

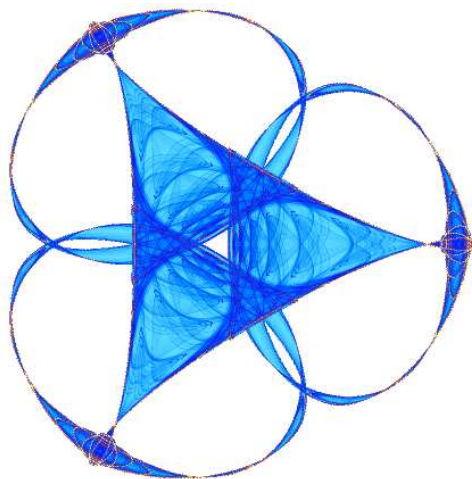
**THE FINITE ELEMENT METHOD FOR A CLASS OF  
DEGENERATE ELLIPTIC EQUATIONS**

By

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# THE FINITE ELEMENT METHOD FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. Consider the degenerate elliptic operator  $\mathcal{L}_\delta := -\partial_x^2 - \frac{\delta}{x^2}\partial_y^2$  on  $\Omega := (0, 1) \times (0, l)$ , for  $\delta > 0, l > 0$ . We prove well-posedness and regularity results for the second-order degenerate elliptic equation  $\mathcal{L}_\delta u = f$  in  $\Omega$ ,  $u|_{\partial\Omega} = 0$  using weighted Sobolev spaces  $K_a^m$ . In particular, by a proper choice of the parameters in the weighted Sobolev spaces  $K_a^m$ , we establish the existence and uniqueness of the solution. In addition, we show that there is no loss of  $K_a^m$ -regularity for the solution of the equation. We then provide an explicit construction of a sequence of finite dimensional subspaces  $V_n$  for the finite element method, such that the optimal convergence rate is attained for  $u_n \in V_n$ , i.e.,  $\|u - u_n\|_{H^1(\Omega)} \leq C \dim(V_n)^{-\frac{m}{2}} \|f\|_{H^{m-1}(\Omega)}$  with  $C$  independent of  $f$  and  $n$ .

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## 1. INTRODUCTION

Let  $\Omega$  be the rectangular domain  $(0, 1) \times (0, l)$  for  $l > 0$ , and define the operator  $\mathcal{L}_\delta$  as follows

$$\mathcal{L}_\delta := -\partial_x^2 - \frac{\delta}{x^2}\partial_y^2, \quad \delta > 0.$$

We then consider a class of degenerate elliptic equations on  $\Omega$  corresponding to  $\mathcal{L}_\delta$  with the Dirichlet boundary condition:

$$(1) \quad \begin{cases} \mathcal{L}_\delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Denote by  $u$  the solution of Equation (1). Let  $\Omega_\xi := (0, \xi) \times (0, l)$  and  $S_\xi := \Omega \setminus \Omega_\xi$  be subsets of  $\Omega$ , depending on  $\xi$ . Denote by  $P_r \subset \Omega$ , an arbitrary open subset containing the neighborhoods of the vertices  $(1, 0)$  and  $(1, l)$ . Then, the strong

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ellipticity of the operator  $\mathcal{L}_\delta$  on  $S_\xi \setminus P_r$  implies that there exists a unique solution  $u \in H^{m+1}(S_\xi \setminus P_r) \cap \{u|_{\partial\Omega} = 0\}$  for any  $f \in H^{m-1}(\Omega)$  [24, 41].

However, it is well known that this result does not extend to the entire domain  $\Omega$  in general, because of the loss of ellipticity at  $x = 0$  and the possible singularities of the solution arising from the corners. In fact, it is generally impossible to find a solution  $u \in H^{m+1}(\Omega)$  for large  $m$ , even if the given data is smooth  $f \in C^\infty(\Omega)$ . Various techniques have been used to investigate different degenerate elliptic equations in the papers of Boimatov [16], Felli and Schneider [25], French [26], Langlais [33], and others. In this paper, inspired by the methods of Babuska and Aziz [8], Bacuta, Nistor, and Zikatanov [14, 15], Kondratiev [30], we shall study the well-posedness and regularity of the solution of Equation (1) in terms of some weighted Sobolev space  $K_a^m$ . The well-posedness of the solution  $u$  in weighted Sobolev spaces will be proved, and consequently, we shall show that there is no loss of  $K_a^m$ -regularity for the solution of Equation (1).

