 \mu_7^2 \mu_{10}^2} \quad (6)$$

and

$$x_{10}^3 = \frac{\mu_8 \mu_1 \mu_4 \mu_2 \mu_{10} \mu_3 \mu_6}{\mu_5^2 \mu_9^2 \mu_2^2}$$

From (3) are gets, using $\sum_{\sigma} x_8(\sigma) = 1$ that

$$b_8 = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_3^2(\sigma) a_7^2(\sigma) a_{10}^2(\sigma)}{a_6(\sigma) a_9(\sigma) a_2(\sigma) a_5(\sigma) a_8(\sigma) a_1(\sigma) a_4(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_6 \lambda_9 \lambda_2 \lambda_5 \lambda_8 \lambda_1 \lambda_{10}}{\lambda_3^2 \lambda_7^2 \lambda_{10}^2}$$

In an analogous way from (6) we get

$$b_9 = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_1^2(\sigma) a_4^2(\sigma) a_8^2(\sigma)}{a_7(\sigma) a_{10}(\sigma) a_3(\sigma) a_6(\sigma) a_9(\sigma) a_2(\sigma) a_5(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_7 \lambda_{10} \lambda_3 \lambda_6 \lambda_9 \lambda_2 \lambda_{10}}{\lambda_1^2 \lambda_4^2 \lambda_8^2} \quad (7)$$

$$b_{10} = \left(\frac{1}{\sum_{\sigma} \left(\frac{a_5^2(\sigma) a_9^2(\sigma) a_2^2(\sigma)}{a_8(\sigma) a_1(\sigma) a_4(\sigma) a_7(\sigma) a_{10}(\sigma) a_3(\sigma) a_6(\sigma)} \right)^{1/3}} \right)^3 = \frac{\lambda_8 \lambda_1 \lambda_4 \lambda_2 \lambda_{10} \lambda_3 \lambda_6}{\lambda_5^2 \lambda_9^2 \lambda_2^2}$$

Multiplying the last three b's we get

$$\lambda_6^3 = b_8 b_9 b_{10}$$

Thus λ_6 is completely determined. Doing the same thing all the others λ 's might be determined. Replacing these values in the corresponding expressions of the x 's the unique completely mixed E -points can be obtained.

As a remark we would like to mention that any point $(\bar{\sigma}, \bar{\sigma}, \bar{\sigma}, \dots, \bar{\sigma})$ is an E -point. Then the both previous games have exactly

$$\sum_{i=1}^m \binom{m}{i} = 2^m - 1$$

E -points. Only one of them is completely mixed.

2. A general study for a $3k + 1$ person game with $k \geq 1$

Now in this section we will study and compute the E -points in a general diagonal game with $3k + 1$ with $k \geq 1$ players. The structure function is given by $d(i) = \{i, i + 1, i + 2, i + 5, i + 6, \dots, 3k + 1, 1, 2, \dots, i - 1\}$ with payoff functions given by

$$A_i(\sigma_1, \dots, \sigma_{3k+1}) = a_i(\sigma_i) \delta(\sigma_i, \sigma_{i+2}, \sigma_{i+3}, \sigma_{i+4}) \pmod{3k+1}$$

Therefore the equation (1) in this case is

$$\bar{\mu}_i - x_{i+2}x_{i+3}x_{i+4} = 0 \pmod{3k+1}.$$

or

$$\mu_{i+2} - x_{i+2}x_{i+3}x_{i+4} = 0 \pmod{3k+1}$$

where $\bar{\mu}_i = \mu_{i+2}$.

From two consecutive two equations of (8) we get

$$x_{i+3} = \frac{\mu_{i+1}}{\mu_i} x_i = S_{i+1} x_i$$

and replacing this

$$x_{i+3} = \prod_{s=1}^r S_{i+1-3s} x_{i-3r} \pmod{3k+1} \quad (9)$$

Now let us permit to arrange the numbers in strips in a natural way

$$\begin{array}{llll} 14 \dots 3k - 2 & \underline{3k + 1} & 3k + 4 \equiv 3 \dots 6k + 1 & 6k + 4 \equiv 2 \dots \\ 25 \dots 3k - 1 & 3k + 2 \equiv 1 & \dots \underline{6k + 2} \equiv 3k + 1 & 6k + 5 \equiv 3 \dots \\ 35 \dots 3k & 3k + 3 \equiv 3 & \dots 6k + 3 \equiv 1 & \dots \\ & & & \\ & \dots & 9k + 4 \equiv 1 & \\ & \dots & 9k + 5 \equiv 2 & \\ & \dots \underline{9k + 3} \equiv k + 1 & 9k + 6 \equiv 3. & \end{array} \quad (10)$$

