TRAVELING WAVE SOLUTIONS ARISING FROM
A COMBUSTION MODEL

BY

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Traveling Wave Solutions Arising From a Combustion Model

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1. Introduction

This paper is concerned with proving the existence of traveling wave solutions of a reaction-diffusion system which arises from the theory of combustion. The equations take the form

\[ U_t = DU_{xx} + F(U) \]  \hspace{1cm} (1.1)

where \( U = (T,Y_1,\ldots,Y_{n-1}) \in \mathbb{R}^n \) and \( D \) is a positive, diagonal matrix. The traveling wave represents a combustion front in a premixed reactive gas. The components of \( U \) specify the dimensionless temperature and the concentration of the reactants. For a background of the physical motivation of these equations see [2], [11].

By a traveling wave solution of (1.1) we mean a nonconstant, bounded solution of the form \( U(x,t) = U(z), z = x + \theta t \). Note that a traveling wave solution satisfies the system of ordinary differential equations

\[ DU'' - \theta U' + F(U) = 0 . \]

In order to motivate our results we make a few remarks concerning the simple reaction \( A \to P \). If we assume that the reaction rate is of mass action-Arrhenius form, \( T \) is the dimensionless temperature, and \( Y \) is the concentration of the reactant \( A \), then \( (T,Y) \) satisfies the system

\[ T_t = d_1 T_{xx} + QBYe^{-E/T} \]
\[ Y_t = d_2 Y_{xx} - BYe^{-E/T} \]
where \( d_1, d_2, Q, B, \) and \( E \) are positive constants. We assume that \((T,Y)\) satisfies the boundary conditions

\[
(T,Y)(-\infty) = (T_-,Y_-) \quad \text{and} \quad (T,Y)(+\infty) = (T_+,0) .
\]

A traveling wave solution then satisfies the system

\[
d_1 T'' - \theta T' = -QBYe^{-E/T} \tag{1.3}
\]
\[
d_2 Y'' - \theta Y' = BYe^{-E/T}
\]

along with the boundary conditions (1.2). Note that to prove the existence of a traveling wave solution we must show that there exists a \( \theta \) for which there exists a solution of (1.2), (1.3). One quickly finds that there cannot exist any traveling wave solutions. This is because the righthand side of (1.3) is zero only when \( Y = 0 \) or \( T = 0 \). Physically, the reason that there cannot exist any traveling wave solutions is that the formulation (1.3) requires that the mixture to be reacting all the way in from \( x = -\infty \). By the time finite \( x \) is reached the combustion would be complete. This is often referred to as the cold boundary difficulty, (see [2]). To overcome this difficulty it is necessary to avoid reaction at a finite rate over an infinite time. One way to do this is as follows. Replace (1.3) by

\[
d_1 T'' - \theta T' = -QYf(T) \tag{1.4}
\]
\[
d_2 Y'' - \theta Y' = Yf(T)
\]

where \( f(T) \) is continuous and satisfies

a) there exists \( T_1 > 0 \) such that \( f(T) = 0 \) for \( T < T_1 \),

\[
\tag{1.5}
\]

b) \( f(T) > 0 \) for \( T > T_1 \).
Berestycki, Niclanenko, and Scheurer [1] prove that there does indeed exist a traveling wave solution of (1.3), (1.2) if \( f(T) \) satisfies (1.5).

In this paper we consider the existence of traveling wave solutions arising from the reactions

\[
A \overset{(i)}{\rightarrow} B \overset{(ii)}{\rightarrow} P.
\]  

(1.6)

If \( T \) is the dimensionless temperature, \( Y_1 \) is the concentration of \( A \), and \( Y_2 \) is the concentration of \( B \), then the traveling wave equations corresponding to this reaction network is

\[
d_0 T' - \theta T' = -Q_1 B_1 Y_1 e^{-E_1/T} - Q_2 B_2 Y_2 e^{-E_2/T} \\
\]

\[
d_1 Y_1' - \theta Y_1' = B_1 Y_1 e^{-E_1/T} \\
\]

(1.7)

\[
d_2 Y_2' - \theta Y_2' = -B_1 Y_1 e^{-E_1/T} + B_2 Y_2 e^{-K_2/T}.
\]

We assume that the unburned state

\[
U_0 = (T_-, Y_1-, Y_2-)
\]  

(1.8)

is prescribed. As before, there cannot exist any traveling wave solutions of this system due to the cold boundary difficulty. Following what was done for the single reaction, we replace (1.7) by

\[
d_0 T' - \theta T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \\
\]

\[
d_1 Y_1' - \theta Y_1' = Y_1 f_1(T) \\
\]

(1.9)

\[
d_2 Y_2' - \theta Y_2' = -Y_1 f_1(T) + Y_2 f_2(T)
\]

and consider two cases. We assume that \( f_1(T) \) and \( f_2(T) \) are continuous and satisfy either
(A) There exists $T_1, T_2$ such that for $i = 1, 2$,

(i) $0 < T_1 < T_2$

(ii) $f_i(T) = 0$ for $T \leq T_1$

(iii) $f_i(T) > 0$ for $T > T_1$.

(B) There exists $T_1, T_2$ such that for $i = 1, 2$,

(i) $0 < T_2 < T_1$

(ii) $f_i(T) = 0$ for $T \leq T_1$

(iii) $f_i(T) > 0$ for $T > T_1$

(1.10)

Following Fife and Nicolaenko [7] we begin by discussing simple flames arising from (1.9). That is, we imagine a simple flame resulting from one of the reactions, the other being artificially suppressed. There are four basic simple flames:

1) F1, produced by reaction (i), see (1.6), only, converting the given unburned state to a partially burned one:

$$U_0 \rightarrow U_+ = (T_1, 0, Y_{1-} + Y_{2-})$$

2) F2, produced by reaction (ii) only

$$U_0 \rightarrow U_2 = (T_2, Y_{1-}, 0)$$

3) F12, produced by (ii) acting on the product (final state) of F1:

$$U_1 \rightarrow U_+ = (T_4, 0, 0)$$

4) F21, acting on the product of F2. One can think of this flame as converting A to B by reaction (i), and B thereupon being almost immediately converted to P by the faster reaction (ii).
The above flames will proceed at different velocities. For example, if the speed of F12, which is built on the products of F1, is slower than F1 itself, then one can imagine two flames both existing, but the distance between them ever increasing. Another, more interesting, phenomenon is if F12 is faster than F1. In this case, the rear flame approaches the forward one. As it does so its effect is to heat up the forward one. One expects that the speed of the forward wave to then increase. One may then expect that what eventually evolves is a single configuration moving with constant velocity. In this paper we prove that such a wave does indeed exist.

In order to formally state our first result we assume, for now, that \( f_1(T) \) and \( f_2(T) \) satisfy (1.10A). For convenience, we assume that the unburned state is given by

\[
U_\varepsilon = (T_\varepsilon, Y_\varepsilon, Y_\varepsilon) = (0, 1, 1),
\]

(1.11)

and

\[
0 < T_1 < Q_1 < T_2 < Q_1 + 2Q_2.
\]

(1.12)

As we now show, (1.12) is needed to guarantee the existence of simple flames. The two simple flames, F1 and F12, are shown as follows:

\[
U_\varepsilon \xrightarrow{F1} U_1 = (T^1, 0, Y^1_2) \xrightarrow{F12} U_+ = (T_+, 0, 0).
\]

We first consider the flame F1. Note that

\[
T^1 = Q_1 \quad \text{and} \quad Y^1_2 = 2.
\]

(1.13)

This is proved as follows. Integrate the second equation of (1.9) for

\[-\varepsilon < z < \varepsilon\]

to obtain
\[ \theta = \int_{-\infty}^{\infty} Y_1 f_1(T) dz \]  

(1.14)

Recall, from (1.10A), that \( f_2(T) = 0 \) if \( T < T_2 \). Hence, if \( T < T_2 \) along \( F_1 \), then we can integrate the first equation of (1.9) for \(-\infty < z < \infty\) to obtain

\[ -\theta T' = -Q_1 \int_{-\infty}^{\infty} Y_1 f_1(T) dz \]

This, together with (1.14), implies that \( T' = Q_1 \). Still assuming that \( T < T_2 \) along \( F_1 \), we integrate the third equation in (1.9) to obtain

\[ -\theta (Y_2' - 1) = -\int_{-\infty}^{\infty} Y_1 f_1(T) dz \]

This, together with (1.14), implies that \( Y_2' = 2 \).

It remains to prove that \( T < T_2 \) along \( F_1 \). However, if \( T < T_2 \), then \( (T, Y_1) \) satisfies

\[
\begin{align*}
\frac{d_0 T'}{d_0} - \theta T' &= -Q_1 Y_1 f_1(T) \\
\frac{d_1 Y_1'}{d_1} - \theta Y_1' &= Y_1 f_1(T) \\
\end{align*}
\]  

(1.14)

\[(T, Y_1)(-\infty) = (0, 1), \; \text{and} \; (T, Y_1)(+\infty) = (Q_1, 0) \].

Berestycki, Nicolaenko, and Scheurer proved that there exists a solution of (1.14). Moreover, the solution is monotone. Hence, \( T(z) < Q_1 < T_2 \) for all \( z \).

Note that after solving (1.14), \( Y_2(z) \) is obtained as the solution of

\[
\begin{align*}
\frac{d_2 Y_2'}{d_2} - \theta Y_2' &= -Y_2 f_1(T) \\
Y_2(-\infty) &= 1, \; Y_2(+\infty) = 2 \\
\end{align*}
\]
We do not know that the solution of (1.14) is unique. Let

\[ \theta^1 = \sup(\theta : \text{There exists a solution of (1.14)} \]

\[ \text{with speed } \theta \} . \]

That is, \( \theta^1 \) is the maximum speed of F1.

Now consider the flame F12. Along F12, \( Y_1(z) = 0 \). Hence, \((T,Y_2)\) satisfies

\[
\begin{cases} 
  d_0 T' - \theta T' = -Q_2 Y_2 f_2(T) \\
  d_2 Y_2' - \theta Y_2' = Y_2 f_2(T)
\end{cases}
\]

(1.15)

\((T,Y_2)(-\infty) = (Q_1,2), (T,Y_2)(+\infty) = (T_+,0)\).

Integrating these equations, as before, we find that

\[ T_+ = Q_1 + 2Q_2 . \]

The results of Berestycki, Nicolaenko, and Scheurer imply that there exists a monotone solution of (1.15). As before, we do not know if the solution of (1.15) is unique. Let

\[ \theta^{12} = \inf(\theta : \text{There exists a solution of (1.15)} \]

\[ \text{with speed } \theta \} . \]

In this paper we are interested in the case \( \theta^1 < \theta^{12} \). We prove

**Theorem 1**: Assume (1.10A) and \( \theta^1 < \theta^{12} \). Then there exists a solution of (1.9), for some speed \( \theta^0 \), which satisfies

\[(T,Y_1,Y_2)(-\infty) = (0,1,1) \text{ and } (T,Y_1,Y_2)(+\infty) = (Q_1 + 2Q_2,0,0) . \]
Moreover, \( T, Y_1, \) and \( Y_2 \) are positive for all \( z \), \( T(z) \) is monotone increasing, and \( Y_1(z) \) is monotone decreasing.

**Remark:** One does not expect \( Y_1(z) \) to be monotone.

We now state other theorems which we can prove in a manner similar to the proof of Theorem 1. The proofs of these results will appear in a future paper.

Assume that (1.10B) holds. We assume (1.10B), and, instead of (1.12), we assume that

\[
0 < T_1 < Q_2 < T_2 < Q_1 + 2Q_2
\]  

(1.16)

Consider the simple flames \( F_2 \) followed by \( F_{21} \):

\[
V_\rightarrow F_2 \Rightarrow U_2 = (T^2, Y_1, 0) F_{21} \Rightarrow (T_+, 0, 0).
\]

Integrating the equations in (1.9) as before we find that

\[
T^2 = Q_2 \quad \text{and} \quad T_+ = Q_1 + 2Q_2.
\]

The assumptions \( T_1 < Q_2 < T_2 \) imply that \( F_2 \) corresponds to a solution of the reduced system

\[
\begin{align*}
T^* - \theta T^* &= -Q_2 Y_2 f_2(T) \\
Y_2^* - \theta Y_2^* &= Y_2 f_2(T)
\end{align*}
\]

(1.17)

\((T, Y_2)(-\infty) = (0, 1), \ (T, Y_2)(+\infty) = (Q_2, 0)\).

We conclude from the results of Beresnycki, Nicolaenko, and Scheurer that there exists a monotone solution of (1.17). Let
\[ \varepsilon^2 = \sup(\theta : \text{There exists a solution of (1.17) with speed } \theta). \]

Note that along F2, \( Y_1(z) = 1 \). Now consider the flame F21. Unlike before, F21 does not correspond to the solution of a reduced system. Instead, it is a solution of (1.9) together with the boundary conditions

\[ (T, Y_1, Y_2)(-) = (Q_2, 1, 0), \quad (T, Y_1, Y_2)(+) = (Q_1 + 2Q_2, 0, 0) \quad (1.18) \]

We can then prove:

**Theorem 2:** Assume (1.10B) and (1.16). Then there exists a solution of (1.9), (1.18) for some \( \theta > 0 \). Moreover, \( T, Y_1, \) and \( Y_2 \) are positive for all \( z \), \( T \) is monotone increasing, and \( Y_1(z) \) is monotone decreasing.

We do not know that the solution of (1.9), (1.18) is unique. Let

\[ \theta^{21} = \inf(\theta : \text{There exists a solution of (1.9), (1.18)} \]

with speed \( \theta \).

We then prove

**Theorem 3:** Assume (1.10B) and \( \theta^2 < \theta^{21} \). Then there exists a solution of (1.9), for some speed \( \theta^0 \), which satisfies

\[ (T, Y_1, Y_2)(-) = (0, 1, 1) \quad \text{and} \]

\[ (T, Y_1, Y_2)(+) = (Q_1 + 2Q_2, 0, 0). \]

Moreover, \( T, Y_1, \) and \( Y_2 \) are positive for all \( z \), \( T(z) \) is monotone increasing, and \( Y_1(z) \) is monotone decreasing.
In remains to consider the cases when (1.12) or (1.16) are not satisfied.

We can then prove

**Theorem 4:** Assume that either (1.10A) is satisfied and

\[ 0 < T_1 < T_2 < Q_1 < Q_1 + 2Q_2, \]

or (1.10B) is satisfied and

\[ 0 < T_1 < T_2 < Q_2 < Q_1 + 2Q_2. \]

Then there exists a solution of (1.19), for some speed \( \theta^0 \), which satisfies

\[ (T, Y_1, Y_2)(-\infty) = (0, 1, 1) \quad \text{and} \quad (T, Y_1, Y_2)(+\infty) = (Q_1 + 2Q_2, 0, 0). \]

Moreover, \( T, Y_1, \) and \( Y_2 \) are positive for all \( z, \) \( T(z) \) is monotone increasing, and \( Y_1(z) \) is monotone decreasing.

In the next section we outline the proof of Theorem 1. The proof is quite geometrical, and the purpose of the next section is to introduce the basic geometrical features of the proof. The proof of the theorems consist of three basic steps. The first step is to obtain a priori bounds for the solution. These are obtained in Sections 3 and 4. The next step in the proof is to prove the theorem for the special case \( d_0 = d_1 = d_2 = 1. \) The proof for this case is based on the Conley index and is carried out in Section 5. The last step of the proof is to continue the solution from the case \( d_0 = d_1 = d_2 = 1. \) The a priori bounds will play a crucial role in the continuation. This is carried out for Theorem 1 in Section 6.
An essential feature in the proof of the Theorems is the Conley index. Other authors have used index arguments to prove the existence of traveling wave solutions. In particular, Conley and Smoller [5] used a method similar to what appears here to prove the existence of traveling wave solutions for systems of the form

$$U_t = U_{xx} + VF(U).$$

In the context of Combustion theory, Gardner [9] used the Conley index to prove the existence of traveling wave solutions. The system he considered is different from what appears here.
2. A Brief Outline of the Proof of Theorem 1

Here we give a brief outline of the proof of Theorem 1. We assume throughout this section that (1.10A) and (1.12) are satisfied. The proof of Theorem 1 is quite geometrical. The purpose of this section is to introduce the basic geometrical features of the proof.

The first step is to reduce (1.9) to a first order system. Let \( T' = q, \ Y'_1 = p_1, \) and \( Y'_2 = p_2. \) Then (1.9) is equivalent to the system

\[
\begin{align*}
T' &= q \\
\dot{q} &= \theta q - Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \\
Y'_1 &= p_1 \\
\dot{p}_1 &= \theta p_1 + Y_1 f_1(T) \\
Y'_2 &= p_2 \\
\dot{p}_2 &= \theta p_2 - Y_1 f_1(T) + Y_2 f_2(T).
\end{align*}
\]

Let

\[
\gamma(z) = (T(z), q(z), Y_1(z), p_1(z), Y_2(z), p_2(z)),
\]

and

\[
A = (0, 0, 1, 0, 1, 0), \ B = (Q_1, 0, 0, 0, 2, 0), \ C = (Q_1 + 2Q_2, 0, 0, 0, 0, 0).
\]

Then \( F_1 \) corresponds to a solution, \( \gamma_1(z), \) of (2.1) which satisfies

\[
\lim_{z \to A} \gamma_1(z) = A \quad \text{and} \quad \lim_{z \to B} \gamma_1(z) = B.
\]

\( F_12 \) corresponds to a solution \( \gamma_{12}(z) \) of (2.1) which satisfies
\[ \lim_{z \to -\infty} \gamma_{12}(z) = B \quad \text{and} \quad \lim_{z \to +\infty} \gamma_{12}(z) = C. \]

For convenience we assume, in this section, that these simple waves are unique. In particular, the speeds of these waves, \( \theta^1 \) and \( \theta^{12} \) respectively, are assumed to be uniquely determined. We wish to prove that if \( \theta^1 < \theta^{12} \), then there exists a solution \( \gamma_0(z) \) of (2.1) which satisfies

\[ \lim_{z \to -\infty} \gamma_0(z) = A \quad \text{and} \quad \lim_{z \to +\infty} \gamma_0(z) = C. \]

In each case the traveling wave solution corresponds to a trajectory in phase space which connects two critical points.

One of the key ideas in the proof of the theorems is to attach to (2.1) the equation

\[ \theta' = \epsilon(\theta - \theta_o)(\theta - \theta_i) \quad (2.2) \]

where \( 0 < \theta_i < \theta_o \) and \( 0 < \epsilon \ll 1 \). Let

\[ A_1 = (A, \theta_o), \quad B_1 = (B, \theta_o), \quad C_1 = (C, \theta_o), \]
\[ A_2 = (A, \theta_1), \quad B_2 = (B, \theta_1), \quad C_2 = (C, \theta_1), \]

\[ \mathcal{I}_A = \{ (\gamma, \theta) : \gamma = A, \theta_1 < \theta < \theta_o \}, \]
\[ \mathcal{I}_B = \{ (\gamma, \theta) : \gamma = B, \theta_1 < \theta < \theta_o \}, \]
\[ \mathcal{I}_C = \{ (\gamma, \theta) : \gamma = C, \theta_1 < \theta < \theta_o \}. \]

First consider the case \( \epsilon = 0 \). When \( \epsilon = 0 \), \( \mathcal{I}_A, \mathcal{I}_B, \) and \( \mathcal{I}_C \) correspond to lines of critical points. Then \( (\gamma_1(z), \theta^1) \) and \( (\gamma_{12}(z), \theta^{12}) \) trace out curves in phase space as shown in Figure 1. In Figure 1A we have shown the case \( \theta^1 > \theta^{12} \), while in Figure 1B we have shown \( \theta^1 < \theta^{12} \).
Now suppose that $\varepsilon > 0$. The lines $\ell_A$, $\ell_B$, and $\ell_C$ now correspond to trajectories which connect the six critical points $A_1$ and $A_2$, $B_1$ and $B_2$, and $C_1$ and $C_2$, respectively. Crucial to the proof of Theorem 1 is the following

**Proposition 2.1:** For each $\varepsilon > 0$, there exists a solution $(\gamma^\varepsilon(z), \theta^\varepsilon(z))$ of (2.1), (2.2) which satisfies

$$\lim_{z \to A} (\gamma^\varepsilon(z), \theta^\varepsilon(z)) = A_1 \quad \text{and} \quad \lim_{z \to C} (\gamma^\varepsilon(z), \theta^\varepsilon(z)) = C_2.$$ 

**Remark:** This proposition is true in both cases, $\theta^1 < \theta^{12}$ and $\theta^1 > \theta^{12}$.

This result is proved later in the paper. We comment on the proof shortly. We first discuss what happens to $(\gamma^\varepsilon(z), \theta^\varepsilon(z))$ as $\varepsilon \to 0$. We shall have a priori bounds so that it will be clear that at least some subsequence of $\{(\gamma^\varepsilon(z), \theta^\varepsilon(z))\}$ converges to something.