Another main result of this paper is regarding the numerical approximation by the finite element method (FEM). Let  $V_n$  be a sequence of finite dimensional subspaces for the FEM. Denote by  $u_n \in V_n$  the corresponding discrete solution. Then, we shall provide a simple, explicit way to construct a sequence of finite dimensional subspaces  $V_n \subset H_0^1(\Omega)$ , such that  $u_n$  satisfies

$$\|u - u_n\|_{H^1(\Omega)} \leq C \dim(V_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)},$$

where  $f \in H^{m-1}(\Omega) \cap K_{-1+\epsilon}^{m-1}(\Omega)$  is arbitrary and  $C$  is a constant that depends on  $\Omega$  and  $m$ , but not  $n$  or  $f$ . Namely, one can recover the optimal rate of convergence that is expected for smooth solutions [8, 17, 19].

In addition, according to the result of Babuska and Aziz [9], the maximum angle of the triangles in the triangulation for the FEM should be bounded away from  $\pi$ , such that a uniform error estimate can be obtained in the usual Sobolev spaces  $H^m$  on each triangle. Otherwise, the energy norm of the error  $|u - u_n|_{H^1}$  on  $\Omega$  might be difficult to control. In our construction of subspaces  $V_n$ , however, thin triangles that violate this maximum-angle condition will appear. In fact, the maximum angle in the triangles will keep increasing with  $\pi$  as the limit. We shall show that the difficulty for the estimates in this case can be overcome by a homogeneity argument in weighted Sobolev spaces.

In Section 2, we shall introduce our weighted Sobolev spaces  $K_a^m(\Omega)$  and some notations that will be used throughout this paper. The properties of  $K_a^m(\Omega)$  that are important for the regularity of the solution will be studied in detail.

In Section 3, we shall prove the well-posedness and regularity of the solution in  $K_a^m(\Omega)$ . Denote by  $(v_1, v_2)$  the inner product of  $v_1, v_2 \in H^0(\Omega) = L^2(\Omega)$ . The corresponding weak solution  $u \in K_1^1(\Omega) \cap \{u|_{\partial\Omega} = 0\}$  of Equation (1) is defined by

$$a(u, v) := (\partial_x u, \partial_x v) + \delta \left( \frac{1}{x} \partial_y u, \frac{1}{x} \partial_y v \right) = \langle f, v \rangle, \quad \forall v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega} = 0\}.$$

Furthermore, we will prove that, for  $m \geq 0$ , the operator  $\mathcal{L}_\delta = -\partial_x^2 - \frac{\delta}{x^2} \partial_y^2$  is an isomorphism from  $K_1^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$  to  $K_{-1}^{m-1}(\Omega)$  and Fredholm  $K_{1+\epsilon}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1+\epsilon}^{m-1}(\Omega)$  iff  $\epsilon$  is away from a specific countable set of values.

In Section 4, we will analyze the numerical solution  $u_n$  for Equation (1). Explicitly, we will look for  $u_n \in V_n$  satisfying

$$a(u_n, v_n) = \langle f, v_n \rangle, \quad \forall v_n \in V_n.$$

To better explain our results, let us recall some basic results in classical approximation theory [17, 19, 22]. Denote by  $\Gamma = (T_i)$  the triangulation of  $\Omega$  with triangles. Let  $V = V(\Gamma, m + 1)$  be the finite element space associated to the degree  $m$  Lagrange triangle [19], such that  $V$  consists of polynomials of degree  $\leq m$  on each triangle  $T_i \in \Gamma$ , in which the nodes are obtained by taking points with barycentric coordinates in  $m^{-1}\mathbb{Z}$ . Let  $u_V \in V$  be the numerical solution of Equation (1). For any continuous solution  $u$ , denote by  $u_I \in V(\Gamma, m + 1)$  the interpolation associated to  $u$ , which is uniquely determined by the condition  $u(x_i) = u_I(x_i)$  for any node. Also, a symmetric bilinear form  $a(\cdot, \cdot)$  induces an equivalent norm  $\|\cdot\|_a$  on a normed space, provided that  $a(\cdot, \cdot)$  is both continuous and coercive on this space. As a result from Section 3, we shall show that  $\|\cdot\|_{K_1^1}$  and  $\|\cdot\|_a$  are equivalent norms on  $\Omega$ . Therefore, based on *Céa's Lemma* [17], we have the following inequality

$$\|u - u_V\|_{K_1^1(\Omega)} \leq C\|u - u_I\|_{K_1^1(\Omega)}.$$

The constant  $C$  in the expression is independent of the triangulation  $\Gamma$  and the solution  $u$ . From our estimates on the interpolation error  $\|u - u_I\|_{K_1^1(\Omega)}$ , we shall construct a class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  of partitions  $\Gamma$  of  $\Omega = (0, 1) \times (0, l)$ ,  $l > 0$ , such that

$$\|u - u_V\|_{H^1(\Omega)} \leq C \dim(V)^{-m/2} \|f\|_{H^{m-1}(\Omega)}, \quad \forall f \in H^{m-1}(\Omega) \cap K_{-1+\epsilon}^{m-1}(\Omega).$$

More details about the notation and proof will be given in Section 4.