Take $p \leq k$ and consider the $3p + 1$ -expression

$$\mu_{3p+1} - x_{3p+1} x_{3p+2} x_{3p+3} = 0 \quad (11)$$

Next we are going to express x_{3p+2} and x_{3p+3} in terms of x_{3p+1} . For this take $3p + 3 + 3k + 1 = 3(p + k) + 4 = i + 3$ or $i = 3(p + k) + 1$ in (9) with $r = k$. Then we have

$$x_{3p+3} = \prod_{s=0}^k S_{3(p+k)+2-3s} x_{3p+1} \quad (12)$$

Now taking $v = 3(p + 2k) + 1$ and $r = 2k$ replacing in (9) one gets

$$x_{3p+2} = \prod_{s=0}^{2k} S_{3(p+2k)+2-3s} x_{3p+1} \quad (12')$$

and replacing these values in (11) it is derived

$$\mu_{3p+1} - x_{3p+1}^3 \prod_{s=0}^k S_{3(p+k)+2-3s} \prod_{s=0}^{2k} S_{3(p+2k)+2-3s} = 0 \quad (13)$$

Now consider the number $3p + 2$ with $p \leq k$. Then for it we have

$$\mu_{3p+2} - x_{3p+2} x_{3p+3} x_{3p+4} = 0 \quad (14)$$

Putting in (9) $i = 3(p + k) + 2$ and $r = k$ we get

$$x_{3p+4} = \prod_{k=0}^k S_{3(p+k)+3-3s} x_{3p+2} \quad (15)$$

and with $i = 3(p + 2k) + 2$ and $r = 2k$

$$x_{3p+3} = \prod_{k=0}^{2k} S_{3(p+2k)+3-3s} x_{3p+2} \quad (16)$$

which replaced in (14) provides

$$\mu_{3p+2} - x_{3p+2}^3 \prod_{s=0}^k S_{3(p+k)+3-3s} \prod_{s=0}^{2k} S_{3(p+2k)+3-3s} = 0 \quad (17)$$

Finally with $i = 3(p + k) + 3$ and $r = k$ on the one hand and $i = 3(p + 2k) + 3$ and on the other using (9) and replacing the corresponding values of x_{3p+4} and x_{3p+5} it is easy to derive

$$\mu_{3p+3} - x_{3p+3}^3 \prod_{k=0}^k S_{3(p+k)+4-3s} \prod_{s=0}^{2k} S_{3(p+2k)+3-3s} = 0 \quad (18)$$

Now replacing the values of S_i in (12') and using the condition $\sum_{\sigma} x_{3p+1}(\sigma) = 1$

it holds

$$\lambda_{3p-1} \frac{\prod_{s=0}^k \lambda_{3(p+k)-1-3s}}{\prod_{s=0}^k \lambda_{3(p+k)-3s}} \frac{\prod_{s=0}^{2k} \lambda_{3(p+2k)-1-3s}}{\prod_{s=0}^{2k} \lambda_{3(p+2k)-3s}} = b_{3p+1} = \quad (19)$$

$$= \left(\frac{1}{\sum_{\sigma} \left(\frac{1}{a_{3p-1}(\sigma)} \frac{\prod_{s=0}^k a_{3(p+k)-3s}(\sigma)}{\prod_{s=0}^{2k} a_{3(p+k)-1-3s}(\sigma)} \frac{\prod_{s=0}^{2k} a_{3(p+2k)-2s}(\sigma)}{\prod_{s=0}^{2k} a_{3(p+2k)-1-3s}(\sigma)} \right)^{1/3}} \right)^3$$

Doing the same thing with (17), one derives

$$\lambda_{3p} \frac{\prod_{s=0}^k \lambda_{3(p+k)-3s}}{\prod_{s=0}^k \lambda_{3(p+k)+1-3s}} \frac{\prod_{s=0}^{2k} \lambda_{3(p+2k)-3s}}{\prod_{s=0}^{2k} \lambda_{3(p+2k)+1-3s}} = b_{3p+2} = \quad (20)$$

$$= \left(\frac{1}{\sum_{\sigma} \left(\frac{1}{a_{3p}(\sigma)} \frac{\prod_{s=0}^k a_{3(p+k)+1-3s}(\sigma)}{\prod_{s=0}^k a_{3(p+k)-3s}(\sigma)} \frac{\prod_{s=0}^{2k} a_{3(p+2k)+1-3s}(\sigma)}{\prod_{s=0}^{2k} a_{3(p+2k)-3s}(\sigma)} \right)^{1/3}} \right)^3$$