Let us, for the moment, consider the case $\theta^1 > \theta^{12}$. It is possible that for $0 < \varepsilon \ll 1$, $(\gamma^\varepsilon(z), \theta^\varepsilon(z))$ is as shown in Figure 2A. That is, $(\gamma^\varepsilon(z), \theta^\varepsilon(z))$
lies close to \( t_A \) for \( \theta^1 < \theta < \theta_0 \), then lies close to \( (\gamma_1(z), \theta^1) \), then lies close to \( t_B \) for \( \theta^{12} < \theta < \theta^1 \), then lies close to \( (\gamma_{12}(z), \theta^{12}) \), and finally lies close to \( t_C \) for \( \theta_1 < \theta < \theta^{12} \). In the limit \( \varepsilon \to 0 \) the curves \((\gamma^\varepsilon(z), \theta^\varepsilon(z))\) converge to the union of the curves \( t_A \) for \( \theta^1 < \theta < \theta_0 \), \((\gamma_1(z), \theta^1)\), \( t_B \) for \( \theta^{12} < \theta < \theta^1 \), \((\gamma_{12}(z), \theta^{12})\), and \( t_C \) for \( \theta_1 < \theta < \theta^{12} \). Hence, \((\gamma^\varepsilon(z), \theta^\varepsilon(z))\) does not converge to a traveling wave solution.

![Figure 2](image)

In the above paragraph we strongly used the assumption that \( \theta^1 > \theta^{12} \). Since \( \theta^1 < 0 \), it follows that if \( \theta^1 < \theta^{12} \), then \((\gamma^\varepsilon(z), \theta^\varepsilon(z))\) cannot converge to a curve which consists of pieces of \( t_A \), \( t_B \), \( t_C \), \((\gamma_1(z), \theta^1)\) and \((\gamma_{12}(z), \theta^{12})\). We prove, later, that the only other possibility is that there exists \( \theta^0 \), and a sequence \((\varepsilon_n)\) such that as \( n \to \infty \), \( \varepsilon_n \to 0 \) and \((\gamma^{\varepsilon n}(z), \theta^{\varepsilon n}(z))\) converges to a curve which consists of three pieces. These are:

a) \( t_A \) for \( \theta^0 < \theta < \theta_0 \)

b) a trajectory \((\gamma_0(z), \theta^0)\) which satisfies (2.1), (2.2) for all \( z \) and \( \varepsilon = 0 \), and satisfies
\[ \lim_{z \to -} \gamma_0(z) = A \quad \text{and} \quad \lim_{z \to +} \gamma_0(z) = C \]

c) \( C \) for \( \theta_1 < \theta < \theta^0 \).

Then \( \gamma_0(z) \) is the desired solution.

We conclude this section with a few remarks concerning the proof of Proposition 2.1. The proof of Proposition 2.1 consists of three main steps. We first prove the result for the case when \( d_0 = d_1 = d_2 = 1 \). The proof in this case is based on Conley's Morse theory. For the proof we will need such notions as isolated invariant sets, the Conley index, and Morse decompositions. Those readers unfamiliar with these notions are referred to [3], [6], and [10].

The second step in the proof of Proposition 2.1 is to obtain a priori bounds on both the variables \( (T,q,Y_1,P_1,Y_2,P_2) \) and the speed of the wave, \( \theta^0 \). The estimates we obtain are similar to those obtained by Berestycki, Nicolaenko, and Scheurer [1]. These estimates are derived in Sections 3 and 4. They are used in the third part of the proof of Proposition 2.1 where we continue the solution from the case \( d_1 = d_2 = d_3 = 1 \). This last step is carried out in Section 6.
3. A priori Bounds, $\theta' \neq 0$

In this and the next sections we derive the a priori bounds necessary for the proof of the theorems. We assume that either (1.10A) or (1.10B) are satisfied, and $\theta_0 > \theta_1 > 0$. For convenience, we assume throughout the remainder of the paper that $d_0 = 1$. Let $\zeta(z) = (T,q,Y_1,p_1,Y_2,p_2,\theta)(z)$ be a solution of

\[
\begin{align*}
T' &= q \\
q' &= \theta q - Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \\
Y_1' &= p_1 \\
1p_1' &= \theta p_1 + Y_1 f_1(T) \\
Y_2' &= p_2 \\
2p_2' &= \theta p_2 - Y_1 f_1(T) + Y_2 f_2(T) \\
\theta' &= \epsilon(\theta - \theta_0)(\theta - \theta_1)
\end{align*}
\]

(3.1)

Together with the boundary conditions

\[
\lim_{z \to -\infty} \zeta(z) = A_1 = (A, \theta_0) \quad \text{and} \quad \lim_{z \to +\infty} \zeta(z) = C_2 = (C, \theta_1).
\]

(3.2)

The reason for studying this system was motivated in the previous section.

**Lemma 3.1:** $Y_1(z) > 0$ for all $z$.

**Proof:** We first show that $Y_1(z) > 0$ for each $z$. If not, then there exists $z_0$ such that

\[
Y_1(z_0) < 0 \quad \text{and} \quad Y_1'(z_0) > 0
\]

(3.3)
Let
\[ \alpha = \sup\{z < z_0 : Y_1'(z) = 0 \} . \]

Certainly \( \alpha \) is well defined. Then \( Y_1(\alpha) = 0, Y_1(z) > 0 \) for \( z \in (\alpha, z_0) \), and \( Y_1(z) < 0 \) on \( (\alpha, z_0) \). Then
\[
d_1 Y_1 - \theta Y_1' = Y_1 f_1(T) \leq 0 \text{ on } [\alpha, z_0] .
\]

On the other hand,
\[
d_1 Y_1 - \theta Y_1' \geq d_1 Y_1 - \theta_0 Y_1' \text{ on } [\alpha, z_0]
\]
Therefore,
\[
d_1 Y_1 - \theta_0 Y_1' \leq 0 \text{ on } [\alpha, z_0] ,
\]
or
\[
(e^{-\theta_0/d_1z_1'})' \leq 0 \text{ on } [\alpha, z_0] .
\]

Integrate this last equation from \( \alpha \) to \( z_0 \) to obtain \( Y_1(z_0) \leq 0 \). This, however, contradicts (3.3).

If \( Y_1(z_0) = 0 \) for some \( z_0 \) then we must have \( Y_1(z_0) = 0 \). This implies that \( Y_1(z) = 0 \) for all \( z \). Since \( Y_1(-\epsilon) = 1 \), this gives a contradiction.

**Lemma 3.2**: \( Y_2(z) > 0 \) for all \( z \).

**Proof**: We first prove that \( Y_2(z) > 0 \) for all \( z \). If not, there exists \( z_0 \) such that
\[
Y_2(z_0) < 0 \text{ and } Y_2'(z_0) > 0 . \tag{3.4}
\]

Let
\[ \alpha = \sup\{z < z_0 : Y_2'(z) = 0 \} . \]
Certainly $\alpha$ is well defined. Then $Y_2'(\alpha) = 0$, $Y'(z) > 0$ in $(\alpha, z_0]$ and $Y_2(z) < 0$ in $[\alpha, z_0]$. Therefore,

$$d_2 Y_2 - \theta Y_2' = -Y_1 f_1(T) + Y_2 f_2(T) \leq 0 \text{ in } [\alpha, z_0].$$

On the other hand,

$$d_2 Y_2 - \theta Y_2' \leq d_2 Y_2 - \theta_0 Y_2' \text{ in } [\alpha, z_0].$$

Therefore

$$d_2 Y_2 - \theta Y_2' \leq 0 \text{ in } [\alpha, z_0],$$

or

$$\left(e^{-\theta_0/d_2 Y_2'}\right)' \leq 0 \text{ in } [\alpha, z_0].$$

Integrate this last equation from $\alpha$ to $z_0$ to obtain $Y_2'(z_0) \leq 0$. This, however, contradicts 3.4.

If $Y_2(z_0) = 0$ for some $z_0$ then $Y_2'(z_0) = 0$ and $Y_2(z_0) \geq 0$. If $T(z_0) \leq T_1$, then this implies that $Y_2(z) = 0$ for all $z$, which is impossible. If $T(z_0) > T_1$, then

$$d_2 Y_2 = -Y_1 f_1(T) < 0,$$

which, again gives a contradiction.

**Lemma 3.3:** $T(z) > 0$ for all $z$.

**Proof:** We first prove that $T(z) \geq 0$ for all $z$. If not then there exists $z_0$ such that

$$T(z_0) < 0 \text{ and } T'(z_0) > 0. \quad (3.5)$$
Let 
\[ \alpha = \sup\{z < z_0 : T'(z) = 0\} \]

Then \( T'(\alpha) = 0, T'(z) > 0 \) on \((\alpha, z_0]\), and \( T(z) < 0 \) on \([\alpha, z_0]). \) Therefore,
\[ T' - \theta T' = -Y_1 f_1(T) - Y_2 f_2(T) \leq 0 \] on \([\alpha, z_0]\).

On the other hand,
\[ T' - \theta T' \geq T' - \theta_0 T' \] on \([\alpha, z_0]\).

Therefore,
\[ T' - \theta_0 T' \leq 0 \] on \([\alpha, z_0]\),
or
\[ (e^{-\theta_0 z} T')'(z) \leq 0 \] on \([\alpha, z_0]\).

Integrate this last equation from \( \alpha \) to \( z_0 \) to obtain \( T'(z_0) < 0 \). This, however, contradicts (3.5).

If \( T(z_0) = 0 \) for some \( z_0 \), then \( T'(z_0) = 0 \) and \( T'(z_0) \geq 0 \). If \( T(z_0) \leq T_1 \), then this implies that \( T(z) = 0 \) for all \( z \), which is impossible.

If \( T(z_0) > T_1 \), then
\[ d_0 T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) < 0 \]

which, again, gives a contradiction.

**Lemma 3.4:**
(a) \( 0 < Y_1(z) \leq \theta_0 / \theta_1 \)
(b) \(-\theta_0 / d_1 \leq Y_1'(z) \leq \theta_0^2 / d_1 \theta_1 \)

**Proof:** Let \( u(z) = d_1 Y_1'(z) - \theta(z) Y_1(z) \). Then
\[ u'(z) = d_1 Y''(z) - \theta(z)Y_1(z) - \theta'(z)Y_1(z) \]
\[ = Y_1 f_1(T) - \theta'(z)Y_1(z) \geq 0 . \]

Hence, \( u(z) \) is an increasing function. Since

\[ u(-\infty) = \theta_o \quad \text{and} \quad u(+\infty) = 0 , \]

it follows that

\[ -\theta_o \leq u(z) \leq 0 \quad \text{for all} \quad z . \tag{3.6} \]

Hence,

\[ d_1 Y'_1 - \theta_1 Y_1 = d_1 Y'_1 - \theta Y_1 = u - \theta_o \quad \text{for all} \quad z . \]

Multiply the left and right sides of this equation by \( e^{-\theta_1/d_1 z} \) and integrate from \(-\infty\) to \( z \) to obtain

\[ -e^{-\theta_1/d_1 z} Y_1(z) \geq -\theta_o/\theta_1 e^{-\theta_1/d_1 z} \]

from which (a) follows.

From our estimates on \( u(z) \) and \( Y_1(z) \) we find that

\[ Y'_1(z) = 1/d_1 u(z) + \theta(z)/d_1 Y_1(z) \leq \theta_o/\theta_1 Y_1(z) \leq \theta_o^2/d_1 \theta_1 . \]

On the other hand,

\[ Y'_1(z) = 1/d_1 u(z) + \theta(z)/d_1 Y_1(z) \leq 1/d_1 u(z) \leq -\theta_o/d_1 . \]

**Lemma 3.5:**

a) \( 0 < Y_2(z) \leq 2\theta_o/\theta_1 \)

b) \( -2\theta_o/\theta_1 \leq Y_2'(z) \leq 2\theta_o^2/d_3 \theta_1 . \)
**Proof:** Let \( v = d_2 Y'_2 - \theta Y_2 \) and \( w = u + v \) where \( u(z) \) was defined in the proof of Lemma 3.4. Then

\[
w' = u' + v' = Y_2 f_2(T) - \theta'(z) Y_1(z) - \theta'(z) Y_2(z) \geq 0.
\]

Hence, \( w(z) \) is an increasing function. Since

\[
w(-\infty) = -2\theta_o \quad \text{and} \quad w(\infty) = 0
\]

it follows that

\[
-2\theta_o \leq w(z) \leq 0 \quad \text{(3.7)}
\]

Since \( v = w - u \), it follows from (3.6) and (3.7) that

\[
-2\theta_o \leq v(z) \leq \theta_o \quad \text{(3.8)}
\]

Hence,

\[
d_2 Y'_2 - \theta_1 Y_2 \leq d_2 Y'_2 - \theta Y_2 = v \geq -2\theta_o.
\]

Multiply the left and right sides of this equation by \( e^{-\theta_1/d_2 z} \) and integrate from \(-\infty\) to \( z \) to obtain

\[
e^{-\theta_1/d_2 z} Y_2(z) \leq -2\theta_o/\theta_1 e^{-\theta_1/d_2 z},
\]

from which (a) follows.

From our estimate for \( v(z) \) and \( Y_2(z) \) we find that

\[
Y'_2(z) = \frac{1}{d_2} v(z) + \theta/d_2 Y_2(z) \leq \theta_0/d_2 + 2\theta_o/d_2 \theta_1.
\]

On the other hand,
\[ Y_2'(z) = \frac{1}{d_2} v(z) + \frac{\theta}{d_2} Y_2(z) + \frac{-\theta_0}{d_2} . \]

**Lemma 3.6:**

a) \( 0 < T(z) \leq \theta_0/\theta_1 (Q_1 + 3Q_2) \)

b) \( -\theta_0(Q_1 + 3Q_2) \leq T'(z) \leq \theta_0/\theta_1 (Q_1 + 3Q_2)(2\theta_0 - \theta_1) \).

**Proof:** Let \( u(z) \) and \( v(z) \) be as in the proofs of Lemmas 3.4 and 3.5.

Note that

\[ T' - \theta T' + (Q_1 + Q_2)u' + Q_2 v' = 0 . \]

Hence

\[ T' - (\theta T)' + (Q_1 + Q_2)u' + Q_2 v' = -\theta'T \geq 0 . \tag{3.9} \]

Integrate this equation from \(-\infty\) to \( z \) to obtain

\[ T' - \theta T + (Q_1 + Q_2)u + Q_2 v \geq -\theta_0(Q_1 + Q_2) - \theta Q_2 . \]

Together with (3.6) and (3.8) this implies that

\[ T' - \theta T \geq -(Q_1 + Q_2)u - Q_2 v - Q_0(Q_1 + Q_2) - \theta_0 Q_2 \]

\[ \geq -Q_2 \theta_0 - \theta_0(Q_1 + Q_2) - \theta_0 Q_2 \]

\[ = -\theta_0 Q_1 - 3Q_2 \theta_0 . \tag{3.10} \]

Therefore,

\[ T' - \theta_1 T \geq T' - \theta T \geq -\theta_0 Q_1 - 3Q_2 \theta_0 . \]

Multiply the left and right sides of this equation by \( e^{-\theta_1 z} \) and integrate from \(-\infty\) to \( z \) to obtain

\[ -e^{-\theta_1 z}T(z) \geq -\theta_0/\theta_1 (Q_1 + 3Q_2)e^{-\theta_1 z} , \]

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from which (a) follows.

From (3.10) we conclude that

\[ T' \leq \theta T - \theta_0 Q_1 - 3Q_2 \theta_0 \leq -\theta_0 (Q_1 + 3Q_2) \]

To obtain an upper bound for \( T'(z) \) we begin with

\[ T'' - \theta T' = T'' - (\theta T)' + \theta' T = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \leq 0 . \]

Hence,

\[ T'' - (\theta T)' \leq -\theta' T . \]

Integrate this last equation from \(-\infty\) to \( z \) to obtain

\[ T'(z) - \theta(z)T(z) \leq -\int_{-\infty}^{z} \theta' Tdz \leq \frac{\theta_0}{\theta_1} (Q_1 + 3Q_2) \int_{-\infty}^{z} -\theta'(s)ds \]

\[ \leq \theta_0/\theta_1 (Q_1 + 3Q_2)(\theta_0 - \theta_1) . \]

Hence,

\[ T'(z) \leq \theta(z)T(z) + \frac{\theta_0}{\theta_1} (Q_1 + 3Q_2)(\theta_0 - \theta_1) \]

\[ \leq \frac{\theta_0^2}{\theta_1} (Q_1 + 3Q_2) + \frac{\theta_0}{\theta_1} (Q_1 + 3Q_2)(\theta_0 - \theta_1) \]

and the proof is complete.

**Lemma 3.7:** \( Y_1'(z) \leq 0 \) for all \( z \).

**Proof:** If not, there exists \( z_0 \) such that \( Y_1'(z_0) = 0 \) and \( Y_1'(z_0) > 0 \). If \( T(z_0) < T_1 \), then \( d_1 Y_1' - \theta Y' = 0 \). Hence, \( Y_1(z) = Y_1(z_0) \) for all \( z \), which is impossible. If \( T(z_0) > T_1 \), then \( Y_1'(z_0) = Y_1 f_1(T) > 0 \) which gives a contradiction.
Lemma 3.8: If $0 < T(z) < Q_1 + 2Q_2$, then $T'(z) > 0$.

Proof: If not, then because $\lim_{z \to u} T(z) = Q_1 + 2Q_2$, there exists a $z_0$ such that $T'(z_0) = 0$ and $T''(z_0) > 0$. If $T(z_0) < T_1$, then we must have $T(z) = T(z_0)$ for all $z$, which is impossible. If $T(z_0) > T_1$, then

$$
T''(z_0) = -Q_1Yf_1(T) - Q_2Yf_2(T) < 0,
$$

which, again, gives a contradiction.
4. A priori Bounds. \( \theta = \text{constant} \)

In this section we assume that \( \theta = \theta^0 \) is constant and consider solutions of

\[
\begin{align*}
T' &= q \\
q' &= \theta^0 q - Q_1 T_1 f_1(T) - Q_2 Y_2 f_2(T) \\
Y_1' &= p_1 \\
d_1 p_1' &= \theta^0 p_1 + Y_1 f_1(T) \\
Y_2' &= p_2 \\
d_2 p_2' &= \theta^0 p_2 - Y_1 f_1(T) + Y_2 f_2(T)
\end{align*}
\]

(4.1a)

together with the boundary conditions

\[
\begin{align*}
\lim_{z \to -\infty} (T, q, Y_1, p_1, Y_2, p_2)(z) &= A \\
\lim_{z \to +\infty} (T, q, Y_1, p_1, Y_2, p_2)(z) &= C .
\end{align*}
\]

(4.1b)

Our goal will be to obtain a lower and upper estimate on \( \theta^0 \).

We first need a preliminary result.

**Lemma 4.1:** There exists \( K > 0 \) such that if \( y > x \) then \( Y_2(y) < KY_2(x) \).

**Proof:** If the lemma is not true, then there exists two sequences, \( (x_n) \) and \( (y_n) \), such that for each \( n \), \( x_n < y_n \) and \( Y_2(y_n) > nY_2(x_n) \). Since \( Y_2(z) \) is bounded, (Lemma 4.1b), it follows that \( \lim_{n \to \infty} Y_2(x_n) = 0 \). However, \( Y_2(z) > 0 \) for each \( z \). Hence, \( \lim_{n \to \infty} x_n = -\infty \). Since \( y_n > x_n \), we conclude...
that $\lim_{n \to \infty} y_n = 0$. We prove that this is impossible by showing that there exist $\eta$ such that if $z > \eta$, then $Y_i(z) < 0$.

This last statement is proved by linearizing (4.1) at $C$. The behavior of $Y_i(z)$ near $z = C$ is determined by the linearized system. This implies that $Y_i(z)$ is "well behaved" near $C$. One consequence being that $Y_i(z) < 0$ for $z$ sufficiently large.

**Lemma:** $Y_i(z) \leq 0$ and $T'(z) \leq 0$ for all $z$.

**Proof:** It follows from Lemma 3.7 that $Y_i(z) \leq 0$ for all $z$. To prove that $T'(z) \leq 0$ for all $z$ we consider the equation

$$T' - \theta^0 T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \leq 0$$

Multiply this equation by $e^{-\theta^0 z}$ to obtain

$$(e^{-\theta^0 z} T'(z))' \leq 0 .$$

Integrate this last equation from $z$ to $\infty$ to obtain

$$-e^{-\theta^0 z} T'(z) \leq 0 ,$$

or $T'(z) \geq 0$.