In Section 5, numerical results will be presented for the operator  $\mathcal{L}_1 := -\partial_x^2 - \frac{1}{x^2}\partial_y^2$  on  $\hat{\Omega} := (0, 1) \times (0, 10)$  with a smooth  $f$ . We will compare the rates of convergence of the numerical solutions for different mesh sizes. The convergence history will verify our theoretical prediction and demonstrate the efficiency of our technique to approximate the solution.

## 2. THE WEIGHTED SOBOLEV SPACES $K_a^m$

As explained above, weighted Sobolev spaces are convenient for the problem since the solution  $u$  may not belong to  $H^{m+1}(\Omega)$  for large  $m$ , even if  $f \in \mathcal{C}^\infty$ . In this section, we shall introduce the weighted Sobolev spaces  $K_a^m(\Omega)$  and establish some properties of them, which are useful for the study of the boundary value problem (1). One can refer to [10, 21, 30, 31, 32] for more information and additional results on Sobolev spaces with weights.

**2.1. Notation.** As usual, the regular Sobolev spaces  $H^m$  on the domain  $\Omega$  for  $m \in \{0, 1, 2, \dots\}$  are defined as follows [24],

$$H^m(\Omega) = \{v \in L^2(\Omega), \partial^\alpha v \in L^2(\Omega), \forall |\alpha| \leq m\},$$

with the multi-index  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ ,  $|\alpha| := \alpha_1 + \alpha_2$  and  $\partial^\alpha := \partial^{\alpha_1} \partial^{\alpha_2}$ . The  $H^m$ -norm of any  $v \in H^m(\Omega)$  is defined by

$$\|v\|_{H^m(\Omega)}^2 := \sum_{|\alpha| \leq m} \|\partial^\alpha v\|_{L^2(\Omega)}^2,$$

where the  $L^2$ -norm is

$$\|v\|_{L^2(\Omega)}^2 = \int_{\Omega} |v|^2 dx.$$

For  $s \in \mathbb{R}_+$ , we define the spaces  $H^s(\Omega)$  by interpolation [1].

Let  $X(x, y) \in \Omega$  be an arbitrary point in the domain  $\Omega$ . To define weighted Sobolev spaces  $K_a^m$  on the domain  $\Omega$ , we denote by  $r_1(x, y) = \text{dist}(X, (1, 0))$  and  $r_2(x, y) = \text{dist}(X, (1, l))$ , the distances from  $X(x, y)$  to the vertices  $(1, 0)$ ,  $(1, l)$  of  $\Omega$  respectively (Fig. 2.1). The above distances are used in the definition of the spaces  $K_a^m$  and play an important role in reflecting the properties of the solution of Equation (1) near the vertices  $(1, 0), (1, l)$  [13]. Denote by  $S = \{(1, 0), (1, l), [0, y], 0 \leq y \leq l\}$  the set containing the vertices and the degenerate boundary of the domain. Then, we define the weighed Sobolev spaces  $K_a^m$  on  $\Omega$  as follows.

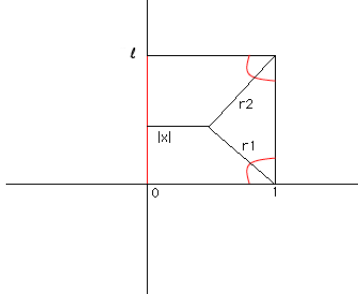


Fig. 2.1 Domain  $\Omega$  and the position of  $X$

**Definition 2.1.** Let  $R := \{x < \frac{1}{2}\} \cup \{r_i < \frac{1}{4}, i = 1, 2\} \subset \Omega$  be a subset of the domain. Let  $r(x, y)$  be a positive smooth function, such that  $r$  stands for the distance from  $X(x, y)$  to the set  $S$  for any  $X(x, y) \in R$ , while  $r$  satisfies that  $r \geq \frac{1}{4}$  for any point in the region  $\Omega \setminus R$ . Therefore,  $r = x$  in the neighborhood of the degenerate boundary;  $r = r_1$  and  $r = r_2$  in the neighborhoods of  $(1, 0)$  and  $(1, l)$  respectively. Then, for  $i, j, m \in \{0, 1, 2, \dots\}$ , the  $m$ th weighted Sobolev space is

$$K_a^m(\Omega) := \{v \in L_{loc}^2(\Omega), r^{-a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j v \in L^2(\Omega), i + j \leq m\}.$$