Finally using (18) it follows

$$\lambda_{3p+1} \frac{\prod_{s=0}^k \lambda_{3(p+k)+1-3s}}{\prod_{s=0}^k \lambda_{3(p+k)+2-3s}} \frac{\prod_{s=0}^{2k} \lambda_{3(p+2k)+1-3s}}{\prod_{s=0}^{2k} \lambda_{3(p+2k)+2-3s}} = b_{3p+1} = \quad (21)$$

$$= \left(\frac{1}{\sum_{\sigma} \left(\frac{1}{a_{3p+1}(\sigma)} \frac{\prod_{s=0}^k a_{3(p+k)+2-3s}(\sigma)}{\prod_{s=0}^k a_{3(p+k)+1-3s}(\sigma)} \frac{\prod_{s=0}^{2k} a_{3(p+2k)+2-3s}(\sigma)}{\prod_{s=0}^{2k} a_{3(p+2k)+1-3s}(\sigma)} \right)^{1/3}} \right)^3$$

Now multiplying the equalities (19), (20) and (21) it holds to

$$b_{3p+1} b_{3p+2} b_{3p+3} = \frac{\lambda_{3p-1}^3 \lambda_{3p} \lambda_{3p+1}}{\lambda_{3(p+k)+2} \lambda_{3(p+2k)+2}}$$

but

$$3p + 1 + 3k + 1 = 3(p + k) + 2 \equiv 3p + 1$$

and

$$3p + 2(3k + 1) \equiv 3(p + 2k) + 2 \pmod{3k + 1}$$

then

$$\lambda_{3p-1} = (b_{3p+1} b_{3p+2} b_{3p+3})^{1/3}. \quad (22)$$

In a similar way or by symmetry it is possible to derive

$$\lambda_{3p} = (b_{3p+2} b_{3p+3} b_{3p+4})^{1/3}$$

and

$$\lambda_{3p+1} = (b_{3p+3} b_{3p+4} b_{3p+5})^{1/3} \quad (23)$$

and in this way we have obtained the explicit value of the λ 's and replacing them in the corresponding place of (13), (14) or (18) one gets the corresponding value of the x 's. Thus the problem is solved positively.

We remind as in [4] that the point $(\bar{\sigma}, \bar{\sigma}, \dots, \bar{\sigma})$ is an E -point. Therefore there are exactly

$$\sum_{i=1}^m \binom{m}{i} = 2^m - 1$$

E -points which might be computed accordingly. Any way there is only one E -point completely mixed. The phenomenon appears in all the games studied here and for this reason we do not repeat the same argument every time.

4. A general study for a $3k + 2$ person game with $k \geq 1$

In this section we are going to study the same game given in the previous paragraph but when the numbers of players are $3k + 2$ with $k \geq 1$. Before to study the general case with an arbitrary k consider the case of $k = 2$. That is to say 8. Therefore the corresponding equation become

$$\lambda_i - a_i x_{i+2} x_{i+3} x_{i+4} = 0 \pmod{8} \quad (24)$$

and from here

$$x_6 = \frac{\mu_2}{\mu_1} x_3, \quad x_7 = \frac{\mu_3}{\mu_2} x_4, \quad x_8 = \frac{\mu_4}{\mu_3} x_5 = \frac{\mu_4 \mu_1 \mu_6}{\mu_3 \mu_8 \mu_5} x_7, \quad x_1 = \frac{\mu_5 \mu_2 \mu_7}{\mu_4 \mu_1 \mu_8} x_8$$

$$x_2 = \frac{\mu_6}{\mu_5} x_7 = \frac{\mu_6 \mu_3 \mu_8}{\mu_4 \mu_1 \mu_6} x_8, \quad x_3 = \frac{\mu_7}{\mu_6} x_8, \quad x_4 = \frac{\mu_8}{\mu_7} x_1, \quad x_5 = \frac{\mu_1}{\mu_8} x_2.$$