Let $K$ be as in Lemma 4.1, and

$$M = (Q_1 + Q_2)(1 - d_1) + d_1 + Q_2 (K^1 - d_2) + d_2$$

(4.2)

**Lemma 4.3:** $(Q_1 + 2Q_2) - T \leq M(Y_1 + Y_2)$ for all $z$.

**Proof:** Let

$$u(z) = T + d_1 (Q_1 + Q_2) Y_1 + d_2 Q_2 Y_2 - (Q_1 + 2Q_2) .$$

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Then
\[ u'' - \theta^0 u' = \theta^0(Q_1 + Q_2)(1 - d_1)Y_1' + \theta^0 Q_2(1 - d_2)Y_2'. \]

Integrate this equation from \(-\infty\) to \(z\) to obtain
\[ u'(z) - \theta^0 u(z) = \theta^0(Q_1 + Q_2)Y_1(z) + \theta^0 Q_2(1 - d_2)Y_2(z). \]

Multiply this last equation by \(e^{-\theta^0 z}\) and integrate from \(z\) to \(\infty\) to obtain
\[ -e^{-\theta^0 z} u(z) = \theta^0(Q_1 + Q_2)(1 - d_1) \int_z^{\infty} e^{-\theta^0 s} Y_1(s) ds \]
\[ + \theta^0 Q_2(1 - d_2) \int_z^{\infty} e^{-\theta^0 s} Y_2(s) ds \]
\[ = \theta^0(Q_1 + Q_2)(1 - d_1) |Y_1(z)| \int_z^{\infty} e^{-\theta^0 s} ds \]
\[ + \theta^0 Q_2(1 - d_2) |KY_2(z)| \int_z^{\infty} e^{-\theta^0 s} ds \]
\[ = (Q_1 + Q_2)(1 - d_1) |Y_1(z)| e^{-\theta^0 z} + Q_2(1 - d_2) |KY_2(z)| e^{-\theta z}. \]

Divide both sides by \(e^{-\theta z}\) to obtain
\[ (Q_1 + 2Q_2) - T - d_1(Q_1 + Q_2)Y_1 - d_2 Q_2 Y_2 \]
\[ = (Q_1 + Q_2)(|Y_1(z)| + Q_2 |KY_2(z)|), \]
or
\[ (Q_1 + 2Q_2) - T(z) \leq (Q_1 + Q_2)(d_1 + |l - d_1|)Y_1(z) \]
\[ + Q_2(d_2 + K|l - d_2|)Y_2(z) \]
\[ \leq M(Y_1(z) + Y_2(z)). \]
In what follows we assume that \( z = 0 \) is chosen so that \( T(0) = T_1 \). For \( z < 0 \), \( T(z) \) satisfies

\[
T'' - \theta^0 T' = 0.
\]

Integrate this equation from \(-\infty\) to 0 to obtain

\[
T'(0) = \theta^0 T_1.
\]  \hspace{1cm} (4.3)

**Lemma 4.4:**

\[
\int_0^\infty |T'(z)|^2 dz < \frac{\theta^0}{2} \left[ (Q_1 + 2Q_2)^2 - T_1^2 \right].
\]

**Proof:** Multiply both sides of the equation

\[
T'' - \theta^0 T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T)
\]

by \( T \) to obtain

\[
(T'T')' - (T')^2 - \frac{\theta^0}{2} \frac{dT^2}{dz} = -Q_1 Y_1 T f_1(T) - Q_2 Y_2 T f_2(T).
\]

Integrate this last equation from 0 to \(-\infty\) to obtain

\[
-\frac{\theta^0}{2} (Q_1 + Q_2)^2 + \frac{\theta^0}{2} T_1^2 - \theta^0 T_1^3 - \int_0^\infty |T'(z)|^2 dz
\]

\[
= -\int_0^\infty Q_1 Y_1 T f_1(T) dz - \int_0^\infty Q_2 Y_2 T f_2(T) dz
\]

\[
\leq 0
\]

from which the result follows.

Let \( f_3(T) = \min(f_1(T), f_2(T)) \) and
\[ \Lambda = \int_{0}^{Q_1 + 2Q_2} (Q_1 + 2Q_2 - s)f_3(s)\,ds . \]

Note that \( \Lambda > 0. \)

**Proposition 4.5:** \( \theta^0 > \left[ \frac{2(Q_1 + Q_2)\Lambda}{(Q_1 + 2Q_2)M} \right]^{1/2} = \theta_{11} . \)

**Proof:** Multiply both sides of the equation

\[ T' - \theta^0 T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \]

by \( T' \) to obtain

\[ -\frac{1}{2}(T')^2 + \theta^0 (T')^2 = Q_1 Y_1 f_1(T)T' + Q_2 Y_2 f_2(T)T' . \quad (4.4) \]

Recall that we are assuming that \( T(0) = T_1, \) from which (4.3) follows.

Integrate both sides of (4.4) from 0 to \( s \) to obtain

\[ \frac{1}{2}(\theta^0)^2 T_1^2 + \theta^0 \int_{0}^{s} |T'(z)|\,dz = \int_{0}^{s} Q_1 Y_1 f_1(T)T' + Q_2 Y_2 f_2(T)T'\,dz \]

\[ \approx \int_{0}^{s} (Q_1 + Q_2) (Y_1 + Y_2) f_3(T)T'(z)\,dz \]

\[ \approx \frac{(Q_1 + Q_2)}{M} \int_{0}^{s} (Q_1 + 2Q_2 - T)f_3(T)T'(z)\,dz \]

\[ = \frac{(Q_1 + Q_2)}{M} \Lambda . \]

We now use Lemma 4.5 to conclude that

\[ \frac{1}{2}(\theta^0)^2 T_1^2 + \frac{1}{2}(\theta^0)^2 [(Q_1 + 2Q_2)^2 - T_1^2] \geq \frac{(Q_1 + Q_2)}{M} \Lambda \]

from which the results follows.
We will also need the following results which are proved similarly to the proof of Proposition 4.5.

**Proposition 4.6:** Fix \( d_1 > 0 \). Then there exists \( \theta_{12} > 0 \), which depends only on \( d_1 \) and \( f_1(T) \), such that if \( d \) is between \( d_1 \) and 1 and \((T(z), Y_1(z))\) is a solution of

\[
\begin{align*}
T' - \theta T' &= -Q_1 Y_1 f_1(T) \\
dY'_1 - \theta Y'_1 &= Y_1 f_1(T),
\end{align*}
\]

\[
\lim_{z \to \infty} (T(z), Y_1(z)) = (0, 1), \quad \lim_{z \to \infty} (T(z), Y_1(z)) = (Q_1, 0),
\]

then \( \theta > \theta_{12} \).

**Proposition 4.7:** Fix \( d_2 > 0 \), \( T_1 < T^* < T_2 \), and \( 0 < Y^* < 2 \). Then there exists \( \theta_{13} > 0 \) which depends only on \( d_2 \) and \( f_2(T) \) such that if \( d \) is between \( d_2 \) and 1 and \((T(z), Y_2(z))\) is a solution of

\[
\begin{align*}
T' - \theta T' &= -Q_2 Y_2 f_2(T) \\
dY'_2 - \theta Y'_2 &= Y_2 f_2(T),
\end{align*}
\]

\[
\lim_{z \to \infty} (T(z), Y_2(z)) = (T^*, Y^*), \quad \lim_{z \to \infty} (T(z), Y_2(z)) = (Q_1 + 2Q_2, 0),
\]

then \( \theta > \theta_{13} \).

Given \( d_1, d_2 \) let

\[
\theta = \min\{\theta_{11}, \theta_{12}, \theta_{13}\}
\]

(4.5)

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We conclude this section by obtaining an upper bound on $\theta^o$. Choose $L$ such that for $i = 1,2,$

$$f_i(T) < LT$$

for $0 < T < Q_1 + 3Q_2$. Let

$$\theta_o = 1 + L(Q_1 + 2Q_2). \tag{4.6}$$

**Proposition 4.8:** $\theta^o < \theta_o$.

**Proof:** Let

$$S = \{(T,q,Y_1,p_1,Y_2,p_2) : 0 < T < q\}.$$

Suppose that $\#(z) = (T,q,Y_1,p_1,Y_2,p_2)(z)$ is a solution of (4.1a,b) with $\theta^o > \theta_o$. We first show that there exists $z_0$, such that $\#(z_0) \in S$. To prove this, note that for $T < T_1$, $(T,q)$ satisfies

$$T' = q$$

$$q' = \theta^o q.$$ 

Hence,

$$(\theta^o T - q)' = 0.$$

Since $T(-\infty) = q(-\infty) = 0$, it follows that

$$q = \theta^o T$$

for all $z$ such that $T(z) < T_1$. Since $\theta^o > \theta_o > 1$ this implies that $\#(z) \in S$ as long as $T(z) < T_1.$
Let 

\[ z_1 = \sup\{z : \#(z) \in S\} . \]

Since \( \lim_{z \to a} \#(z) = C \notin S \), it follows that \( z_1 < a \).

Let \( n = (1, -1) \). Since \( \#(z) \) is leaving \( S \) at \( z = z_1 \),

\[ n \cdot (T'(z_1), q'(z_1)) > 0 . \]

However, using Lemma (4.1), and the fact that \( q(z_1) = T(z_1) \),

\[ n \cdot (T'(z_1), q'(z_1)) = q(z_1) - \theta^0 q(z_1) + Q_1 Y_1 f_1(T) + Q_2 Y_2 f_2(T) \]

\[ = q(z_1) \left[ 1 - \theta^0 + \frac{Q_1 Y_1 f_1(T) + Q_2 Y_2 f_2(T)}{q(z_1)} \right] \]

\[ = q(z_1) \left[ 1 - \theta^0 + \frac{Q_1 Y_1 f_1(T) + Q_2 Y_2 f_2(T)}{T(z_1)} \right] \]

\[ = q(z_1)[1 - \theta^0 + L(Q_1 + 2Q_2)] \]

\[ < 0 . \]

This gives the desired contradiction. In the above calculation we used Lemma 3.4 and 3.5 which imply, because here \( \theta^0 = \theta_1 \), \( Y_1 \neq 1 \) and \( Y_2 \neq 2 \).
5. The Case \( d_0 = d_1 = d_2 = 1 \)

A. Reduction of Order

Our goal is to prove Proposition 2.1 for the case \( d_0 = d_1 = d_2 = 1 \). Hence, we consider solutions of

\[
T'' - \theta T' = -Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T)
\]

\[
Y_1'' - \theta Y_1' = Y_1 f_1(T)
\]  \hspace{1cm} (5A.1)

\[
Y_2'' - \theta Y_2' = -Y_1 f_1(T) + Y_2 f_2(T)
\]

together with the boundary conditions

\[
(T, Y_1, Y_2)(-\infty) = (0, 1, 1) \quad \text{and} \quad (T, Y_1, Y_2)(+\infty) = (Q_1 + 2Q_2, 0, 0). \quad (5A.2)
\]

We begin by showing that one can easily reduce this six dimensional system to a four dimensional one.

If we multiply the second equation in (5A.1) by \( Q_1 + Q_2 \), the third by \( Q_2 \), add the resulting equations to the first and let

\[
z = T + (Q_1 + Q_2)Y_1 + Q_2 Y_2
\]

we find that

\[
z'' - \theta z' = 0
\]

Since \( z'(-\infty) = z'(+\infty) = 0 \), it follows that
\[ T + (Q_1 + Q_2)Y_1 + Q_2 Y_2 = \text{constant} . \]

By assumption \( z(-\infty) = Q_1 + 2Q_2 \). Therefore,

\[ T + (Q_1 + Q_2)Y_1 + Q_2 Y_2 = Q_1 + 2Q_2 , \]

or

\[ Y_2 = \frac{1}{Q_2} [Q_1 + 2Q_2 - T - (Q_1 + Q_2)Y_1] . \]  \hspace{1cm} (5A.3)

Plugging (5A.3) into (5A.1) we obtain

\[
\begin{cases}
T'' - \theta T' = -Q_1 Y_1 f_1(T) - [Q_1 + 2Q_2 - T - (Q_1 + Q_2)Y_1] f_2(T) \\
Y_1'' - \theta Y_1' = Y_1 f_1(T) .
\end{cases}
\]  \hspace{1cm} (5A.4)

(5A.2) reduces to

\[ (T, Y_1)(-\infty) = (0,1), \ (T, Y_1)(+\infty) = (Q_1 + 2Q_2,0) . \]  \hspace{1cm} (5A.5)

**B. Perturbations of the Equations**

Let

\[ \phi = \{(T, Y_1) : T \geq 0, Y_1 \geq 0\} . \]

Of course, we are only interested in values of \((T, Y_1) \in \phi\). The rest points of (5A.4) in \(\phi\) are
\[
\{(T, Y_1) : 0 \leq T \leq T_1, \ Y_1 < 0\} \cup \{(T, Y_1) : Y_1 = 0, \ 0 < T < T_2\} \cup
\]
\[
\{(Q_1 + 2Q_2, 0)\}.
\]

This is a very large set; it is much too large to work with. Recalling the discussion in Section 2, we eventually want to reduce the problem to one in which there are only three critical points. This is done in a number of steps. We begin by perturbing (5A.4) twice so that the resulting system has five critical points.

For the first perturbation let \( f(\varepsilon_1)(T) \) be any function which satisfies for each \( \varepsilon_1 > 0 \),

a) \( f(\varepsilon_1)(T) \in C(\mathbb{R}) \)

b) \( f(\varepsilon_1)(0) = f(\varepsilon_1)(T_1) = 0 \)

c) \( -\varepsilon_1 < f(\varepsilon_1)(T) < 0 \) for \( 0 < T < T_1 \)

d) \( f(\varepsilon_1)'(0) = -\varepsilon_1 \)

e) \( f(\varepsilon_1)(T) = f_1(T) \) for \( T > T_1 \).

For \( \varepsilon_1 > 0 \), let (5A.4 \( \varepsilon_1 \)) equal to (5A.4) with \( f_1(T) \) replaced by \( f(\varepsilon_1)(T) \). Let

\[
F(\varepsilon_1)(T, Y_1) = Q_1 Y_1 f(\varepsilon_1)(T) + [Q_1 + 2Q_2 - T - (Q_1 + Q_2)Y_1]f_2(T).
\]

Then the phase plane for the reaction flow

\[
\dot{T} = F(\varepsilon_1)(T, Y_1) \]

\[
\dot{Y}_1 = -Y_1 f(\varepsilon_1)(T) \]

is as shown in Figure 3.
The bold lines in Figure 3 indicate the critical points of $(5B.2e_1)$. These are

\[ \{(T,Y_1) : T = 0, Y_1 > 0\} \cup \{(T,Y_1) : T = T_1, Y_1 > 0\} \cup \]

\[ \cup \{(T,Y_1) : 0 \leq T \leq T_2, Y_1 = 0\} \cup \{(Q_1 + 2Q_2, 0)\} \, . \]

We now perturb the equations again so that the new reaction flow, which we write as

\[ \dot{T} = f(e_1,e_2)(T,Y_1) \]

\[ \dot{Y}_1 = g(e_1,e_2)(T,Y_1) \, , \]

looks like what is shown in Figure 4.
The critical points of this new system are

\[ a = (0,1), \quad \beta = (T_1, \frac{1}{Q_1}(T_1 - Q_1)), \quad \gamma = (Q_1 + 2Q_2, 0), \]

\[ D = (0,0), \quad \text{and} \quad E = (T_1, 0) \quad (5B.3) \]

The precise properties needed for \( f(\varepsilon_1, \varepsilon_2)(T,Y_1) \) and \( g(\varepsilon_1, \varepsilon_2)(T,Y_1) \) are for each \( \varepsilon_1, \varepsilon_2 \) sufficiently small,

a) \( f(\varepsilon_1, \varepsilon_2)(T,Y_1) \) and \( g(\varepsilon_1, \varepsilon_2)(T,Y_1) \in C(\mathbb{R}^2) \)

b) \( \| (f(\varepsilon_1, \varepsilon_2)(T,Y_1), g(\varepsilon_1, \varepsilon_2)(T,Y_1)) - (F(\varepsilon_1)(T,Y_1), -Y_1 f(\varepsilon_1)(T)) \| < \varepsilon_2 \)

for all \( (T,Y_1) \),

c) if \( T = 0 \), then \( f(\varepsilon_1, \varepsilon_2)(T,Y_1) = 0 \) and \( g(\varepsilon_1, \varepsilon_2)(T,Y_1) = \varepsilon_2 Y_1 (1 - Y_1) \),

if \( T = T_1 \), then \( f(\varepsilon_1, \varepsilon_2)(T,Y_1) = 0 \) and

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\[ g(\varepsilon_1, \varepsilon_2)(T, Y_1) = -\varepsilon_2 (Y_1 - 1/Q_1 (T_1 - Q_1)) Y_1, \]

if \( Y_1 = 0, T < T_2 \), then \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) = \frac{\varepsilon_2}{2T_2(T_2 - T_1)} T(T - T_1) \)

and \( g(\varepsilon_1, \varepsilon_2)(T, Y_1) = 0 \),

d) if \( T + Q_1 Y_1 > T_2 + \varepsilon_2 \), then \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) = F(\varepsilon_1)(T, Y_1) \),

and \( g(\varepsilon_1, \varepsilon_2)(T, Y_1) < -Y_1 f(\varepsilon_1)(T, Y_1) \), \hspace{1cm} (5B.4)

e) if \( T + Q_1 Y_1 < Q_1 \), then \( g(\varepsilon_1, \varepsilon_2)(T, Y_1) > -Y_1 f(\varepsilon_1)(T) \)

and \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) + Q_1 g(\varepsilon_1, \varepsilon_2)(T, Y_1) > 0 \),

f) if \( Y_1 < \varepsilon_1/Q_1 \) and \( T < T_1 \), then \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) < F(\varepsilon_1)(T, Y_1) \),

g) if \( Y_1 < \varepsilon_1/Q_1 \) and \( T_1 < T < Q_1 - Q_1 Y_1 \), then

\[ f(\varepsilon_1, \varepsilon_2)(T, Y_1) > F(\varepsilon_1)(T, Y_1) \]

h) if \( Y_1 > \varepsilon_1/Q_1 \), then \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) = F(\varepsilon_1)(T, Y_1) \)

In what follows we consider the system

\[
T' = q \\
q' = \theta q - f(\varepsilon_1, \varepsilon_2)(T, Y_1) \hspace{1cm} (5B.5) \\
Y_1' = p_1 \\
p_1' = \theta p_1 - g(\varepsilon_1, \varepsilon_2)(T, Y_1)
\]

for each \( \varepsilon_1, \varepsilon_2 > 0 \) sufficiently small. After proving the existence of the desired connecting orbit, we let \( \varepsilon_1, \varepsilon_2 \) approach zero.
C. An Isolating Neighborhood

In this section we construct an isolating neighborhood which contains the critical points of interest. Note that $N$, a compact subset of phase space, is an isolating neighborhood if each trajectory on the boundary of $N$ leaves $N$ in forwards or backwards time. The reason that isolating neighborhoods are important is that one can assign an index (the Conley index) to the maximal invariant set inside of them. Moreover, isolating neighborhoods remain isolating neighborhoods under perturbations of the equations.

To begin with fix $\delta, \delta_1$, and let

$$ D = \{(T, Y_1) - \delta \leq T \leq -Q_1 Y_1 + Q_1 + 2Q_2 + \delta, \ Y_1 \leq \frac{1}{Q_1} (T_1 + 2\delta)\} $$

$$ \cup \{(T, Y_1) : -Q_1 Y_1 + T_1 + \delta \leq T \leq -Q_1 Y_1 + Q_1 + 2Q_2 + \delta, \ Y_1 \leq \frac{1}{Q_1} (T_1 + 2\delta)\} $$

$$ 0 \leq Y_1 \leq \frac{1}{Q_1} (T_1 + 2\delta) \} $$

$$ \cup \{(T, Y_1) : -QY_1 + T_1 + \delta \leq T \leq Q_1 + 2Q_2 + \delta, \ Y_1 \leq 0\} $$

Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ be the sides of $D$ as shown in Figure 5.
Lemma 5C.1: $\mathcal{D}$ is positively invariant for the flow defined by (5B.5) if $\delta, \delta_1,$ and $\varepsilon_2$ are sufficiently small.

Remark: To say that $\mathcal{D}$ is positively invariant means that any solution of (5B.5) which lies in $\mathcal{D}$ for some $z_0$ remains in $\mathcal{D}$ for all $z > z_0$.

Proof: We prove that the vector field defined by the right side of (5B.5) points into $\mathcal{D}$ for $\delta, \delta_1,$ and $\varepsilon$ sufficiently small. We treat the five sides of $\mathcal{D}$ separately.

a) On $\sigma_1$, $f(\varepsilon_1, \varepsilon_2)(T, Y_1) = F(\varepsilon_1)(T, Y_1)$ because of (5B.4h). Hence, $T' = f(\varepsilon_1, \varepsilon_2)(T, Y_1) = Q_1 Y_1 f(\varepsilon_1)(T) > 0$.

b) Suppose that $(T, Y_1) \in \sigma_2$. First assume that $T < T_3$. By (5B.4d), $f(\varepsilon_1, \varepsilon_2)(T, Y_1) = F(\varepsilon_1)(T, Y_1) = Q_1 Y_1 f(\varepsilon_1)(T)$ and $g(\varepsilon_1, \varepsilon_2)(T, Y_1) < -Y_1 f(\varepsilon_1)(T, Y_1)$. Let $n = (1, Q_1)$ be an outward normal to $\sigma_2$. Then
\[ n \cdot (\hat{T}, \hat{Y}_1) = f(\varepsilon_1, \varepsilon_2)(T, Y_1) + Q_1 g(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[ < F(\varepsilon_1)(T, Y_1) - Q_1 Y_1 f(\varepsilon_1)(T, Y_1) \]

\[ = Q_1 Y_1 f(\varepsilon_1)(T, Y_1) - Q_1 Y_1 f(\varepsilon_1)(T, Y_1) \]

\[ = 0 , \]

which is what we want.