The  $K_a^m$ -norm for any function  $v \in K_a^m(\Omega)$  is

$$\|v\|_{K_a^m(\Omega)}^2 := \sum_{i+j \leq m} \|r^{-a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j v\|_{L^2(\Omega)}^2.$$

By  $L_{loc}^2(\Omega)$ , we mean the space that includes all the functions on  $\Omega$ , whose restriction to any compact subset  $A \subset \Omega$  is in  $L^2(A)$ . Also, we denote by  $H_c^m(\Omega)$  the space of functions that belong to  $H^m(A)$  for any compact subset  $A \subset \Omega$ . In addition, we set  $\Omega_\xi := (0, \xi) \times (0, l) \subset \Omega$ , and  $S_\xi := \Omega \setminus \Omega_\xi$  to be two particular subsets that will be used very often in the text.

Since  $r^{-a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j v \in L^2(\Omega)$  for  $\forall v \in K_a^m(\Omega)$ , the completeness of the space  $K_a^m(\Omega)$  then follows the completeness of  $L^2(\Omega)$  and integration by parts with standard arguments in [1, 24]. Moreover,  $K_a^m(\Omega)$  is a Hilbert space associated with the inner product

$$(u, v)_{K_a^m} = \sum_{i+j \leq m} \int_{\Omega} r^{-2a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u (r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j v.$$

Recall that for regular Sobolev spaces  $H^m(\Omega)$  [17, 24, 41],  $m > 0$ , the space  $H^{-m}(\Omega)$  is the dual space of  $H_0^m(\Omega)$ , where  $H_0^m(\Omega)$  is defined as follows,

$$H_0^m(\Omega) = \{v \in H^m(\Omega), \partial^i v|_{\partial\Omega} = 0, i < m\}.$$

Similarly, we denote by  $K_a^{-m}(\Omega) \cong (K_a^m(\Omega) \cap \{v|_{\partial\Omega} = 0\})'$  the dual space of  $K_a^m(\Omega) \cap \{v|_{\partial\Omega} = 0\}$  with respect to the pivot space  $L^2(\Omega)$ ,

$$\|w\|_{K_a^{-m}(\Omega)} := \sup_{v \in K_a^m(\Omega) \cap \{v|_{\partial\Omega} = 0\}} \frac{\int_{\Omega} vw}{\|v\|_{K_a^m(\Omega)}}.$$

We also agree that if  $\|v\|_{K_a^m(\Omega)} = \infty$ , then  $v$  is not in  $K_a^m(\Omega)$ .

*Remark 2.2.* The parameters in the spaces  $K_a^m(\Omega)$  are related to the distance to the vertices  $(1,0)$ ,  $(1,l)$  and to the degenerate boundary  $x = 0$ , where one will see singularities of the solution. It is clear that Equation (1) is strongly elliptic in the small neighborhoods of  $(1,0)$  and  $(1,l)$  with the coefficient frozen in  $x$  at those points. In fact, near the vertices  $(1,0)$ ,  $(1,l)$ , the spaces  $K_a^m(\Omega)$  are the usual weighted Sobolev spaces for elliptic equations on corner singularities [8, 14, 30]; while in the neighborhood of  $x = 0$ ,  $K_a^m(\Omega)$  can be considered as the usual weighted spaces in polar coordinates by setting  $r = x$  and  $\theta = y$ . Our weighted Sobolev spaces are invented in the way that is based on the property of the operator  $\mathcal{L}_{\delta}$  and the geometry of the domain. For this reason, some properties of  $K_a^m(\Omega)$  are important for the study of the regularity of the solution and for the construction of the finite subspaces in the FEM.

Throughout this paper, our notation will follow what we have introduced in this subsection. Based on the definition of the weighted Sobolev space, we shall give some observations and lemmas for  $K_a^m(\Omega)$ .