Thus replacing in (24) with $i = 5$

$$x_7^3 = \frac{\mu_3^2 \mu_8^2 \mu_5^2}{\mu_1 \mu_2 \mu_4 \mu_6 \mu_7}$$

By symmetry

$$x_8^3 = \frac{\mu_6^2 \mu_4^2 \mu_1^2}{\mu_2 \mu_5 \mu_7 \mu_8 \mu_3} \quad (25)$$

$$x_1^3 = \frac{\mu_7^2 \mu_2^2 \mu_5^2}{\mu_3 \mu_6 \mu_8 \mu_1 \mu_4}$$

$$x_2^3 = \frac{\mu_8^2 \mu_3^2 \mu_6^2}{\mu_4 \mu_7 \mu_1 \mu_2 \mu_5}$$

$$x_3^3 = \frac{\mu_1^2 \mu_4^2 \mu_7^2}{\mu_5 \mu_8 \mu_2 \mu_3 \mu_6}$$

$$x_4^3 = \frac{\mu_2^2 \mu_5^2 \mu_8^2}{\mu_6 \mu_1 \mu_3 \mu_4 \mu_7}$$

$$x_5^3 = \frac{\mu_3^2 \mu_6^2 \mu_1^2}{\mu_7 \mu_2 \mu_4 \mu_5 \mu_8}$$

$$x_6^3 = \frac{\mu_4^2 \mu_7^2 \mu_2^2}{\mu_8 \mu_3 \mu_5 \mu_6 \mu_1}$$

Introducing the corresponding values b 's it is possible to obtain

$$\lambda_i^3 = b_{i+2} b_{i+3} b_{i+4} \quad (26)$$

This expresses the fact that in the general case with $3k + 2$ players we should obtain an analogous solution as in the previous general case with $3k + 1$ players.

Indeed consider the general equation for the general case $3k + 2$:

$$\bar{\mu}_i - x_{i+2} x_{i+3} x_{i+4} = 0 \quad \text{mod } 3k + 2$$

where $\bar{\mu}_i = \lambda_i / a_i$, $\bar{\mu}_i = \mu_{i+2}$:

$$\mu_{i+2} - x_{i+2} x_{i+3} x_{i+4} = 0 \quad \text{mod } 3k + 2 \quad (27)$$

$$\mu_{3p+2} - X_{3p+2}^3 \prod_{s=0}^k S_{3(p+k)+3-3s} \prod_{s=0}^{2k+1} S_{3(p+2k)+6-3s} = 0 \quad (36)$$

Finally by symmetry it is easy to get

$$\mu_{3p+3} - X_{3p+3}^3 \prod_{s=0}^k S_{3(p+k)+4-3s} \prod_{s=0}^{2k+1} S_{3(p+2k)+7-3s} = 0 \quad (37)$$

Using the (32), (36) and (37) we get

$$\begin{aligned} & \lambda_{3p-1} \frac{\prod_{s=0}^k \lambda_{3(p+k)-1-3s}}{\prod_{s=0}^k \lambda_{3(p+k)-3s}} \frac{\prod_{s=0}^{2k+1} \lambda_{3(p+2k)+2-3s}}{\prod_{s=0}^{2k+1} \lambda_{3(p+2k)+3-3s}} = b_{3p+1} \\ & = \left(\frac{1}{\sum_{\sigma} \left(\frac{1}{a_{3p+1}(\sigma)} \frac{\prod_{s=0}^k a_{3(p+k)-3s}(\sigma)}{\prod_{s=0}^k a_{3(p+k)-1-3s}(\sigma)} \frac{\prod_{s=0}^{2k+1} a_{3(p+2k)+3-3s}(\sigma)}{\prod_{s=0}^{2k+1} a_{3(p+2k)+2-3s}(\sigma)} \right)^{1/3}} \right)^3 \quad (38) \\ & \lambda_{3p} \frac{\prod_{s=0}^k \lambda_{3(p+k)-3s}}{\prod_{s=0}^k \lambda_{3(p+k)+1-3s}} \frac{\prod_{s=0}^{2k+1} \lambda_{3(p+2k)+3-3s}}{\prod_{s=0}^{2k+1} \lambda_{3(p+2k)+4-3s}} = b_{3p+2} \\ & \lambda_{3p+1} \frac{\prod_{s=0}^k \lambda_{3(p+k)+1-3s}}{\prod_{s=0}^k \lambda_{3(p+k)+2-3s}} \frac{\prod_{s=0}^{2k+1} \lambda_{3(p+2k)+4-3s}}{\prod_{s=0}^{2k+1} \lambda_{3(p+k)+5-3s}} = b_{3p+3} \end{aligned}$$

where the values b_{3p+2} and b_{3p+3} are just analogous to b_{3p+1} .

By multiplying one obtains

$$b_{3p+1} b_{3p+2} b_{3p+3} = \lambda_{3p-1}^3$$

and in this way we have obtained explicitly the unique completely mixed E -point in the game.

As a final remark we would like to say that in the case that the number of players is $3k$ one obtains difficult problems for solving it which we expect to consider in a further paper.

7. Bibliography

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