Now suppose that \((T, Y_1) \in \sigma_2\) and \(T > T_2\). Again, by (5B.4d),
\[ g(\varepsilon_1, \varepsilon_2)(T, Y_1) < -Y_1 f(\varepsilon_1)(T, Y_1) \]
and \[ f(\varepsilon_1, \varepsilon_2)(T, Y_1) = F(\varepsilon_1)(T, Y_1) \]. Let \(n = (1, Q_1)\) be as before. Then
\[ n \cdot (\hat{T}, \hat{Y}_1) = f(\varepsilon_1, \varepsilon_2)(T, Y_1) + Q_1 g(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[ < Q_1 Y_1 f_1(T) + [Q_1 + 2Q_2 - T - (Q_1 + Q_2) Y_1] f_2(T) - Q_1 Y_1 f_1(T) \]

\[ = [Q_1 + 2Q_2 - T - (Q_1 + Q_2) Y_1] f_2(T) . \]

On \(\sigma_2\), \(T + Q_1 Y_1 = Q_1 + 2Q_2 + \delta\). Therefore,
\[ n \cdot (\hat{T}, \hat{Q}) < [Q_1 + 2Q_2 - Q_2 Y_1 - Q_1 - 2Q_2 - \delta] f_2(T) \]

\[ = -[Q_2 Y_1 + \delta_3] f_2(T) \]

\[ < 0 , \]

which is what we wish.

c) On \(\sigma_3\), the outward normal is \((-1, Q_1)\). First assume that \(Y_1 \neq \varepsilon_1/Q_1\). Then, by (5B.4e),
\[ n \cdot (\hat{T}, \hat{Q}) = -f(\varepsilon_1, \varepsilon_2)(T, Y_1) - Q_1 g(\varepsilon_1, \varepsilon_2)(T, Y_1) < 0 \]

Now assume that \( Y_1 < \varepsilon_1/Q_1 \). Then, on \( \sigma_3 \), \( T > T_1 \) if \( \delta < Q_1 - T_1 \). From (5B.4e,g) we conclude that

\[
n \cdot (\hat{T}, \hat{Y}_1) = -f(\varepsilon_1, \varepsilon_2)(T, Y_1) - Q_1 g(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[
< -Q_1 Y_1 f_1(T) - Q_1 Y_1 f_1(T) \]

\[
= 0. \]

d) Suppose that \((T, Y_1) \in \sigma_4\). When \( \varepsilon_2 = 0 \), the unperturbed flow gives \( Y_1 = -Y_1 f_1(T) > 0 \). Hence, this remains true for \( \varepsilon_2 \) sufficiently small.

e) Finally, suppose that \((T, Y_1) \in \sigma_5\). When \( \varepsilon_2 = 0 \), the unperturbed flow gives

\[
T' = Q_1 Y_1 f_1(\varepsilon_1)(T) + [Q_1 + 2Q_2 - T - (Q_1 + Q_2)Y] f_2(T) \]

\[
< [Q_1 + 2Q_2 - T - (Q_1 + Q_2)Y] f_2(T) \]

\[
< [Q_1 + 2Q_2 - (Q_1 + 2Q_2 + \delta) + (Q_1 + Q_2)\delta_1] f_2(T) \]

\[
= [-\delta + (Q_1 + Q_2)\delta_1] f_2(T) \]

\[
< 0 \]

if \( \delta > (Q_1 + Q_2)\delta_1 \) which we assume to be true. Hence, on \( \sigma_5 \), \( T' < 0 \) if \( \varepsilon_2 \) is sufficiently small.

Throughout the remainder of the paper we assume that \( \varepsilon_2, \delta \) and \( \delta_1 \) are chosen so that the Lemma is true.
We now construct an isolating neighborhood for the four dimensional flow defined by (5B.5). Fix \( V > 0 \) and let

\[
\mathcal{D}_1 = \{ (T,q,Y_1,p_1) : (T,Y_1) \in \mathcal{D}, |q| < V, |p_1| < V \}.
\]

Let \( \theta_0 \) be as in (4.4).

**Lemma 5C.2:** There exists \( V \) such that \( \mathcal{D}_1 \) is an isolating neighborhood for each \( \theta, |\theta| < \theta_0 + 1 \).

Recall that we must prove that each trajectory on \( \partial \mathcal{D} \) must leave \( \mathcal{D}_1 \) in forwards or backwards time.

**Proof:** Assume that \( \gamma(z_0) = (T,q,Y_1,p_1)(z_0) \in \partial \mathcal{D}_1 \). First assume that \( |q(z_0)| < L \) and \( |p(z_0)| < L \).

Let \( n \) be an outward normal to \( \partial \mathcal{D} \) at \( (T(z_0),Y_1(z_0)) \). If \( \gamma(z) \) is not tangent to \( \partial \mathcal{D} \) at \( z_0 \), then there is nothing to prove. If \( \gamma(z) \) is tangent to \( \partial \mathcal{D}_1 \) at \( z_0 \) then \( (T(z),Y_1(z)) \) must be tangent to \( \partial \mathcal{D} \) at \( z_0 \). Hence,

\[
0 = n \cdot (q(z_0), p_1(z_0))
\]

which implies that at \( z = z_0 \),

\[
n \cdot (q'(z), p_1'(z)) = \theta n \cdot (q(z), p(z)) - \\
-n \cdot (f(\varepsilon_1, \varepsilon_2)(T,Y_1), g(\varepsilon_1, \varepsilon_2)(T,Y_1))
\]

\[
= -n \cdot (f(\varepsilon_1, \varepsilon_2)(T,Y_1), g(\varepsilon_1, \varepsilon_2)(T,Y_1))
\]

\[
> 0
\]

because of Lemma 5c.1. Hence, \( (T(z),Y_1(z)) \) is outwardly tangent to \( \partial \mathcal{D}_1 \) at \( z_0 \). This implies the desired result.
It remains to consider the cases \(|q(z_0)| = L\) and \(|p(z_0)| = L\). We assume that \(q(z_0) = L\) for some \(z_0\). The other cases are similar. For convenience we assume that \(z_0 = 0\). Choose \(K\) such that

\[|f(\varepsilon_1, \varepsilon_2)(T, Y_1)| + |g(\varepsilon_1, \varepsilon_2)(T, Y_1)| < K\text{ in } D.\]

Then,

\[q' = \theta q - f(\varepsilon_1, \varepsilon_2)(T, Y_1) > -(\theta_0 + 1)q - K\]

if \(q(z) > 0\). Hence,

\[(e^{(\theta_0+1)t}q)' > -Ke^{(\theta_0+1)t} q.\]

Integrate this equation from 0 to \(t\) to obtain

\[q(t) > e^{-(\theta_1+1)t}v - K/\theta_0 + 1 [1 - e^{(\theta_0+1)t}]

> e^{-(\theta_0+1)t}v - K/(\theta_0 + 1)\]

> \(V/2\)

if \(0 \leq t \leq 1\) and \(V\) is sufficiently large. Therefore, if \(0 \leq t \leq 1\), then

\[T' = q > \frac{V}{2} .\]

This implies that

\[T(q) > T(0) + \frac{V}{2} > Q_1 + 2Q_2 + \delta\]
for \( V \) sufficiently large. In particular,

\[(T(1), Y_1(1)) \notin \mathcal{D}\]

which implies that \( \gamma(1) \notin \mathcal{D}_1 \).

For the remainder of the paper we assume that \( V \) is chosen so that the Lemma holds.

**D. A Morse Decomposition**

We begin by stating the definition of a Morse decomposition. A detailed discussion of Morse decompositions can be found in [3], [6], and [10].

**Definition 5D.1 (Morse Decomposition):** Assume that \( S \) is a compact, invariant subset of phase space. A Morse decomposition of \( S \) is a finite collection \( \{M_i\}_{i \in \mathbb{P}} \) of subsets \( M_i \subset S \) which are disjoint, compact and invariant, and can be ordered \( \{M_1, M_2, \ldots, M_n\} \) so that for every \( \gamma \in S \setminus \bigcup_{i \neq j} M_j \), there are indices \( i < j \) such that

\[ \omega(\gamma) \subset M_i \quad \text{and} \quad \omega^*(\gamma) \in M_j. \]

**Remark:** By \( \omega(\gamma) \) and \( \omega^*(\gamma) \) we mean the \( \omega \)-limit and \( \omega^* \)-limit sets of \( \gamma \), respectively.

Let \( \theta_0 \) be as in (4.5) and fix \( \varepsilon, 0 < \varepsilon << 1 \). Attach to (5B.5) the equation

\[ \theta' = -\varepsilon (\theta_0^2 - \theta^2) \quad \text{(5D.1)} \]
The reason for doing this was motivated in Section 2.

Remark: In Section 2 we had $0 < \theta_1 < \theta < \theta_0$. We shall now consider $|\theta| < \theta_0$. The reason for doing this is that it will be easier to compute the Conley index in the set $|\theta| < \theta_0$. Eventually, we will continue the case $|\theta| < \theta_0$ to the case $\theta_1 < \theta \leq \theta_0$. The reason that we need to consider $0 < \theta_1 < \theta$ is because of the a priori bounds which were derived in Section 3. For those bounds, we required that $\theta_1 < \theta \leq \theta_0$.

Let

$$N = \{(\gamma, \theta) : \gamma \in D_1, |\theta| < \theta_0 + 1\}.$$

There are six critical points of (5B.5), (5D.1). These are

$$\alpha_1 = (0, 0, 1, 0, \theta_0), \quad \alpha_2 = (0, 0, 1, 0, -\theta_0),$$

$$\beta_1 = (T_1, 0, 1/Q_1(T_1 - Q_1), 0, \theta_0), \quad \beta_2 = (T_1, 0, 1/Q_1(T_1 - Q_1), 0, -\theta_0),$$

$$\gamma_1 = (Q_1 + 2Q_2, 0, 0, 0, \theta_0), \quad \gamma_2 = (Q_1 + 2Q_2, 0, 0, 0, -\theta_0).$$

Our immediate goal is to prove that for each $\epsilon > 0$ there exists a solution of (5B.5), (5D.1) which satisfies

$$\lim_{z \to -\infty} (\gamma(z), \theta(z)) = \alpha_1 \quad \text{and} \quad \lim_{z \to -\infty} (\gamma(z), \theta(z)) = \gamma_2.$$
The first step will be to define a Morse decomposition for the maximal invariant set in $N$. We begin by defining a certain neighborhood of

$$\alpha_0 = (0,0,1,0).$$

Fix $\lambda$, $0 < \lambda \ll 1$, and $m > 0$. Let $V$ be as in Lemma 5c.2 and

$$G_1 = \{(T,q) - \lambda \leq T \leq \frac{1}{m} q + \lambda; \ 0 \leq q \leq V\} \cup \{(T,q) : \frac{1}{m} q - \lambda \leq T \leq \lambda, -V \leq q \leq 0\},$$

$$G_2 = \{(Y_1,p_1) : 1 - \lambda \leq Y_1 \leq \frac{1}{m} p_1 + 1 + \lambda, 0 \leq p_1 \leq V\} \cup \{(Y_1,p_1) : \frac{1}{m} p_1 + 1 - \lambda \leq Y_1 \leq 1 + \lambda, -V \leq p_1 \leq 0\}.$$

$$G_{\alpha} = \{(T,q,Y_1,p_1) : (T,q) \in G_1 \text{ or } (Y_1,p_1) \in G_2\}.$$

**Lemma 5D.2:** $\lambda$ and $m$ can be chosen so that if $|\theta - \theta_0| \leq 1$, then trajectories can only leave $G_{\alpha}$ through the sides $|q| = V$ or $|p_1| = V$.

**Proof:** We only consider $G_1$. The proof for $G_2$ is similar. We prove that on $\partial G_1 \cap \{(T,q) : |q| < V\}$, the vector field defined by the right hand side of (5B.5) points into $G_1$, if $\lambda$ and $m$ are chosen appropriately and $|\theta - \theta_0| \leq 1$. We treat each side of $\partial G_1$ separately.

a) Assume that $T = -\lambda$ and $q > 0$. Then $T' = q > 0$.

b) Assume that $T = \lambda$ and $q < 0$. Then $T' = q < 0$.

c) Assume that $T = 1/m q + \lambda$ and $q > 0$. Then the outward normal to $\partial G_1$ is $n = (1,-1/m)$, and
\[ n \cdot (T', q') = q - \frac{1}{m} \theta q + \frac{1}{m} f(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[ < (1 - \frac{\theta}{m})q \]

because if \( m \) is sufficiently large and \( \lambda \) small then \( T < T_1 \) in \( G_1 \).

Hence, \( f(\varepsilon_1, \varepsilon_2)(T, Y_1) < 0 \). Therefore,

\[ n \cdot (T', q') < (1 - \theta/m)q \]

\[ < 0 \]

if \( m < \theta \). Since \( |\theta - \theta_0| < 1 \) we assume that \( \theta_0 > 2 \) and \( m = \theta_0 - 2 \).

\( d) \) Assume that \( T = 1/m q - \lambda \) and \( q < 0 \). Then the outward normal to \( \partial G_1 \) is \((-1, 1/m)\), and

\[ n \cdot (T', q') = -q + \frac{1}{m} \theta q - \frac{1}{m} f(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[ < -(1 - 1/m \theta)q \]

since \( T < 0 \). Hence,

\[ n \cdot (T', q') < 0 \]

if, as before, \( m = \theta_0 - 2 \).

We assume throughout that \( \lambda \) and \( m \) are chosen so that Lemma 5D.2 is valid.
Let

\[ G_\alpha = \{(T, q, Y_1, p_1, \theta) : (T, q, Y_1, p_1) \in G', |\theta - \theta_0| \leq 1\} .\]

Note that \( \alpha_1 \in G_\alpha \).

In a similar fashion we construct a neighborhood of \( \gamma_2 \). We briefly sketch the construction. Let

\[ \gamma_0 = (Q_1 + 2Q_2, 0, 0, 0) , \]

\[ H_1 = \{(T, q) : Q_1 + 2Q_2 - \lambda \leq T \leq \frac{1}{m} q + Q_1 + 2Q_2 + \lambda , 0 \leq q \leq V\} \]

\[ \cup \{(T, q) : \frac{1}{m} q + Q_1 + 2Q_2 - \lambda \leq T \leq Q_1 + 2Q_2 + \lambda , -V \leq q \leq 0\} , \]

\[ H_2 = \{(Y_1, p_1) : -\lambda \leq Y_1 \leq \frac{1}{m} p_1 + \lambda , 0 \leq q \leq V\} \]

\[ \cup \{(T, q) : \frac{1}{m} p_1 - \lambda \leq Y_1 \leq \lambda , -V \leq p_1 \leq 0\} , \]

\[ H'_\gamma = \{(T, q, Y_1, p_1) : (T, q) \in H_1 \text{ and } (Y_1, p_1) \in H_2\} , \]

and

\[ H_\gamma = \{(T, q, Y_1, p_1, \theta) : (T, q, Y_1, p_1) \in H'_\gamma , |\theta + \theta_0| \leq 1\} . \]

As in Lemma 5D.2 we have

**Lemma 5D.3:** If \(|\theta + \theta_0| < 1\), then a solution of (5B.5) can only enter \( H'_\gamma \) through the set \( \{(T, q, Y_1, p_1) : |q| = V \text{ or } |p_1| = V\} \).
We are now ready to define the Morse decomposition. Let

\[ N_1 = \text{cl} \{(\gamma, \theta) : \gamma \in S_1 \setminus G'_a, |\theta - \theta_o| \leq 1\}, \]

\[ N_2 = G_a \cup \{(\gamma, \theta) : \gamma \in S_1, |\theta| \leq \theta_o - 1\} \cup H_\gamma, \]

\[ N_3 = \text{cl} \{(\gamma, \theta) : \gamma \in S_1 \setminus H_\gamma, |\theta + \theta_o| \leq 1\}. \]

By \( \text{cl} x \) we mean the topological closure of the set \( x \). Let \( M_0 \) equal to the maximal invariant set in \( N \) and \( M_i, i = 1, 2, 3 \), equal to the maximal invariant set in \( N_i \). The picture we have in mind is

![Diagram of Morse decomposition](image)

**Figure 6**

We claim that \((M_3, M_2, M_1)\) defines a Morse decomposition of \( M_0 \).

**Lemma 5D.5:** \( N \) is an isolating neighborhood.
Proof: Suppose that \((\gamma(z_0),\theta(z_0)) \in \partial N\). If \(\theta(z_0) = \theta_0 + 1\), then \(\theta'(z_0) = 0\) so that \((\gamma(z),\theta(z))\) leaves \(N\) in forwards time. If \(\theta'(z_0) = -\theta_0 - 1\), then \(\theta'(z_0) > 0\) so \((\gamma(z),\theta(z))\) leaves \(N\) in backwards time. Now suppose that \(\gamma(z) \in \partial J\). Since \(J\) is an isolating neighborhood for each \(\theta, |\theta| < \theta_0 + 1\) (Lemma 5c.2) it follows that \(\gamma(z)\) must leave \(J\) in forwards or backwards time. Hence, \((\gamma(z),\theta(z))\) must leave \(N\) in forwards or backwards time.

Lemma 5D.6: Each \(N_i\) is an isolating neighborhood.

Proof: We prove the Lemma for \(N_1\). The proofs for \(N_2\) and \(N_3\) are similar. Suppose that \((\gamma(z_0),\theta(z_0)) \in \partial N_1\). There are various possibilities. If \(\theta = \theta_0 + 1\), then \(\theta' > 0\) so \((\gamma(z),\theta(z))\) leaves \(N_1\) in forwards time. If \(\theta = \theta_0 - 1\), then \(\theta' < 0\) so, again, \((\gamma(z),\theta(z))\) leaves \(N_1\) in forwards time. If \((\gamma(z),\theta(z)) \in \partial N\), then, because \(N\) is an isolating neighborhood, \((\gamma(z),\theta(z))\) must leave \(N\), and therefore \(N_1\), in forwards or backwards time. Finally, suppose that \((\gamma(z_0),\theta(z_0)) \in \partial G_\alpha\). From Lemma 5D.2, \((\gamma(z),\theta(z))\) must enter \(N_2\), and hence leave \(N_1\), in forwards time.

Proposition 5D.7: \((M_1,M_2,M_3)\) defines a Morse decomposition for \(M_0\).

Proof: Note that trajectories which begin in \(N_2\) cannot enter \(N_1\), and trajectories which begin in \(N_3\) cannot enter \(N_2\). This is because the vector field on \(\partial N_1 \cap \partial N_2\) points into \(N_2\), while the vector field on \(\partial N_2 \cap \partial N_3\) points into \(N_3\).

Choose \((\vec{\gamma},\vec{\theta}) \in M_0 \setminus (M_1 \cup M_2 \cup M_3)\) and let \((\gamma(z),\theta(z))\) be the trajectory which satisfies \((\gamma(0),\theta(0)) = (\vec{\gamma},\vec{\theta})\). First suppose that \((\vec{\gamma},\vec{\theta}) \in M_1\). The above comments imply that \((\gamma(z),\theta(z)) \notin N_2 \cup N_3\) for all \(z < 0\). Hence,
\[ \omega^*(\gamma(z), \theta(z)) \in M_1. \] Certainly, \( \omega(\gamma(z), \theta(z)) \notin M_1 \) since this would imply that \( (\overline{\gamma}, \overline{\theta}) \in M_1 \) which we are assuming is not true. Hence, \( \omega(\gamma(z), \theta(z)) \in M_2 \cup M_3 \).

A similar argument shows that if \( (\overline{\gamma}, \overline{\theta}) \in N_2 \), then \( \omega^*(\gamma(z), \theta(z)) \in M_1 \cup M_2 \) and \( \omega(\gamma(z), \theta(z)) \in M_2 \cup M_3 \). Similarly, if \( (\overline{\gamma}, \overline{\theta}) \in N_3 \), then \( \omega^*(\gamma(z), \theta(z)) \in M_1 \cup M_2 \) and \( \omega(\gamma(z), \theta(z)) \in M_3 \). Hence, \((M_3, M_2, M_1)\) defines a Morse decomposition.

E. Computation of the Indices

We prove that \( h(M_0) = h(M_1) = h(M_2) = h(M_3) = \overline{0} \). Here, \( h \) is the Conley (homotopy) index and \( \overline{0} \) is the Conley index of the empty set.

**Lemma 5E.1:** \( h(M_0) = \overline{0} \).

**Proof:** We continue the flow defined by (5B.5), (5D.1) to one in which the maximal invariant set in \( N_0 \) is the empty set.

Consider the family of flows, parametrized by \( r \),

\[ T' = q \]

\[ q' = \theta q - f(\varepsilon_1, \varepsilon_2)(T, Y_1) \]

\[ Y_1' = p_1 \]

\[ p_1' = \theta p_1 - g(\varepsilon_1, \varepsilon_2)(T, Y_1) \]  \hspace{1cm} (5E.1r)

\[ \theta' = -\varepsilon(\theta^2_0 - \theta^2) + r. \]

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When \( r = 0 \), (5E.1) reduces to (5B.5), (5D.1). As in Lemma (5D.5) one easily checks that \( N \) is an isolating neighborhood for each \( r \). If \( r_0 \) is sufficiently large, then

\[
\theta' = -\varepsilon (\theta^2_0 - \theta^2) + r_0 > 0
\]

for all \( \theta \). Hence, (5E.1r_0) cannot have any bounded solutions. If \( I(N(r)) \) is the maximal invariant set in \( N \) for the flow (5E.1r), then we have shown that \( I(N(r_0)) = \emptyset \) if \( r_0 \) is sufficiently large. Hence, \( h(I(N(r_0))) = \emptyset \). Since \( I(N(0)) \) and \( I(N(Ir_0)) \) are related by continuation (see [3]) it follows that \( h(I(N(0))) = \emptyset \) which is the desired result.

**Lemma 5B.2:** \( h(M_1) = \emptyset \).

**Proof:** Clearly \( M_1 \) lies in the set \( \{(\gamma, \theta) : \theta = \theta_0\} \). Let \( h_1(M_1) \) equal to the index of \( M_1 \) considered as an isolated invariant set of the flow defined by (5B.5) in the set \( \{(\gamma, \theta) : \theta = \theta_0\} \). We prove that \( h_1(M_1) = \emptyset \). This immediately implies that \( h(M_1) = \emptyset \).

The first step is to note that for \( \theta_0 \) large, the bounded solutions of (5B.5) have projections onto \( (T,Y_1) \) space lying close to the bounded solutions of the reaction system

\[
\begin{align*}
\dot{T} &= f(\varepsilon_1, \varepsilon_2) (T, Y_1) \\
\dot{Y}_1 &= g(\varepsilon_1, \varepsilon_2) (T, Y_1)
\end{align*}
\]

(5B.2)

This is proved in Conley and Gardner [4]. Moreover, their proof shows that if \( P \) is equal to the projection of \( N_1 \) onto \( (T,Y_1) \) space and \( M \) is the maximal invariant set in \( P \), then

\[
h_1(M) = h(M) \wedge \Sigma^1,
\]

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the smash product of the index of $M$, as an invariant set for (5E.2), and $\Sigma^1$, the pointed one sphere. Because $\overline{0} = \overline{0} \wedge \Sigma^1$ it suffices to prove that $h_1(M) = \overline{0}$.

Now $P$ is equal to $\mathcal{B}$, which was defined in Section 5C, minus a small neighborhood of $a$. Note that $a$ is an attractor for the equations (5E.2). Hence we can find a compact neighborhood $H$ of $a$ such that $H \subset \mathcal{B}$, $M \cap H = \tau$, and on $\partial H$ the vector field defined by the right hand side of (5E.2) points into $H$.

It follows that $\partial \mathcal{B} \backslash H$ is an isolating neighborhood, and the maximal invariant set in $\partial \mathcal{B} \backslash H$ is $M$. Note that trajectories enter $\partial \mathcal{B} \backslash H$ along its outward boundary (Lemma 5C.1), and enter $\partial \mathcal{B} \backslash H$ along $\partial H$. This is shown in Figure 7. It is easy to imagine a vector field which enters $\partial \mathcal{B} \backslash H$ along its outward boundary, leaves $\partial \mathcal{B} \backslash H$ along $\partial H$, and has no invariant sets. It then follows that $h(P) = \overline{0}$. Of course, one could also compute $h(P)$ directly, using the isolating neighborhood $\partial \mathcal{B} \backslash H$.
**Lemma 5E.3:** $h(M_3) = 0$.

**Proof:** The proof of this result is very similar to the one just given so we do not give the details.

In the remainder of this section we prove

**Lemma 5E.4:** $h(M_2) = 0$. We begin by presenting some aspects of Conley's Morse theory.

Let $S$ be a compact invariant set in phase space.

**Definition 5E.5:** (attractor, repeller). A subset $A \subset S$ is called an attractor (relative to $S$) if there is a neighborhood $U$ of $A$ in $S'$ such that $\omega(U) = A$. Similarly, a set which is the $\omega^*$-limit set of a neighborhood of itself is called a repeller.

Given an attractor $A$, let $A^*$ be the set of $\gamma \in S$ such that $\omega(\gamma) \cap A = \emptyset$. Then $(A, A^*)$ is a Morse decomposition of $S$. We refer to $(A, A^*)$ as an attractor-repeller pair for $S$.

The following result is proved in [6]

**Proposition 5E.6:** Suppose that $(A, A^*)$ is an index pair for $S$. Then there exists an increasing sequence of compact sets

$$S_1 \subset S_2 \subset S_3$$

such that

a) $(S_3, S_1)$ is an index pair for $S$

b) $(S_3, S_2)$ is an index pair for $A^*$

c) $(S_2, S_3)$ is an index pair for $A$.  

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Remark: For the definition of index pair see [3]. We point out that if \( S \) is an isolated invariant set an \((S_1, S_2)\) is an index pair for \( S \) then

\[
h(S) = [S_1, S_2],
\]

the homotopy type of a pointed topological space

\[
S_1/S_2 = (S_1 \setminus S_2) \cup [S_2], [S_2]) .
\]

Let \( H(S_i, S_j), i > j, \) be the cohomology with coefficients in some fixed ring for the compact pair \((S_i, S_j)\).

Remark: We use cohomology because we are following Conley and Zender [6].

Then (5E.4) induces the exact sequence

\[
0 \rightarrow H^0(S_2, S_1) \rightarrow H^0(S_3, S_1) \rightarrow H^0(S_3, S_2) \xrightarrow{\delta^0} H^1(S_2, S_1) \rightarrow \\
\rightarrow H^1(S_3, S_2) \rightarrow H^1(S_3, S_2) \xrightarrow{\delta^1} H^2(S_2, S_1) \rightarrow \cdots
\]

or, from (5E.5),

\[
0 \rightarrow H^0(h(A^*)) \rightarrow h^0(h(S)) \rightarrow H^0(h(A)) \xrightarrow{\delta^0} H^1(h(A^*)) \\
\rightarrow H^0(h(S)) \rightarrow H^1(h(A)) \xrightarrow{\delta^1} H^2(h(A^*)) \rightarrow \cdots
\]  

(5E.5)

An immediate consequence of (5E.5) is:

**Proposition 5E.7:** a) If \( h(S) = h(A^*) = 0 \), then \( h(A) = 0 \).

b) If \( h(S) = h(A) = 0 \), then \( h(A^*) = 0 \).

We now return to the proof of Lemma 5E.4. Let \( N_{12} = N_1 \cup N_2 \) and \( M_{12} \) equal to the maximal invariant set in \( N_{12} \). Then \((M_3, M_{12})\) defines an
attractor-repellor pair for \( M_0 \). From Lemmas 5E.1, 5E.3 and Proposition 5E.7a, we conclude that \( h(M_1) = \bar{0} \).

Now \((M_1,M_2)\) defines an attractor repellor pair for \( M_{1,2} \). From Lemma 5E.2 and Proposition 5E.7b, we conclude that \( h(M_2) = \bar{0} \).

F. Existence of a Connecting Orbit for \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \)

In this section we prove

**Proposition 5F.1**: There exists a solution \( \mathbf{t}(z) \) of (5B.5), (5D.1) which satisfies \( \lim_{z \to +} \mathbf{t}(z) = \alpha_1 \) and \( \lim_{z \to -} \mathbf{t}(z) = \gamma_2 \).

We begin by presenting some more aspects of Conley's Morse theory.

The index of an isolated invariant set is a pointed topological space. It is possible to take the sum, \( \nu \), of any two pointed spaces \((A,a)\) and \((B,b)\). It is defined as \((A \cup B)/(a,b)\) or, in otherwords, it is the pointed space obtained by taking the union of \( A \) and \( B \) and identifying the two distinguished points \( a \) and \( b \). The sum is denoted by \((A,a) \nu (B,b)\). It is proved in [3]

**Proposition 5F.1**: If \( S_1 \) and \( S_2 \) are isolated invariant sets and \( S_1 \cap S_2 = \emptyset \), then \( S_1 \cup S_2 \) is an isolated invariant set and

\[ h(S_1 \cup S_2) = h(S_1) \nu h(S_2). \]

It is also proved in [3],

**Proposition 5F.2**: If \( h(S_1) \nu h(S_2) = \bar{0} \), then \( h(S_1) = h(S_2) = \bar{0} \).

Together with Proposition 5F.1 this implies
**Corollary 5F.3:** Suppose that $N$ is an isolating neighborhood and $S_1$ and $S_2$ are isolated invariant sets in $N$ such that either $h(S_1) \neq \bar{0}$ or $h(S_2) \neq \bar{0}$. Let $I(N)$ equal to the maximal invariant set in $N$. If $h(I(N)) = \bar{0}$, then $I(N) \neq S_1 \cup S_2$.

We now return to the proof of Proposition 5F.1. Note that $\alpha_1$ and $\gamma_2$ are isolated critical points in $N_2$. By linearizing (5B.5), (5D.1) at $\alpha_1$ and $\gamma_2$, we find that the linearized equations have three negative eigenvalues at $\alpha_1$ and two negative eigenvalues at $\gamma_2$. Hence,

$$h(\alpha_1) = \Sigma^3 \text{ and } h(\gamma_2) = \Sigma^2$$

where $\Sigma^n$ is the pointed $n$-sphere. Since $h(M_2) = \bar{0}$ we conclude from Corollary 5F.3 that

$$M_2 \neq \alpha_1 \cup \gamma_2 \quad (5F.1)$$

To complete the proof of Proposition 5F.1 we show that on any bounded nonconstant orbit in $N_2$, $\theta' < 0$. This implies that the only bounded orbits in $N_2$ (or $M_2$) are critical points or trajectories which connect two critical points. Since $\alpha_1$ and $\gamma_2$ are the only critical points in $N_2$, Proposition 5F.1 now follows from (5F.1). So, it remains to prove

**Lemma 5F.4:** Suppose that $\psi(z) = (\gamma(z), \theta(z))$ is a bounded, nonconstant solution which lies entirely in $N_2$. Then $\theta'(z) < 0$ for all $z$.

**Proof:** If $\theta'(z_0) = 0$ for some $z_0$, then either $\theta(z_0) \in P_1 = \{(\gamma, \theta) : \theta = 0\}$ or $\theta(z_0) \in P_2 = \{(\gamma, \theta) : \theta = -\theta_0\}$. Both of these sets are invariant, however. From the construction it is clear that the only bounded solutions in $N_2 \cap P_1$ or $N_2 \cap P_2$ are the critical points $\alpha_1$ or $\gamma_2$. Since we are assuming that $\psi(z)$ in nonconstant, this is impossible.
Suppose that $\theta'(z_0) > 0$ for some $z_0$. Then either $\theta(z_0) > \theta_0$ or $\theta(z_0) < -\theta_0$. It is impossible for $\theta(z_0) > \theta_0$ because $\theta'(z) > 0$ for all $z > z_0$. Hence, $\theta(z)$ must become unbounded in forward time. Similarly, it is impossible for $\theta(z_0) < -\theta_0$. Since $\theta'(z) > 0$ for all $z < -\theta_0$, it would follow that $\theta(z)$ must become unbounded in negative time.

G. The Flow Near $\alpha_1$

It is necessary to change the flow near $\alpha_1$. The reason is that when $\varepsilon_1 = \varepsilon_2 = 0$, (5B.5) has a continuum of critical points in every neighborhood of $\alpha_1$. If $S(\varepsilon_1,\varepsilon_2)$ is the set of solutions of (5B.5), (5D.1), then $S(0,0)$ will not be an isolated invariant set. Hence, the Conley index of $S(0,0)$ will not be defined. We change the flow near $\alpha_1$ so that $\alpha_1$ is an isolated critical point for all nonnegative $\varepsilon_1$ and $\varepsilon_2$. Then $S(\varepsilon_1,\varepsilon_2)$ will remain an isolated invariant set as $\varepsilon_1$ and $\varepsilon_2$ approach zero. The new flow will be defined in such a way that it agrees with the old one in a neighborhood of the unstable manifold at $\alpha_1$. This will imply that for each $\varepsilon_1,\varepsilon_2$, $S(\varepsilon_1,\varepsilon_2)$ for the new flow agrees with $S(\varepsilon_1,\varepsilon_2)$ for the old flow. Hence, once we prove the existence of a connecting orbit for the new flow, we will automatically have a connecting orbit for the old one. Moreover, since $h(S(\varepsilon_1,\varepsilon_2)) = \overline{0}$ for each $\varepsilon_1,\varepsilon_2 > 0$ (as was shown in the last section), it will follow that for the new flow $h(S(0,0)) = \overline{0}$.

We now define the neighborhood of $\alpha_1$ in which that flow will be changed. Let $\lambda$ be as in Lemma 5D.2. Let

$$P_1 = \{(T,q,Y_1,p_1,\theta) : |q| \leq \frac{1}{4}(T - |\theta - \theta_0|), |\theta - \theta_0| \leq T - \frac{1}{4}\lambda\}$$

and

$$\{(T,q,Y_1,p_1,\theta) : |q| \leq \frac{1}{4}(T + |\theta - \theta_0|), -\frac{1}{4}\lambda \leq T \leq |\theta - \theta_0|\},$$
\[ P_2 = \{ (T, q, Y, P_1, \Theta) : |p_1| = \frac{1}{4}(Y_1 - 1 - |\Theta - \Theta_o|), |\Theta - \Theta_o| \leq Y_1 - 1 \leq \frac{1}{4} \lambda \} \]
\[ \cup \{ (T, q, Y, P_1, \Theta) : |p_1| = -\frac{1}{4}(Y_1 - 1 + |\Theta - \Theta_o|), -\frac{1}{4} \lambda \leq Y_1 - 1 \leq -|\Theta - \Theta_o| \} \]
\[ Q_1 = \{ (T, q, Y, P_1, \Theta) : |q| = \frac{1}{2}(T - |\Theta - \Theta_o|), |\Theta - \Theta_o| \leq T \leq \frac{1}{2} \lambda \} \]
\[ \cup \{ (T, q, Y, P_1, \Theta) : |q| = -\frac{1}{2}(T + |\Theta - \Theta_o|), -\frac{1}{2} \lambda \leq T \leq -|\Theta - \Theta_o| \} \]
\[ Q_2 = \{ (T, q, Y, P_1, \Theta) : |p_1| = \frac{1}{2}(Y_1 - 1 - |\Theta - \Theta_o|), |\Theta - \Theta_o| \leq Y_1 - 1 \leq \frac{1}{2} \lambda \} \]
\[ \cup \{ (T, q, Y, P_1, \Theta) : |p_1| = -\frac{1}{2}(Y_1 - 1 + |\Theta - \Theta_o|), -\frac{1}{2} \lambda \leq Y_1 - 1 \leq -|\Theta - \Theta_o| \} \]

\[ \mathcal{N}_1 = P_1 \cap P_2 \]
\[ \mathcal{N}_2 = Q_1 \cap Q_2 \]

The picture we have in mind is

![Diagram](image)

**Figure 8**
**Remark 1:** Note that $N_1 \subset N_2 \subset G_\alpha$, which was defined in Section 5D. Hence, $N_1 \subset N_2 \subset \text{int } N_2$.

**Remark 2:** By linearizing (5B.5), (5D.1) at $\alpha_1$, and computing the eigenvalues and eigenvectors, we find that if $W^s_\alpha$ and $W^u_\alpha$ are, respectively, the stable and unstable manifolds at $\alpha_1$, then near $\alpha_1$, $N_2$ is a neighborhood of $W^s_\alpha$ and $W^u_\alpha \cap N_2 = \emptyset$.

Let $\psi(T,q,Y_1,p_1,\theta)$ be any function which satisfies

a) $\psi = 1$ in $N_1$

b) $\psi = 0$ in the complement of $N_2$

c) $\psi(\alpha_1) = 0$ \hspace{1cm} (5G.1)

d) $0 \leq \psi \leq 1$ for all $(T,q,Y_1,p_1,\theta)$

e) $\psi$ is continuous except at $\alpha_1$.

Let $Y(\varepsilon_1,\varepsilon_2)$ be the vector field defined by the right hand side of (5B.5) and (5D.1). Choose $\varepsilon_0 > 0$ so that the constructions in the preceding sections are valid for $\varepsilon_1,\varepsilon_2 < \varepsilon_0$. For $\varepsilon_1,\varepsilon_2 < \varepsilon_0$, consider the new vector field

$$\chi(\varepsilon_1,\varepsilon_2) =\psi Y(\varepsilon_0,\varepsilon_0) + (1-\psi) Y(\varepsilon_1,\varepsilon_2).$$

(5G.2)

**Lemma 5G.1:** $\chi(\varepsilon_1,\varepsilon_2)$ is continuous for all $(T,q,Y_1,p_1,\theta)$.

**Proof:** The only possible problem is at $\alpha_1$. However, $Y(\varepsilon_1,\varepsilon_2)$ and $Y(\varepsilon_0,\varepsilon_0)$ are both zero at $\alpha_1$. Hence, given $\delta$ there exists $\eta$ such that if $\| (T,q,Y_1,p_1,\theta) - \alpha_1 \| < \eta$, then $\| Y(\varepsilon_1,\varepsilon_2) \| < \delta$ and $\| Y(\varepsilon_0,\varepsilon_0) \| < \delta$. From (5G.2) it follows that if $\| (T,q,Y_1,p_1,\theta) - \alpha_1 \| < \eta$, then $\| \chi(\varepsilon_1,\varepsilon_2) \| < \delta$, and the proof is complete.

Let $S_0$ be the set of solutions of (5B.5), (5D.1) which satisfy

$$\lim \psi(z) = \alpha_1 \quad \text{and} \quad \lim \psi(z) = \gamma_2$$

(5G.3)
Let $S_1$ equal to the set of solutions of

$$
\psi'(z) = \chi(\varepsilon_1, \varepsilon_2)(z)
$$

(5G.4)

which satisfy 5G.3. From Remark 2 of this section we conclude that

**Lemma 5G.2:** For each $\varepsilon_1, \varepsilon_2$ sufficiently small, $S_0 = S_1$.

To conclude this section we point out that because $N_2 \subset \text{Int} N_2$, the index arguments given in the preceding section remain uneffected by the change of flow from (5B.5), (5D.1) to (5G.4). This is because the index of the maximal invariant set in $N_2$ only depends on the vector field on the boundary of $N_2$. In what follows we only consider solutions of (5G.4). Because of Lemma 5G.2, once we prove the existence of a solution of (5G.4), (5G.3) we automatically have a solution of (5B.5), (5D.1), (5G.3).

**H. The Limits $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$**

We have now proved the existence of a solution $\psi(z)$ of (5G.3), (5G.4) for all sufficiently small, positive $\varepsilon$, $\varepsilon_1$, and $\varepsilon_2$. To emphasize the dependence of $\psi(z)$ on $\varepsilon$, $\varepsilon_1$, and $\varepsilon_2$ we write $\psi(\varepsilon, \varepsilon_1, \varepsilon_2)(z)$. In this section we let $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, thus proving the existence of a solution of (5A.4), (5D.1), and (5G.3). Making the substitutions (5A.3) this will prove the existence of functions $(T, Y_1, Y_2, \theta)(z)$ which satisfy (5A.1), (5D.1),

$$
\lim_{z \to \infty} (T, Y_1, Y_2, \theta)(z) = (0, 1, 1, \theta_0), \quad \text{and}
$$

$$
\lim_{z \to \infty} (T, Y_1, Y_2, \theta)(z) = (Q_1 + 2Q_2, 0, 0, -\theta_0).
$$

First we consider $\varepsilon_2$. Let $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$ be sufficiently small and let

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\( \dot{\psi}^k(z) = \dot{\psi}(z, \varepsilon, 1/k \varepsilon) (z) \).

We assume that

\( \dot{\psi}^k(z) = (\gamma^k(z), \theta^k(z)) = (T^k(z), q^k(z), \gamma_1^k(z), p_1^k(z), \theta^k(z)) \)

and choose the translation so that

\[ \theta^k(0) = 0 \text{ for each } k. \quad (5H.1) \]

Since \( \dot{\psi}^k(z) \in N \), a compact set, for each \( k \) and \( z \), it follows that there exists a subsequence \( \{ \lambda_k \} \) such that \( \lambda_k \to * \) as \( k \to * \) and \( \dot{\psi}^{\lambda_k}(0) \) converges to say \( \dot{\psi}_0 \) as \( k \to * \). For convenience we assume that \( \lambda_k = k \) for each \( k \).

Let \( \dot{\psi}(z) \) be the solution of (5G.4) with \( \varepsilon = 0 \) which satisfies \( \dot{\psi}(0) = \dot{\psi}_0 \). Clearly \( \theta(0) = 0 \). Moreover, from the continuous dependence of solutions on initial data we conclude

**Lemma 5H.1:** Given \( \delta, T \) there exists \( M \) such that if \( k > M \), then

\[ |\dot{\psi}^k(z) - \dot{\psi}(z)| < \delta \text{ for } |z| < T. \]

Consider the \( \omega \)-limit and \( \omega^* \)-limit sets of \( \dot{\psi}(0) \). Because \( \theta^*(z) < 0 \) for \( |\theta| < \theta_0 \) it follows that

\[ \omega(\dot{\psi}(0)) \in \{(\gamma, \theta) : \theta = -\theta_0 \} \cap N_2 = D_1 \]

and

\[ \omega^*(\dot{\psi}(0)) \in \{(\gamma, \theta) : \theta = +\theta_0 \} \cap N_2 = D_2 \]

Because the only invariant sets in \( D_1 \) and \( D_2 \) are critical points, and \( \alpha_1 \) and \( \gamma_2 \) are isolated critical points it follows that
\[ \omega^z(\theta(0)) = \alpha_1 \quad \text{and} \quad \omega(\theta(0)) = \gamma_3 \quad (5H.2) \]

We now have a solution of (5G.3), (5G.4) with \( \epsilon_2 = 0 \). An analysis similar to that just given shows that there is no problem in letting \( \epsilon_1 \to 0 \).

**I. Continuation from \( |\theta| \leq \theta_o \) to \( 0 < \theta, \theta \leq \theta_o \)**

Let \( \theta_1 \) be as in (4.4) and, for \( 0 \leq r \leq 1 \),

\[ \theta_r = r\theta_1 + (1 - r)\theta_o. \]

Let \( \theta(z) = (\gamma(z), \theta(z)) \) and rewrite (5G.4) in the form

\[ \gamma'(z) = \Lambda(z) \]
\[ \theta'(z) = -\epsilon(\theta^2 - \theta_o^2). \]

In this section we consider the one parameter family of equations

\[ \gamma'(z) = \Lambda(z) \]
\[ \theta'(z) = -\epsilon(\theta_o - \theta)(\theta - \theta_r). \quad (5I.1r) \]

We prove that for each \( r \) there exists a solution of (5I.1r) which satisfies the boundary conditions

\[ (T, q, Y_1, p_1, \theta)(-\omega) = (0, 0, 1, 0, \theta_o) = \alpha_1 \]
\[ (T, q, Y_1, p_1, \theta)(+\omega) = (Q_1 + 2Q_2, 0, 0, 0, \theta_r) = \gamma_r. \quad (5I.2r) \]
The reason for continuing solutions of (51.1r), (51.2r) from \( r = 0 \) to \( r = 1 \) is that the apriori bounds derived in Sections 3 and 4 are only for \( \theta_1 \leq \theta \leq \theta_0 \).

In what follows we assume that \( \mathbf{t}_r(y)(z) \) is the solution of (51.1r) which satisfies \( \mathbf{t}_r(y)(0) = y \). For \( 0 \leq r \leq 1 \), let

\[
S'_r = \{ y : \lim_{z \to +} \mathbf{t}_r(y)(z) = \alpha \quad \text{and} \quad \lim_{z \to -} \mathbf{t}_r(y)(z) = \gamma \},
\]

\[
S_r = S'_r \cup \{ \alpha \} \cup \{ \gamma \}.
\]

Let

\[
I = \{ r \in [0,1] : S_r \text{ is an isolated invariant set and } h(S_r) = \bar{0} \}.
\]

**Remark 1:** To prove that \( S_r \) is an isolated invariant set, we must show that \( S_\lambda \), the maximal invariant set is some compact set \( P \), and \( S \subset \text{int } P \).

**Remark 2:** Since \( h(\alpha) \neq \bar{0} \) and \( h(\gamma) \neq \bar{0} \), it follows that if \( h(S_r) = \bar{0} \), then \( S_r \neq \{ \alpha \} \cup \{ \gamma \} \). Hence, if \( h(S_r) = \bar{0} \), then \( S_r = S \).

**Lemma 51.1:** \( I \) is nonempty.

**Proof:** In the preceding sections we proved that \( S_0 \) is the maximal invariant set in \( N_2 \), \( S_0 \subset \text{int } N_2 \), and \( h(S_0) = \bar{0} \). Hence, \( 0 \in I \).

**Lemma 51.2:** \( I \) is open

**Proof:** Assume that \( r \in I \) and \( S_r \) is the maximal invariant set in the isolating neighborhood \( P \). An isolating neighborhood remains one upon perturbations of the equations. Therefore, there exists \( \delta > 0 \) such that \( P \) is an isolating neighborhood for (51.1\eta) for \( |r - \eta| < \delta \). Because \( P \) is
isolating it follows that \( S_\eta \subset \text{int} \, P \) for \(|\eta - r| < \delta\). It remains to prove that if \(|\eta - r| < \delta\), then \( S_\eta \) is the maximal invariant set in \( P \).

For each \( \eta \), \(|\eta - r| < \delta\), let \( M_\eta \) equal to the maximal invariant set in \( P \) for the flow (5I.1\eta). Clearly \( S_\eta \subset M_\eta \). For \( y \in M_\eta \), \( y \neq a_1 \) or \( \gamma_\eta \), suppose that \( \dagger(y)(z) = (\gamma(y)(z), \theta(y)(z)) \). Because \( \theta(y)'(z) < 0 \), it follows that \( \dagger(y)(z) \) must converge, as \( z \to \infty \), to invariant sets in \((\gamma, \theta) : \theta = \theta_\eta \) and \((\gamma, \theta) : \theta = \theta_\eta \), respectively. However, \( a_1 \) and \( \gamma_\eta \) are the only invariant sets in \((\gamma, \theta) : \theta = \theta_0 \) \cap P \) and \((\gamma, \theta) : \theta = \theta_\eta \) \cap P \). Therefore, \( \lim_{z \to \infty} \dagger(y)(z) = a_1 \) and \( \lim_{z \to \infty} \dagger(y)(z) = \gamma_\eta \). Hence, \( y \in S_\eta \).

The index of an isolated invariant set does not change under perturbations. Hence, \( h(S_\eta) = 0 \). This implies that, \( \eta \in I \) and \( I \) is open.

It remains to prove that \( I \) is closed. We must first prove some preliminary results. Until stated otherwise, fix \( r \in [0,1] \), and let \( \dagger(z) = (T, q, Y_1, p_1, \theta)(z) \) be a solution of (5I.1r), (5I.2r).

Note that \( N \) is an isolating neighborhood for all \( r \). Hence, \( \dagger(z) \in N \) and \( (T(z), Y_1(z)) \in \mathcal{O} \) for all \( z \). This immediately implies, from the definition of \( \mathcal{O} \),

**Lemma 5I.3:** Let \( \delta \) be as in the definition of \( \mathcal{O} \) in Section 5C. If \( 0 < T(z) < T_1 + \delta/2 \), then \( Y_1(z) > 1/Q_1 \, (T_1 + \delta/2) = L \).

Choose \( M \) so that \( f_1(T) > M(T - T_1) \) for \( T_1 < T < T_2 \).

**Lemma 5I.4:** Choose \( z_0 \) so that \( T(z_0) = T_1 \). There exists positive constants \( \bar{\theta} \) and \( V \) such that if \( \theta(z_0) < \bar{\theta} \), then \( q(z_0) > V \).

**Proof:** Suppose that \( z > z_0 \) and \( T_1 < T(z) < T_2 \). Then, by Lemma 3.8, \( T'(z) > 0 \), and

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\[ T^*(z) = \theta(z)T'(z) - Q_1Y_1f_1(T) \]  
\[ \leq \theta(z_0)T'(z) - Q_1LM(T - T_1) \]  

Consider solutions of the equation 

\[ x''(z) - \theta(z_0)x'(z) = -Q_1LM(x(z) - T_1) \]  

Suppose that \( \bar{\theta} = 1/2 \sqrt{Q_1LM} \). If \( \theta(z_0) < \bar{\theta} \), then the eigenvalues of the linear equations of (51.5) at \((x, x') = (T_1, 0)\) are complex and \((T_1, 0)\) is a repelling spiral. Hence, there exists \( V > 0 \) such that if \( \theta(y_0) < \bar{\theta} \), \( x(y_0) = T_1 \), and \( x'(y_0) = V \) then there exists \( y_1 > y_0 \) such that \( x(y_1) < T_2 \), \( x'(y_1) = 0 \), and \( 0 < x'(y) < V \) for \( y \in (y_0, y_1) \). (See Figure 9)

We denote the curve traced out by this trajectory as \( x' = G(x) \) and consider the region.
\[ R = \{ (T,q) : T_1 \leq T \leq x(y_1), \ 0 \leq q \leq G(T) \} . \]

The boundary of \( R \) consists of the three edges:

\[ e_1 = \{ (T,q) : T = T_1, \ 0 \leq q \leq V \} , \]
\[ e_2 = \{ (T,q) : T_1 \leq T \leq x(y_1), \ q = G(T) \} , \]
\[ e_3 = \{ (T,q) : T_1 \leq T \leq x(y_1), \ q = 0 \} . \]

Using (51.4) it follows that if \( T(z) \) is a solution of (51.3), then \( (T(z),q(z)) \) enters \( R \) along \( e_1 \) and \( e_2 \), and leaves \( R \) along \( e_3 \).

In \( R \), \( T' > 0 \). Hence trajectories which begin on \( e_1 \) must enter \( R \) and then leave through \( e_3 \). This proves that if \( \theta < \bar{\theta} \), \( T(z_0) = T_1 \), and \( T'(z_0) < V \), then there exists \( z > z_0 \) such that \( T'(z) < 0 \). This, however, contradicts Lemma 3.8, and the proof of Lemma 51.4 is complete.

**Lemma 51.5:** Choose \( z_0 \) such that \( T(z_0) = T_1 \), and assume that \( \bar{\theta} \) and \( V \) are as in Lemma 51.4. Then \( \theta(z_0) \geq \min(\bar{\theta}, V/4, T_1) \).

**Proof:** Assume that \( \theta(z_0) < \min(\bar{\theta}, V/4, T_1) \). From Lemma 51.4 it follows that \( q(z_0) > V \). Let

\[ R = \{ (T,q) : 0 \leq T \leq T_1, \ (VT/4 T_1) + V/2 \leq q \} \]

The sides of \( R \) are

\[ e_1 = \{ (T,q) : T = 0, \ V/2 \leq q \} , \]
\[ e_2 = \{ (T,q) : T = T_1, \ (3/4)V \leq q \} , \]
\[ e_3 = \{ (T,q) : 0 \leq T \leq T_1, \ q = (VT/4 T_1) + V/2 \} . \]
Note that \((T(z_0), q(z_0)) \in e_3\). Moreover, in \(R\), \(T' = q > 0\). Hence, if we follow \((T(z), q(z))\) backward from \((T(z_0), q(z_0))\) it must leave \(R\), in backwards time, through either \(e_1\) or \(e_3\). We claim that \((T(z), q(z))\) cannot leave \(R\) in backwards time, or enter \(R\) in forwards time, through \(e_3\). Let \(n = (v/4 \ T_1, -1)\) be a vector normal to \(e_3\) which points away from \(R_3\). Let \(v = (q, \theta q)\) be the vector field of the system in \(R\):

\[
T' = q \\
q' = \theta q .
\]

(51.6)

Then

\[
v \cdot n = v/4 \ T_1 q - \theta q \\
= (v/4 \ T_1 - \theta)q \\
> 0
\]

because of the assumption that \(\theta < v/4 \ T_1\). Hence \((T(z), q(z))\) must leave, in backwards time, \(R\) through either \(e_1\). However, if this were true, then \(T(z)\) would have to become negative which contradicts Lemma 3.3.

Let \(\hat{\theta} = 1/2 \ \min(\theta, v/4 \ T_1)\).

**Lemma 51.6:** If \(0 < T(z) < T_1\), then \(q(z) > \hat{\theta} T(z)\).

**Proof:** Let

\[
\varnothing = \{ (T, q) : 0 < T < T_1, \ q > \hat{\theta} T \}
\]

The sides of \(\varnothing\) are
\[ e_1 = \{ (q, T) : T = 0, \ q \geq 0 \}, \]
\[ e_2 = \{ (q, T) : T = T_1, \ q \geq \hat{T}_1 \}, \]
\[ e_3 = \{ (q, T) : 0 < T < T_1, \ q = \hat{T} \}. \]

Since \( T' = q \) trajectories enter \( \mathcal{G} \) along \( e_1 \). Let \( n = (\hat{\theta}, -1) \) be a vector outwardly normal to \( e_3 \), and let \( v = (q, \hat{\theta}q) \) be the vector field given by (51.6). Then

\[ n \cdot v = \hat{\theta}q - \theta q < 0 \]

because Lemma 51.5 implies that \( \theta(z) > \hat{\theta} \) as long as \( T(z) < T_1 \). Hence trajectories enter \( \mathcal{G} \) along \( e_3 \). To conclude the proof, we show that the unstable manifold at \( \alpha_1 \) points into \( \mathcal{G} \). Since \( (T(z), q(z)) \) must then leave \( \mathcal{G} \) along \( e_2 \), always lying above \( e_3 \), this will give the desired result.

From (51.6) it is clear that \( (T(z), q(z)) \) approaches \( (T, 0) \) as \( z \to -\infty \) tangent to the vector \( (1, \theta_0) \). This vector points into \( \mathcal{G} \) because \( \theta_0 > \hat{\theta} \), which we assume to be true. This implies that \( (T(z), q(z)) \in \mathcal{G} \) for \( z \) sufficiently large, and the proof is complete.

We now return to the proof that \( I \) is closed. We begin by showing

**Proposition 51.7:** Each \( S_r \) is closed.

**Proof:** Fix \( r \in [0, 1] \). Choose \( \{y_n\} \) such that for each \( n \), \( y_n \in S_r \), and suppose that \( \lim_{n \to \infty} y_n = y_0 \) exists. Of course, we wish to prove that \( y_0 \in S_r \). Let

\[ \dot{y}_n(z) = (T_n', q_n', Y_1n', P_1n', \theta_n')(z) \]

be the solution of (51.1r) which satisfies \( \dot{y}_n(0) = y_n \), and
\[ \tau(z) = (T,q,Y_1,p_1,\theta)(z) \]

the solution of (51.1r) which satisfies \( \tau(0) = \gamma_0 \). We assume that each \( \tau_n(z) \) is not identically \( \alpha_1 \) or \( \gamma_r \) (or else the proof is trivial). We choose the translation so that \( T_n(0) = T_1 \) for each \( n \). Then, clearly, \( T(0) = T_1 \).

Because \( Y'_1(z) \neq 0 \) for all \( z \) (Lemma 3.7) it follows that \( \omega^*(\tau(0)) \) and \( \omega(\tau(0)) \) are critical points. From Lemma 51.6 we conclude that

\[ \omega^*(\tau(0)) \in \{(T,q,Y_1,p_1,\theta) : T = 0, \theta = \theta_0 \}. \]

Because \( \alpha_1 \) is an isolated critical point and \( \lim_{z \to -\infty} \tau_n(z) = \alpha_1 \) for all \( n \), we conclude that \( \omega^*(\tau(0)) = \alpha_1 \). It remains to prove that \( \omega(\tau(0)) = \gamma_r \). We consider a few cases.

First suppose that \( \theta(0) = \theta_0 \). Because

\[ W_1 = \{(T,q,Y_1,p_1,\theta) : \theta = \theta_0 \} \]

is invariant, it follows that \( \tau(z) \in W_1 \) for all \( z \). Hence, \( \tau(z) \) is a bounded solution which lies in \( W_1 \) and satisfies \( \lim_{z \to -\infty} \tau(z) = \alpha_1 \). However, we showed earlier that each trajectory in the unstable manifold of \( \alpha_1 \), which lies completely in \( W_1 \), becomes unbounded. This gives a contradiction. Hence, \( \tau(0) \notin W_1 \).

Now suppose that

\[ \tau(0) \in W_2 = \{(T,q,Y_1,p_1,\theta) : \theta = \theta_r \}. \]  \hspace{1cm} (51.7)

Because \( W_2 \) is invariant it follows that \( \tau(z) \in W_2 \) for all \( z \). Since \( \omega^*(\tau(0)) \) is a critical point, it has the form
\[ \omega^*(\theta(0)) = (\hat{T}, 0, \hat{Y}, 0, \theta_r) . \]

From Lemma 51.6 it follows that \( \hat{T} > T_1 \). This implies that \( \dot{Y}_1 = 0 \) and either \( T_1 < \hat{T} < T_2 \) or \( \hat{T} = Q_1 + 2Q_2 \). Therefore, \( \hat{T}(z) \) corresponds to a solution of

\[
\begin{align*}
    T' &= q \\
    q' &= \theta q - Q_1 Y_1 f_1(T) - [Q_1 + 2Q_2 - T - (Q_1 + Q_2) Y_1] f_2(T) \\
    Y_1' &= p_1 \\
    p_1' &= \theta p_1 + Y_1 f_1(T)
\end{align*}
\]  

(5I.8a)

with \( \theta = \theta_r \), together with the boundary conditions

\[
\lim_{z \to \infty} (T, q, Y_1, p_1)(z) = (0, 0, 1, 0) \tag{5I.8b}
\]

and

\[
\lim_{z \to \infty} (T, q, Y_1, p_1)(z) = (T^*, 0, 0, 0)
\]

where either \( T_1 < T^* = T_2 \) or \( T^* = Q_1 + 2Q_2 \). First suppose that \( T_1 < T^* < T_2 \). Then integrating the equations in (5I.8a) as in the first section we find that \( T^* = Q_1 \). However, by Proposition 4.6, any solution of (5I.8) must have \( \theta > \theta_1 \). Since \( \theta_r < \theta_1 \), this gives the desired contradiction. If \( T^* = Q_1 + 2Q_2 \), then Proposition 4.5 implies that for any solution of (5I.8) we must have \( \theta > \theta_1 \). As before, this gives the desired contradiction.

We have now shown that \( \theta_r < \theta(0) < \theta_0 \), and \( \omega^*(\theta(0)) \) and \( \omega(\hat{T}(0)) \) are critical points. It remains to show that \( \omega(\hat{T}(0)) = \gamma_r \). Suppose that this is not true. Clearly \( \omega(\hat{T}(0)) \in \mathcal{W}_3 \), which was defined in (5I.7). Suppose that

\[ \omega(\hat{T}(0)) = (\hat{T}, 0, \hat{Y}, 0, \theta_r) \]

where \( \hat{T} \neq Q_1 + 2Q_2 \). Because \( T'(z) > 0 \) for \( T < T_2 \) (Lemma 3.8) and \( T(0) = T_1 \), we have \( T_1 < \hat{T} < T_2 \) which implies that \( \dot{Y}_1 = 0 \).
Choose \( z_n \) such that for each \( n \), \( T_n(z_n) = 1/2(Q_1 + 2Q_2 + T_3) \). The \( t_n(z_n) \) are bounded so at least some subsequence of \( \{t_n(z_n)\} \) converges to, say, \( y_1 \). We assume for convenience, that the entire sequence \( \{t_n(z_n)\} \) converges to \( y_1 \). Let \( \hat{t}(z) = (\hat{T}, \hat{q}, \hat{y}, \hat{p}, \hat{\theta})(z) \) be the solution of (51.4r) which satisfies \( \hat{t}(0) = y_1 \). Because \( \theta_n(z) < 0 \) for each \( n \), and \( \omega(t(0)) \in W_2 \) it follows that \( y_1 \in W_2 \). Because \( W_2 \) is invariant, \( \omega(y_1) \) and \( \omega(y_1) \) are critical points which lie in \( W_2 \). Now \( \omega(y_1) \) is a critical point with \( T > 1/2(T_2 + Q_1 + 2Q_2) \). Hence, \( \omega(y_1) = \gamma \). Suppose that \( \omega(y_1) = (T^*, 0, Y^*, 0, \theta^*) \). Because \( T_n(z) > 0 \) if \( T_n(z) < Q_1 + 2Q_2 \) (Lemma 3.8) we conclude that \( T^* > T > T_1 \). Hence, \( Y^* = 0 \). (Because if \( (T, 0, Y, 0, \theta) \) is a critical point with \( T > T_1 \), then \( Y = 0 \).) Because \( Y_n(z) < 0 \) for each \( n \) (Lemma 3.7) we have that \( \hat{Y}_1(z) = 0 \) for all \( z \). Hence \( (\hat{T}(z), \hat{q}(z)) \) satisfies the system

\[
\hat{T}' = \hat{q} \\
\hat{q}' = \theta \hat{q} - [Q_1 + 2Q_2 - \hat{T}]f_2(\hat{T})
\]

together with the boundary conditions

\[
\lim_{z \to -} (T(z), q(z)) = (T^*, 0) \quad \text{and} \quad \lim_{z \to +} (T(z), q(z)) = (Q_1 + 2Q_2, 0).
\]

However, from Proposition 4.7 we conclude that on any such solution, \( \theta > \theta_1 \). Because \( \theta \theta < \theta_1 \), we have the desired contradiction, and the proof of the Lemma is complete.

**Corollary 51.13:** Each \( S_\lambda \) is compact.

**Proof:** Because of Lemma 51.12 it suffices to show that each \( S_\lambda \) is bounded. However each \( S_\lambda \) lies in the isolating neighborhood \( N \).
**Lemma 51.14**: For each $\lambda$ there exists a compact set $N_\lambda$ such that $N_\lambda$ is an isolating neighborhood, $S_\lambda \subset \text{int } N_\lambda$, and $S_\lambda$ is the maximal invariant set in $N_\lambda$.

**Proof**: We begin by discussing the behavior of (51.1$\lambda$) near $\gamma_\lambda$. If we linearize (51.1$\lambda$) at $\gamma_\lambda$ we find that there are three negative eigenvalues, say $-\lambda_1,-\lambda_2,-\lambda_3$, and two positive eigenvalues, say $\lambda_4$ and $\lambda_5$. Choose new coordinates $(x_1,x_2,x_3,x_4,x_5)$ so that near $\gamma_\lambda$, (51.1$\lambda$) is given in the new coordinates by:

\[
\begin{align*}
x_1' &= -\lambda_1 x_1 + g_1(x_1,x_2,x_3,x_4,x_5) \\
x_2' &= -\lambda_2 x_2 + g_2(x_1,x_2,x_3,x_4,x_5) \\
x_3' &= -\lambda_3 x_3 + g_3(x_1,x_2,x_3,x_4,x_5) \\
x_4' &= \lambda_4 x_4 + g_4(x_1,x_2,x_3,x_4,x_5) \\
x_5' &= \lambda_5 x_5 + g_5(x_1,x_2,x_3,x_4,x_5)
\end{align*}
\]

(51.13)

For each $i$, $g_i(x_1,x_2,x_3,x_4,x_5) = O(|x|^2)$ where $|x|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$. We assume that (51.13) holds in $C = \{(x_1,x_2,x_3,x_4,x_5) : |x_i| \leq \delta$ for each $i\}$.

Let

\[
C_1 = \{(x_1,x_2,x_3,x_4,x_5) \in C : |x_1| = \delta, \text{ or } |x_2| = \delta, \text{ or } |x_3| = \delta\}
\]

\[
C_2 = \{(x_1,x_2,x_3,x_4,x_5) \in C : |x_4| = \delta, \text{ or } |x_5| = \delta\}
\]

$W^u_\gamma = \text{unstable manifold at } \gamma_\lambda$

$W^s_\gamma = \text{stable manifold at } \gamma_\lambda$

Then $W^u_\gamma \cap \partial C = C_1$ and $W^s_\gamma \cap \partial C = C_2$. For $p \in C$, let $\#(p)(z)$ be the solution of (51.13) which satisfies $\#(p)(0) = p$. If $\delta$ is sufficiently small,
then trajectories enter $C$ along $C_2$ and leave $C$ along $C_1$. Moreover, if $\delta$ is sufficiently small then if $p \in C$ then either $\#(p)(z) \in W_\gamma^B$ or there exists $z_1 > 0$ such that $\#(p)(z) \in C$ for $z \in (0,z_1)$ and $\#(p)(z_1) \in C_1$.

A similar analysis holds near $\alpha_1$. If we linearize (5I.1\lambda) at $\alpha_1$ we find that there are two negative eigenvalues, say $-\gamma_1$ and $-\gamma_2$, and three positive eigenvalues, say $\gamma_3$ and $\gamma_5$. Choose coordinates $(y_1,y_2,y_3,y_4,y_5)$ so that near $\alpha_1$, (5I.1\lambda) is given in the new coordinates by:

\begin{align*}
y'_1 &= -\gamma_1 y_1 + h_1(y_1,y_2,y_3,y_4,y_5) \\
y'_2 &= -\gamma_2 y_2 + h_2(y_1,y_2,y_3,y_4,y_5) \\
y'_3 &= \gamma_3 y_3 + h_3(y_1,y_2,y_3,y_4,y_5) \quad \text{(5I.14)} \\
y'_4 &= \gamma_4 y_4 + h_4(y_1,y_2,y_3,y_4,y_5) \\
y'_5 &= \gamma_5 y_5 + h_5(y_1,y_2,y_3,y_4,y_5)
\end{align*}

For each $i$, $h_i(y_1,y_2,y_3,y_4,y_5) = 0(|y|^2)$ where $|y|^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2$. We assume that (5I.14) holds in

$$E = \{(y_1,y_2,y_3,y_4,y_5) : |y_i| \leq \eta \text{ for each } i\}.$$ 

Let

\begin{align*}
E_1 &= \{(y_1,y_2,y_3,y_4,y_5) \in E : |y_1| = \eta \text{ or } |y_2| = \eta\}, \\
E_2 &= \{(y_1,y_2,y_3,y_4,y_5) \in E : |y_3| = \eta, \text{ or } |y_4| = \eta, \text{ or } |y_5| = \eta\}, \\
W_{\alpha}^u &= \text{unstable manifold at } \alpha_1, \\
W_{\alpha}^s &= \text{stable manifold at } \alpha_1.
\end{align*}

Then $W_{\alpha}^u \cap E \subseteq E_1$ and $W_{\alpha}^s \cap E \subseteq E_2$. For $p \in E$, let $\#(p)(z)$ be the solution of (5I.14) which satisfies $\#(p)(0) = p$. If $\eta$ is sufficiently small, then
trajectories enter $E$ through $E_1$ and leave $E$ through $E_1$. Moreover, if $\eta$ is sufficiently small, then if $p \in E$ then either $\tau(p)(z) \in W_a u$ or $\tau(p)(z)$ leaves $E$ in backwards time through $E_2$.

We now construct $N_\lambda$ which was mentioned in the statement of the Lemma. For each $p \in S_\lambda$ we shall define a compact neighborhood $N(p)$ such that $p \in \text{int} \ N(p)$. Then $\bigcup_{p \in S_\lambda} \text{int} \ N(p)$ will form an open cover of $S_\lambda$. Since $S_\lambda$ is compact, there exists a finite subcover of $S_\lambda$. Then $N_\lambda$ will be the union of the compact sets in the subcover. The $N(p)$ are defined as follows.

If $p \in C$, then let $N(p) = C$. If $p \in E$, then let $N(p) = E$. So suppose that $p \notin C \cup E$. Since $p \in S_\lambda$, it follows that $\tau(p)(z_1) \in C$ for some $z_1 > 0$ and $\tau(p)(z_2) \in E$ for some $z_2 < 0$. By the continuous dependence of solutions on initial data we may choose $N(p)$ so that if $q \in N(p)$, then $\tau(q)(z_3) \in C$ for some $z_3$, and $\tau(q)(z_4) \in E$ for some $z_4$.

Now it is clear that $S_\lambda \subset \bigcup_{p \in S_\lambda} \text{int} \ N(p)$. Since $S_\lambda$ is compact, there exists $p_1, p_2, \ldots, p_n$ such that $S_\lambda = \bigcup_{i=1}^{n} \text{int} \ N(p_i)$. Let $N_\lambda = \bigcup_{i=1}^{n} N(p_i)$. Then, clearly, $N_\lambda$ is a compact set and $S_\lambda \subset \text{int} \ N_\lambda$. It remains to prove $S_\lambda$ is the maximal invariant set in $N_\lambda$. We prove that if $q \in N_\lambda \setminus S_\lambda$, then $\tau(q)(z)$ leaves $N_\lambda$ in either forwards or backwards time.

Because $q \notin S_\lambda$ it follows that either $q \notin W_\gamma$ or $q \notin W_a u$. First assume that $q \notin W_\gamma$. From the construction of $N(\lambda)$ it is clear that there exists $z_1$ such that $\tau(q)(z_1) \in C$. Because $q \notin W_\gamma$ it follows, from the construction of $C$, that there exists $z_2 > z_1$ such that $\tau(q)(z) \in C$ for $z \in (z_1, z_2)$ and $\tau(q)(z_2) \in C_1$. However, trajectories which lies on $C_1$ must leave $C$. Once $\tau(q)(z)$ leaves $C$, it leaves $N_\lambda$. 

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It remains to consider the case \( q \in W_\alpha^B \) but \( q \notin W_\alpha^U \). Then there exists \( z_1 \) such that \( \tau(q)(z_1) \in \mathcal{E} \). Because \( q \notin W_\alpha^U \), \( \tau(q)(z) \) must leave \( \mathcal{E} \) through \( \mathcal{E}_2 \) in backwards time. Once \( \tau(q)(z) \) leaves \( \mathcal{E} \), it also leaves \( N_\lambda \) and the proof is complete.

**Lemma 5.1.6**: \( I \) is closed

**Proof**: Suppose that \( (\lambda_n) \subseteq I \) and \( \lim_{n \to \infty} \lambda_n = \lambda_0 \). Of course, we wish to prove that \( \lambda_0 \in I \). From the preceding Lemma, there exists an isolating set \( N_\lambda \) such that \( S_{\lambda_0} \subseteq \text{Int} \, N_\lambda \) and \( S_{\lambda_0} \) is the maximal invariant set in \( N_\lambda \). So it remains to prove that \( h(S_{\lambda_0}) = \emptyset \). However, \( N_\lambda \) remains an isolating neighborhood upon perturbations of the equations. Therefore, just as in the proof that \( I \) is open, it follows that there exists a \( \delta > 0 \) such that if \( |\lambda_0 - \eta| < \delta \), then \( h(S_{\lambda_0}) = h(S_{\eta}) \). Choose \( n \) so that \( |\lambda_0 - \lambda_n| < \delta \). Then \( h(S_{\lambda_0}) = h(S_{\lambda_n}) = \emptyset \).

This completes the proof that there exists a solution of (51.1), (51.2). Making the substitution (5A.3) this also completes the proof of Proposition 2.1 for the case \( d_0 = d_1 = d_2 = 1 \).
6. Proof of Proposition 2.1

We no longer assume that \( d_0 = d_1 = d_2 = 1 \). For convenience, and without loss of generality, we assume that \( d_0 = 1 \). For \( 0 \leq r \leq 1 \), let

\[
d_1^r = rd_1 + (1 - r) \quad \text{and} \quad d_2^r = rd_2 + (1 - r) .
\]

Let \( \theta_0 \) and \( \theta_1 \) be as in (4.5) and (4.6). Consider the one parameter family of equations:

\[
\begin{align*}
T' &= q \\
n' &= \theta q - Q_1 Y_1 f_1(T) - Q_2 Y_2 f_2(T) \\
Y_1' &= p_1 \\
p_1' &= 1/d_1^r(\theta p_1 + Y_1 f_1(T)) \\
Y_2' &= p_2 \\
p_2' &= 1/d_2^r(\theta p_2 - Y_1 f_1(T) + Y_2 f_2(T)) \\
\theta' &= -\varepsilon (\theta - \theta_0)(\theta - \theta_1)
\end{align*}
\]

(6.1r)

together with the boundary conditions

\[
\lim_{z \to \infty} (T, q, Y_1, p_1, Y_2, p_2, \theta)(z) = A_1 \ast (0, 0, 1, 0, 1, 0, \theta_0) 
\]

and

\[
\lim_{z \to \infty} (T, q, Y_1, p_1, Y_2, p_2, \theta)(z) = C_2 \ast (Q_1 + 2Q_2, 0, 0, 0, 0, 0, \theta_1) .
\]

(6.2)

We prove that there exists a solution of (6.1r), (6.2) for each \( \varepsilon > 0 \) by continuing from the case \( d_1 = d_2 = 1 \). The argument we give is very similar to that in Section 5.1.
It is necessary to change the flow near \( A_1 \) and \( C_2 \) in a way similar to what was done in Section 5G. The reason is that we need to have \( A_1 \) and \( C_2 \) to be isolated critical points. As in Section 5G, we change the flow so that the unstable manifold at \( A_1 \) and the stable manifold at \( C_2 \) are not affected. Hence, any solution of (6.2) for the new flow is also a solution of (6.1r). Because the construction of the new flow is so similar to the construction in Section 5I, we only sketch the details.

We first define the neighborhood of \( A_1 \) in which the flow will be changed. Let \( \lambda \) be as in Section 5D.2, and \( \gamma = (T,q,Y_1,p_1,Y_2,p_2) \). Let

\[
P_1 = \{ (\gamma, \theta) : |q| < \frac{1}{4}(T - |\theta - \theta_o|), |\theta - \theta_o| \leq T \leq \frac{1}{4}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |q| > -\frac{1}{4}(T + |\theta - \theta_o|), -\frac{1}{4}\lambda \leq T \leq -|\theta - \theta_o| \}
\]

\[
P_2 = \{ (\gamma, \theta) : |p_1| \leq \frac{1}{4}(Y_1 - 1 - |\theta - \theta_o|), |\theta - \theta_o| \leq Y_1 - 1 \leq \frac{1}{4}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |p_1| \leq -\frac{1}{4}(Y_1 - 1 + |\theta - \theta_o|), -\frac{1}{4}\lambda \leq Y_1 - 1 \leq -|\theta - \theta_o| \}
\]

\[
P_3 = \{ (\gamma, \theta) : |p_2| \leq \frac{1}{4}(Y_2 - 1 - |\theta - \theta_o|), |\theta - \theta_o| \leq Y_2 - 1 \leq \frac{1}{4}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |p_2| \leq -\frac{1}{4}(Y_2 - 1 + |\theta - \theta_o|), -\frac{1}{4}\lambda \leq Y_2 - 1 \leq -|\theta - \theta_o| \}
\]

\[
Q_1 = \{ (\gamma, \theta) : |q| < \frac{1}{2}(T - |\theta - \theta_o|), |\theta - \theta_o| \leq T \leq \frac{1}{2}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |q| > -\frac{1}{2}(T + |\theta - \theta_o|), -\frac{1}{2}\lambda \leq T \leq -|\theta - \theta_o| \}
\]

\[
Q_2 = \{ (\gamma, \theta) : |p_1| \leq \frac{1}{2}(Y_1 - 1 - |\theta - \theta_o|), |\theta - \theta_o| \leq Y_1 - 1 \leq \frac{1}{2}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |p_1| \leq -\frac{1}{2}(Y_1 - 1 + |\theta - \theta_o|), -\frac{1}{2}\lambda \leq Y_1 - 1 \leq -|\theta - \theta_o| \}
\]

\[
Q_3 = \{ (\gamma, \theta) : |p_2| \leq \frac{1}{2}(Y_2 - 1 - |\theta - \theta_o|), |\theta - \theta_o| \leq Y_2 - 1 \leq \frac{1}{2}\lambda \}
\]

\[
\cup \{ (\gamma, \theta) : |p_2| \leq -\frac{1}{2}(Y_2 - 1 + |\theta - \theta_o|), -\frac{1}{2}\lambda \leq Y_2 - 1 \leq -|\theta - \theta_o| \}
\]

\[
\mathcal{N}_1 = P_1 \cap P_2 \cap P_3
\]

\[
\mathcal{N}_2 = Q_1 \cap Q_2 \cap Q_3
\]
The neighborhood of \( C_2 \) in which we change the flow is defined as follows. Let

\[
J_1 = \{ (\gamma, \theta) : \| (\gamma, \theta) - (T, 0, 0, 0, 0, 0) \| \leq \frac{1}{4} (T - (Q_1 + 2Q_2) - |\theta - \theta_1|), \| \theta + \theta_1 \| \leq T - (Q_1 + 2Q_2) \leq \frac{1}{4} \lambda \}
\]

\[ \cup \{ (\gamma, \theta) : \| (\gamma, \theta) - (T, 0, 0, 0, 0, 0) \| \leq -\frac{1}{4} (T - (Q_1 + 2Q_2) + |\theta - \theta_1|), -\frac{1}{2} \lambda \leq T - (Q_1 + 2Q_2) \leq -|\theta - \theta_1| \}\],

\[
J_2 = \{ (\gamma, \theta) : \| (\gamma, \theta) - (T, 0, 0, 0, 0, 0) \| \leq \frac{1}{2} (T - (Q_1 + 2Q_2) - |\theta - \theta_1|), \| \theta + \theta_1 \| \leq T - (Q_1 + 2Q_2) \leq \frac{1}{2} \lambda \}
\]

\[ \cup \{ (\gamma, \theta) : \| (\gamma, \theta) - (T, 0, 0, 0, 0, 0) \| \leq -\frac{1}{2} (T - (Q_1 + 2Q_2) + |\theta - \theta_1|), -\frac{1}{2} \lambda \leq T - (Q_1 + 2Q_2) \leq -|\theta - \theta_1| \}\}.
\]

The picture we have in mind for \( \mathcal{N}_1, \mathcal{N}_2, J_1, \) and \( J_2 \) are similar to what is shown in Figure 8.

Let \( \psi(\gamma, \theta) \) be any function which satisfies

a) \( \psi = 1 \) in \( \mathcal{N}_1 \cup J_1 \),

b) \( \psi = 0 \) in the complement of \( \mathcal{N}_2 \cup J_2 \),

c) \( \psi(A_1) = \psi(C_2) = 0 \),

d) \( 0 \leq \psi \leq 1 \) for all \( (\gamma, \theta) \),

e) \( \psi \) is continuous except of \( A_1 \) and \( C_2 \).

We now define a new vector field in \( \mathcal{N}_2 \cup J_2 \). Care is taken in the definition so that we anticipate what is to follow. Throughout this section we assume that \( \gamma = (T, q, Y_1, p_1, Y_2, p_2) \) and \( \zeta = (T, q, Y_1, p_1) \).

Let \( \Lambda \) be the flow which appeared in (5A.3), and let
\[ Z = T + (Q_1 + Q_2)Y_1 + Q_2 Y_2 \]  

(6.3)

be as in Section 5A. The new vector field in \( \mathcal{H}_2 \cup J_2 \) is defined as

\[ \xi'(z) = \Lambda(z) \]

\[ z' = \xi \]

(6.4)

\[ \xi' = \theta \xi - (z - (Q_1 + 2Q_2)) \]

\[ \theta' = \varepsilon(\theta_0 - \theta)(\theta_1 - \theta) \]

Let \( \eta(r) \) be the vector field defined by the right-hand side of (6.1r) and \( \chi \) the vector field defined by the right-hand side of (6.4). For \( 0 \leq r \leq 1 \), let \( V(r) \) be the vector field

\[ V(r) = \varphi \eta(r) + (1 - \varphi) \chi. \]

(6.5)

The following lemmas are proved exactly as in Section 5G.

**Lemma 6.1:** Given \( r \), \( V(r) \) is continuous for all \( (\gamma, \theta) \).

**Lemma 6.2:** Given \( r \), the only critical points in

\[ \{(\gamma, \theta) : |T| \leq \frac{1}{2} \lambda, \, |1 - Y_1| \leq \frac{1}{2} \lambda, \, |1 - Y_2| \leq \frac{1}{2} \lambda\} \]

\[ \cup \{(\gamma, \theta) : |T - (Q_1 + 2Q_2)| \leq \frac{1}{2} \lambda, \, |Y_1| \leq \frac{1}{2} \lambda, \, |Y_2| \leq \frac{1}{2} \lambda\} \]

are \( A_1 \) and \( C_2 \).

**Lemma 6.3:** Let \( \tilde{\sigma}_r \) be the set of solutions of (6A.1r) which satisfy (6.2). Let \( \sigma'_r \) be the set of solutions of the system

\[ \xi'(z) = V(r) \]

(6.6r)

which satisfy (6.2). Then \( \tilde{\sigma}_r = \sigma'_r \).
**Remark:** We do not prove these three lemmas because their proofs are so similar to the proofs given in Section 5I. The key idea in Lemma 6.3 is that the unstable manifold at \( A_1 \) and the stable manifold at \( C_2 \) lie in the compliment of \( X_2 \cup J_2 \).

In the remainder of Section 6 we show that \( \sigma_r^* \) is nonempty for each \( r \) by continuing from \( r = 0 \). The continuation is similar to what was done in Section 5I. Let \( \sigma_r^* \) be as above and

\[
\sigma_r = \sigma_r^* \cup (A_1) \cup (C_2)
\]

(6.2)

Let

\[
I = \{ r = [0,1] : \sigma_r^* \text{ is an isolated invariant set and } h(\sigma_r^*) = 0 \}.
\]

**Remark:** As in Section 5I, it follows that if \( h(\sigma_r) = 0 \), then \( \sigma_r^* \neq 0 \). This is because \( h(A_1) \neq 0 \) and \( h(C_2) \neq 0 \). Hence, if \( h(\sigma_r) = 0 \), then \( \sigma_r \neq (A_1) \cup (C_2) \).

**Lemma 6.4:** \( I \) is nonempty.

**Proof:** We shall prove that \( 0 \in I \). The results of Section 5 imply that \( \sigma_0^* \) is nonempty. This is because if we let \((T,q,Y_1,p_1,\theta)\) be a solution of (5I.1), (5I.2), define \( Y_2(z) \) by (5A.3), and let \( p_2(z) = Y_2^*(z) \), then \((T,q,Y_1,p_1,Y_2,p_2,\theta)\) is a solution of (6.6), (6.2) with \( r = 0 \).

The proof that \( 0 \in I \) is carried out in two steps. First we consider solutions of (6.4). We show that the set of solutions of (6.4), (6.2) is isolated with index \( 0 \). We then continue the system (6.4) to (6.6), showing that the set of solutions of (6.2) remains isolated with index \( 0 \) along the continuation.

Let us now consider (6.4). It is essentially the product of (5I.1\(\lambda \)), with \( \lambda = 1 \), and the two dimensional system.
\[ z' = \xi \]  
\[ \xi' = \theta \xi - (z - (Q + 2Q_2)) . \]  

(6.8)

The only bounded solution of (6.8) is the critical point \( P = (Q_1 + 2Q_2,0) \). The linearized equations of (6.8) has two positive eigenvalues at \( P \), so \( \{P\} \) is an isolated invariant set with index \( \Sigma \), the pointed two sphere.

Let \( \Lambda(y)(z) \) be the solutions of (6.4) which satisfying \( \Lambda(y)(0) = y \). Let
\[ s' = \{y : \lim_{z \to z_1} \Lambda(y)(z) = A_1 \text{ and } \lim_{z \to z_2} \Lambda(y)(z) = C_2\} \]
and
\[ s = s' \cup \{A_1\} \cup \{C_2\} . \]

Finally, let \( S_1 \) be as in Section 5I. Then \( s \) is equal to the product of \( S_1 \) and \( \{p\} \). Because the product of two isolated invariant sets is isolated and has index equal to the (smash) product of its indices, we conclude that \( s \) is an isolated invariant set and

\[ h(s) = h(S_1) \wedge h(P) = 0 \wedge \Sigma = 0 . \]

To complete the proof that \( 0 \in I \) we continue from the new system (6A.4) to (6A.6r), with \( r = 0 \). For \( 0 \leq j \leq 1 \), let \( \varphi_j(\gamma,\theta) \) be any set of functions which satisfy

a) \( \varphi_j = 1 \) in \( N_1 \cup J_1 \)
b) \( \varphi_j = j \) in the complement of \( N_2 \cup J_2 \)
c) \( \varphi_j(A_1) = \varphi_j(C_2) = 0 \)
d) \( 0 \leq \varphi_j \leq 1 \) for all \( (\gamma,\theta) \)
e) \( \varphi_j \) is continuous except at \( A_1 \) and \( C_2 \)
f) \( \varphi_0(\gamma,\theta) = \varphi(\gamma,\theta) \) for all \( (\gamma,\theta) \)
g) \( \varphi_1(\gamma,\theta) = 1 \) for all \( (\gamma,\theta) \).
Let $V_j$ be the vector field defined by the right-hand side of the equations

$$\xi'(t) = \Lambda(t)$$

$$z' = \xi$$

$$\xi' = \theta \xi - \psi_j(z - (Q_1 + 2Q_2))$$

$$\theta' = \varepsilon(\theta_0 - \theta)(\theta_1 - \theta)$$

(6.10)

**Remark:** Note that $V_0 = V(0)$ and $V_1 = \chi$, the vector field defined in (6.4).

Consider the family of systems

$$\dot{\xi}'(z) = V_j(z)$$

(6.11)

Given $y$, let $\dot{y}_j(y)(z)$ be the solution of (6B.5) which satisfies $\dot{y}_j(y)(0) = 0$. Let

$$\sigma_j' = \{y : \lim_{z \to -\infty} \dot{y}_j(y)(z) = A_1 \text{ and } \lim_{z \to +\infty} \dot{y}_j(y)(z) = C_2\} ,$$

$$\sigma_j = \sigma_j' \cup \{A_1\} \cup \{C_2\} ,$$

and

$$J = \{j : \sigma_j \text{ is an isolated invariant set for the flow (6B.5)}$$

$$\text{and } h(\sigma_j) = \bar{0}\}$$

We wish to prove that $J = [0,1]$. We have already shown that $1 \in J$. Hence, $J$ is nonempty. The proof that $J$ is open and closed is very similar to the proof given in Section 5I, so we do not give the details. This completes the proof of Lemma 6.4.
It remains to prove that $I$ is both open and closed. The proofs of these statements are very similar to the proofs of Lemmas 51.2 and Lemma 51.12, so we do not give the details.
7. The Limit $\varepsilon \to 0$

Let $\tau_\varepsilon(z) = (T_\varepsilon, q_\varepsilon, Y_1\varepsilon, p_1\varepsilon, Y_2\varepsilon, p_2\varepsilon, \theta_\varepsilon)(z)$ be a solution of (3.1), (3.2). Choose $\varepsilon = 0$ so that $T_\varepsilon(0) = T_1$ for all $\varepsilon$. By compactness there exists a subsequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $\tau_{\varepsilon_k}(0)$ converges to some

$$\tau^* = (T^*, q^*, Y_1^*, p_1^*, Y_2^*, p_2^*, \theta^*) .$$

Let $\tau(z)$ be the solution of (3.1) with $\varepsilon = 0$ which satisfies $\tau(0) = \tau^*$. Of course, $\tau(z)$ lies in the surface $\{\theta - \theta^*\}$.

Recall that for each $\varepsilon$, $Y_1\varepsilon'(z) < 0$ (Lemma 3.7). Hence, $Y_1^\varepsilon(z) < 0$ for all $z$. Because $\tau(z)$ is bounded this implies

Lemma 7.1: There exist critical points $K_1$ and $K_2$ of (3.1) with $\varepsilon = 0$ such that

$$\lim_{z \to -\infty} \tau(z) = K_1 \quad \text{and} \quad \lim_{z \to +\infty} \tau(z) = K_2 .$$

It is not hard to figure out what $K_1$ and $K_2$ are.

Lemma 7.2: $K_1 = (A, \theta^*)$.

Proof: A proof similar to that of Lemma 5.I.G shows that for each $\varepsilon$, and $z < 0$, $|q_\varepsilon(z)| > \theta_1 T_\varepsilon(z)$. Moreover, $|p_1\varepsilon(z)| > \theta_1 |Y_1\varepsilon(z) - 1|$ and $|p_2\varepsilon(z)| > \theta_1 |Y_2\varepsilon(z) - 1|$. We also have that for each $\varepsilon > 0$, $z < 0$, $T_\varepsilon(z)$, $Y_1\varepsilon(z)$ and $Y_2\varepsilon(z)$ are monotone functions of $z$. This is because, if $z < 0$, then $T_\varepsilon$, $Y_1\varepsilon$, and $Y_2\varepsilon$ all satisfy equations of the form $dy' - \theta y' = 0$. From all this we conclude that for $z < 0$, $|q(z)| > \theta_1 |T(z)|$, $|p_1(z)| > \theta_1 |Y_1(z) - 1|$, and $|p_2(z)| > \theta_1 |Y_2(z) - 1|$. Moreover, for $z < 0$, $T$, $Y_1$ and...
$Y_3$ are all monotone. This is enough to prove that \( \lim_{z \to -\infty} (T, Y_1, Y_2) = (0, 1, 1) \)
and \( \lim_{z \to -\infty} \quad ^{\dagger}(z) = \Lambda. \)

**Lemma 7.2:** Either \( K_2 = (B, \theta^*) \) or \( K_2 = (C, \theta^*). \)

**Proof:** We consider two cases. These are, either \( T(z) < T_2 \) for all \( z \)
or \( T(z) > T_2 \) for some \( z \). First assume that \( T(z) < T_2 \) for all \( z \). Then \( f_2(T(z)) = 0 \) for all \( z \), and \( (T, Y_1) \) satisfy the equations

\[
\begin{align*}
T^* &= \theta^* T^* + Q_1 Y_1 f_1(T) = 0 \\
\dot{Y}_1 &= \theta^* Y_1^* - Y_1 f_1(T) = 0 \\
\lim_{z \to -\infty} (T, Y_1)(z) &= (0, 1) \quad \text{and} \quad \lim_{z \to +\infty} (T, Y_1)(z) = (K_T, K_Y)
\end{align*}
\tag{7.1}
\]

where \( K_T \) and \( K_Y \) are, respectively, the \( T \) and \( Y_1 \) components of \( K_2 \).

Clearly \( K_T > T_1 \). This implies that \( K_Y = 0 \). Integrate the second equation in (7.1) from \( -\infty \) to \( +\infty \) to obtain

\[ \theta^* = \int_{-\infty}^{\infty} Y_1 f_1(T) dz. \tag{7.2} \]

Now integrate the first equation in (7.1) to obtain

\[ \theta^* K_T = Q_1 \int_{-\infty}^{\infty} Y_1 f_1(T) dz. \tag{7.3} \]

It follows from (7.2) and (7.3) that \( K_T = Q_1 \).

Note that \( Y_2(z) \) satisfies the equation

\[
\begin{align*}
\dot{Y}_2^* &= \theta^* Y_2^* - Y_2 f_1(T) = 0 \\
Y_2(-\infty) &= 1.
\end{align*}
\]
Integrate the first equation from $-\infty$ to $\infty$ to obtain

$$\theta^*(Y_2(+\infty) - 1) = \int_{-\infty}^{\infty} Y_1 f_1(T) dT.$$ 

Together with (7.2) this implies that $Y_2(+\infty) = 2$. Hence, $K_2 = (B, \theta^*)$.

Now suppose that $K_2 \neq (B, \theta^*)$. Then we must have that $T(z) > T_2$ for some $z$. Then $T, Y_1,$ and $Y_2$ satisfy the equations

$$T' - \theta^* T' + Q_1 Y_1 f_1(T) + Q_2 Y_2 f_2(T) = 0,$$

$$d_1 Y_1' - \theta^* Y_1' = Y_1 f_1(T) = 0 \quad (7.4)$$

$$d_2 Y_2' - \theta^* Y_2' + Y_1 f_1(T) - Y_2 f_2(T) = 0,$$

$$\lim_{z \to -\infty} (T, Y_1, Y_2) = (0, 1, 1).$$

Suppose that the $T, Y_1,$ and $Y_2$ components of $K_2$ are, respectively, $K^1, K^2,$ and $K^3$. In Section 4 we showed that if $(T, Y_1, Y_2)$ is a solution of (7.4) which connects two critical points, then $T'(z) > 0$. We are assuming that $T(z) > T_2$ for some $z$. Hence, $K_1 > T_2$. This implies that $K_2 = 0$.

If we integrate the second equation in (7.4) from $-\infty$ to $\infty$ we find that (7.2) holds. Now if we integrate the third equation in (7.4) from $-\infty$ to $\infty$ we find that

$$\int_{-\infty}^{\infty} Y_2 f_2(T) dz = 2\theta^* \quad (7.5)$$

Finally, if we integrate the first equation in (7.4) we conclude that $K_1 = Q_1 + 2Q_2$. This implies that $K_2 = (C, \theta^*)$.

Note that if $K_2 \neq (B, \theta^*)$ then we have proved Theorem 1. That is, there exists a solution of (1.9) which satisfies
\((T,Y_1,Y_2)(\pi) = (0,1,1)\) and \((T,Y_1,Y_2)(\pi^\ast) = (Q_1 + 2Q_2,0,0)\).

Hence, we now assume that \(K_2 = (B,\theta^\ast)\).

Let \(t_k\) be as in the beginning of this section. That is, \(t_k(0)\) converges to \(\pi^\ast\) as \(k \to \infty\). For each \(k\), reparametrize the trajectories \(t_k(z)\) so that instead of \(-\pi < z < \pi\) we have \(0 < s \leq 1\). Call the new curves \(\Lambda_k(s)\). As \(k \to \infty\), the curves \(\Lambda_k(s)\) will converge to a curve which we denote as

\[\Lambda(s) = (T(s),q(s),Y_1(s),p_1(s),Y_2(s),p_2(s),\theta(s)) = (\gamma(s),\theta(s))\).

A key result is

**Proposition 7.3:** Suppose that \(K_2 = (B,\theta^\ast)\). Then there exists \(0 < s_1 < s_2 < s_3 < s_4 < 1\) such that

a) \(\gamma(s) = A\) for \(0 < s < s_1\)
b) \(\theta(s) = \theta^\ast\) for \(s_1 < s < s_2\)
c) \(\gamma(s) = B\) for \(s_2 < s < s_3\)
d) \(\theta(s) = \theta^\ast_1\), a constant, for \(s_3 < s < s_4\)
e) \(\gamma(s_4) = (C,\theta^\ast)\)

**Proof:** We have already proved that there exists \(0 < s_1 < s_2\) such that \(\gamma(s) = A\) for \(0 < s < s_1\), \(\theta(s) = \theta^\ast\) for \(s_1 < s < s_2\), and \(\gamma(s_2) = B\). It remains to determine the behavior of \(\Lambda(s)\) for \(s > s_2\). To do this we must first study the local behavior of the flow (3.1) near the points \((B,\theta)\), \(\theta_1 < \theta < \theta_0\).

Suppose, for the moment, that \(\varepsilon = 0\) in (3.1). The for each \(\theta \in (\theta_1,\theta_0)\), \((B,\theta)\) is a critical point. The local behavior near \((B,\theta)\) is determined by linearizing (3.1) at \((B,\theta)\).
The linearized equations at \((B, \theta)\) for \(\varepsilon = 0\) has two positive

eigenvalues, two negative eigenvalues, and a triple zero eigenvalue. The
triple zero eigenvalue is because near \((B, \theta)\) there is a three dimensional
subset of critical points. This subset of critical points is of the form

\[
\left\{ (T, q, Y_1, p_1, Y_2, p_2, \theta) : Y_1 = q_1 = p_1 = p_2 = 0 \right\}.
\]

Through \((B, \theta)\) there is a two dimensional stable and a two dimensional
unstable manifold.

Choose new coordinates so that in the new coordinates (3.1), with \(\varepsilon = 0\),
becomes

\[
\begin{align*}
x'_1 &= -\lambda_1 x_1 + g_1(\eta) \\
x'_2 &= -\lambda_2 x_2 + g_2(\eta) \\
y'_1 &= g_3(\eta) \\
y'_2 &= g_4(\eta) \\
z'_1 &= \lambda_3 z_1 + g_5(\eta) \\
z'_2 &= \lambda_4 z_2 + g_6(\eta) \\
\theta' &= 0
\end{align*}
\]

(7.7)

Here, \(\eta = (x_1, x_2, y_1, y_2, z_1, z_2, \theta)\) and \(g_i(\eta) = 0 \parallel \eta \parallel\) for each \(i\). Moreover we
may assume that for \(\parallel \eta \parallel\) sufficiently small, \(\theta_1 \leq \theta \leq \theta_0\),

\[
\begin{align*}
g_1(x_1, x_2, 0, 0, 0, 0, \theta) &= g_2(x_1, x_2, 0, 0, 0, 0, \theta) = 0 , \\
g_3(0, 0, y_1, y_2, 0, 0, \theta) &= g_4(0, 0, y_1, y_2, 0, 0, \theta) = 0 , \\
g_5(0, 0, 0, 0, z_1, z_2, \theta) &= g_6(0, 0, 0, 0, z_1, z_2, \theta) = 0 . \tag{7.8}
\end{align*}
\]
We assume that in the new coordinates \((B, \theta) = (0,0,0,0,0,0,\theta)\). Choose \(\delta_0 > 0\) so that (7.7) and (7.8) hold for \(|\eta| \leq \delta_0\). For \(\delta < \delta_0\) let

\[
N_\delta = \{ \eta : |x_1| \leq \delta, |x_2| \leq \delta, |z_1| \leq \delta_0, |z_2| \leq \delta_0, |Y|^2 \leq \delta(|z|^2 + 1), \theta_1 \leq \theta \leq \theta_0 \}.
\]

Here, \(|Y|^2 = y_1^2 + y_2^2\) and \(|z| = z_1^2 + z_2^2\).

We now show that for each \(\delta < \delta_0, \delta_0\) sufficiently small, trajectories can only leave \(N_\delta\) through the sides \(|z_1| = \delta_0\) or \(|z_2| = \delta_0\). To prove this we show that on the other sides of \(N_\delta\) the vector field given by (7.7) points into \(N_\delta\).

If \(x_1 = \delta\), then \(x'_1 = -\lambda_1 x_1 + g(\eta) < 0\) if \(\delta_0\) is sufficiently small. If \(x_1 = -\delta\), then \(x'_1 = -\lambda_1 x_1 + g_1(\eta) > 0\) if \(\delta_0\) is sufficiently small. A similar analysis shows that if \(|x_2| = \delta|\), there the vector field given by (7.7) points into \(N_\delta\).

Now suppose that \(|Y|^2 = \delta(|z|^2 + 1)\). Let

\[
n = (0,0,2y_1,2y_2,-2\delta^2 z_1,-2\delta^2 z_2,\theta)
\]

be a vector outwardly normal to \(N_\delta\) at \(\eta\). If \(x\) is the vector given by the right side of (7.7), then

\[
n \cdot x = 2Y_1 g_3(\eta) + 2Y_2 g_4(\eta) - 2\delta^2 \lambda_3 z_1 - 2\delta^2 \lambda_4 z_2 - 2\delta^2 g_3(\eta) - 2\delta^2 g_4(\eta)
\]

< 0

if \(\delta_0\) is sufficiently small. This is what we wished to prove.

Let

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\[ \mathcal{E}_\delta = \{ \eta : |z_1| = \delta_\eta \text{ or } |z_2| = \delta_\eta \} \cap N_\delta. \]

We have shown that any solution of (3.1) for \( \varepsilon = 0 \) which lies in \( N_\delta \) for some time can only leave \( N_\delta \) through \( \mathcal{E}_\delta \). By continuity of the flow with respect to the parameter \( \varepsilon \), we conclude that given \( \delta < \delta_0 \) there exists \( \varepsilon_\delta \) such that if \( 0 < \varepsilon < \varepsilon_\delta \) and \( \eta(z) \) is a solution of (3.1) which lies in \( N_\delta \) for some \( z \), then \( \eta(z) \) can only leave \( N_\delta \) through \( \mathcal{E}_\delta \).

We are assuming that \( \lim_{z \to \infty} \eta(z) = (B, \theta^*) \). Hence, for each \( \delta \), \( \eta(z) \in N_\delta \) for \( z \) sufficiently large. Another way to say this is that given \( \delta \), \( \Lambda(s) \in N_\delta \) for \( s \) sufficiently close to \( s_2 \). This implies that given \( \delta \), there exists \( K_\delta \) such that if \( k > K_\delta \), then \( \Lambda_k(s) \in N_\delta \) for some \( s \). Now \( \varepsilon_\delta(z) \to C_2 \) as \( z \to \infty \) for each \( \varepsilon > 0 \). Therefore, if \( k > K_\delta \), then \( \Lambda_k(s) \) must leave \( N_\delta \). By choosing \( K_\delta \) larger, if necessary, we conclude that if \( k > K_\delta \), then \( \Lambda_k(s) \) leaves \( N_\delta \) through \( \mathcal{E}_\delta \). For \( k > K_\delta \), choose \( s_k \) so that \( \Lambda_k(s_k) \in \mathcal{E}_\delta \). The \( \Lambda_k(s_k) \) will converge to some point \( \Lambda_0 \in \mathcal{E}_k \). However, \( \mathcal{E}_k \) lies on \( W(B, \theta)^u \), the unstable manifold of (3.1) with \( \varepsilon = 0 \) at the point \( (B, \theta) \) for some \( \theta \in (\theta_1, \theta_0) \). Let

\[ s_3 = \sup \{ s > s_2 : \Lambda(s) = (B, \theta) \text{ for some } \theta \}, \]

and choose \( \sigma_0 \) so that \( \Lambda(\sigma_0) = \Lambda_0 \). Clearly, \( \sigma_0 > s_3 \). Let \( \theta^* = \theta(s_3) \). We have now shown that \( \gamma(s) = B \) for \( s_2 < s < s_3 \) and \( \Lambda(s) \in W(B, \theta)^u \) for \( s \in (s_3, \sigma_0) \).

We must now analyze what happens for \( s > \sigma_0 \).

Let \( \Gamma(z) \) be the solution of (3.1) with \( \varepsilon = 0 \) and \( \theta = \theta^* \) such that \( \Gamma(0) = \Lambda_0 \). Because \( Y_{1\varepsilon}(z) \) is decreasing for each \( \varepsilon \) and \( z \), (Lemma 3.7), we conclude that the \( Y_1 \) component of \( \Gamma(z) \) is decreasing. Hence, there
exists a critical point \( K_3 \) of (3.1) with \( \varepsilon = 0 \) and \( \theta = \theta^* \) such that \( \lim_{z \to \infty} \Gamma(z) = K_3 \). We wish to show that \( K_3 = (C, \theta^*) \). This will complete the proof of Proposition 7.3.

Suppose that \( \Gamma(z) = (T, q, Y_1, p_1, Y_2, p_2, \theta)(z) \). Since \( \Gamma(z) \in W(B, \theta^*)^U \), the \( Y_1 \) component of \( B \) is zero, and \( Y_1(z) \) is nonincreasing, it follows that \( Y_1(z) = 0 \) for all \( z \). Hence, \( (T(z), Y_2(z)) \) satisfy the equations

\[
\begin{align*}
T' - \theta^* T' + Q_2 Y_2 f_2(T) &= 0 \\
\frac{d}{dz} Y_2' - \theta^* Y_2' - Y_2 f_2(T) &= 0 \\
\lim_{z \to -\infty} (T(z), Y_2(z)) &= (Q_1, 2) \\
\lim_{z \to +\infty} (T(z), Y_2(z)) &= (r_1, r_2)
\end{align*}
\tag{7.9}
\]

for some \( r_1, r_2 \). Of course, we wish to prove that \( (r_1, r_2) = (Q_1 + 2Q_2, 0) \).

For \( Q_1 \leq T \leq T_2, f_2(T) = 0 \). Hence, while \( Q_1 \leq T(z) \leq T_2, T(z) \) satisfies

\[
T' - \theta^* T' = 0 .
\]

If we integrate this equation from \( -\infty \) to \( z \) we find that

\[
T' = \theta^* (T - Q_1) > 0 .
\]

Hence, \( T(z) \) is strictly increasing while \( Q_1 \leq T(z) \leq T_2 \). This implies that \( r_1 > T_2 \). This, however, implies that \( r_2 = 0 \).

Now integrate the second equation in (7.9) to obtain

\[
\theta^* = \frac{1}{2} \int_{-\infty}^{\infty} Y_2 f_2(T) dz .
\]

Integrate the first equation in (7.9) to obtain
\[-\theta_*(r_1 - Q_1) = -Q_2 \int_{-\infty}^{\infty} Y_2 f_2(T)dz = -2\theta_1 Q_2,
\]
or

\[r_1 = Q_1 + 2Q_2.\]

The proof of Proposition 7.3 is now complete.

The proof of Theorem 1 now follows. Suppose that \( \theta^1 < \theta^{12} \) where \( \theta^1 \) and \( \theta^1 \) are as in Theorem 1. From Lemma 7.2, either \( K_2 = (B, \theta^2) \) or \( K_2 = (C, \theta^2) \). If \( K_2 = (C, \theta^2) \) then \( \psi(z) \) corresponds to the desired solution. We now prove that if \( \theta^1 < \theta^{12} \) then it is impossible to have \( K_2 = (B, \theta^2) \).

If \( K_2 = (B, \theta^2) \), then we conclude from Proposition 7.3 that there exists a solution of (1.14) with speed \( \theta^2 \), and there exists a solution of (1.15) with speed \( \theta^2 \). Moreover \( \theta^2 < \theta^* \). From the definition of \( \theta^1 \) and \( \theta^{12} \) this is impossible if \( \theta^1 < \theta^{12} \). The proof of Theorem 1 is now complete.
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