**2.2. Lemmas.** We here summarize several properties for the spaces  $K_a^m(\Omega)$  that are useful for the development of the theorems in Section 3 and Section 4. Most of the properties are derived from straightforward calculation based on the definition of  $K_a^m(\Omega)$ . For the reason that most of functions we are interested in are defined on  $\Omega$ , we shall omit  $\Omega$  in the notation  $K_a^m(\Omega)$  and  $H^m(\Omega)$ , which are used often below, to simplify the expression. Therefore,  $K_a^m = K_a^m(\Omega)$  and  $H^m = H^m(\Omega)$  for the rest of this paper. Moreover,  $a \simeq b$  means that there exist constants  $C_1, C_2 > 0$ , such that  $C_1 b \leq a \leq C_2 b$ . An isomorphism in the text means a continuous bijection between two Banach spaces.

The first lemma claims an alternative definition for the space  $K_a^m$ .

**Lemma 2.3.** *Denote by  $P_x \subset \Omega$ ,  $P_{r_1} \subset \Omega$  and  $P_{r_2} \subset \Omega$  the small neighborhoods of  $x = 0$  and the vertices  $(1,0)$ ,  $(1,l)$  respectively. Then, for  $P_r = P_{r_1} \cup P_{r_2}$ , we have*

$$K_a^m = \{u \in H_c^m(\Omega), r^{-a}(r\partial_r)^i \partial_{\theta}^j u \in L^2(P_r), r^{-a}(r\partial_r)^i \partial_y^j u \in L^2(P_x), \forall i + j \leq m\}.$$

*Proof.* On the outside of  $P = P_x \cup P_r$ ,  $u \in H^m(\Omega \setminus P)$  is equivalent to  $u \in K_a^m(\Omega \setminus P)$ , since  $r, r_1, r_2$  and  $x$  are all bounded from above and 0 by the definition.

On the region  $P_x$ , we notice  $r = x$  and  $r_1, r_2$  are bounded. Therefore

$$r^{-a}(r_1 r_2)^{i+j} (x\partial_x)^i \partial_y^j u \simeq r^{-a}(r\partial_r)^i \partial_y^j u$$

in the norm of  $L^2$ .

On  $P_{r_i}$ ,  $i = 1, 2$ , we freeze the coefficient of  $\mathcal{L}_{\delta}$  in  $x$  at the vertex and change the variables  $x, y$  into  $r, \theta$  for the polar coordinates centered at the vertex,

$$\begin{aligned} \partial_x &= \cos(\theta)\partial_r - \frac{\sin(\theta)}{r}\partial_{\theta} \\ \partial_y &= \sin(\theta)\partial_r + \frac{\cos(\theta)}{r}\partial_{\theta}, \end{aligned}$$

where  $r = r_1$  on  $P_{r_1}$  and  $r = r_2$  on  $P_{r_2}$ . The proof then follows that

$$r^{-a}r^{i+j}(\cos(\theta)\partial_r - \frac{\sin(\theta)}{r}\partial_\theta)^i(\sin(\theta)\partial_r + \frac{\cos(\theta)}{r}\partial_\theta)^j u \simeq r^{-a}(r\partial_r)^i\partial_\theta^j u$$

in  $L^2$ -norm on  $P_r$ .  $\square$

**Lemma 2.4.** *The function  $r^{-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b$  is bounded on  $\Omega$ .*

*Proof.* On the region  $P_r$  where  $r = r_1$  or  $r = r_2$ , we follow the notation in Lemma 2.3, and change to the polar coordinates centered at the vertices. Since  $(r\partial_r)^k r^b = b^k r^b$  and  $\partial_\theta r = 0$ , we have

$$\begin{aligned} |r^{-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b| &\leq C_1|r^{-b}r^{i+j}\partial_x^i\partial_y^j r^b| \\ &= C_1|r^{-b}r^{i+j}(\cos(\theta)\partial_r - \frac{\sin(\theta)}{r}\partial_\theta)^i(\sin(\theta)\partial_r + \frac{\cos(\theta)}{r}\partial_\theta)^j r^b| \\ &\leq C|r^{-b}\sum_{k+h\leq i+j}(r\partial_r)^k\partial_\theta^h r^b| \\ &\leq C|r^{-b}\sum_{k\leq i+j}(r\partial_r)^k r^b| = C|\sum_{k\leq i+j}b^k|. \end{aligned}$$

Therefore  $r^{-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b$  is bounded on  $P_r$ .

On the region  $P_x$  where  $r = x$ , we have  $(x\partial_x)^i r^b = b^i r^b$  and  $\partial_y r = 0$ . Thus, the proof follows

$$\begin{aligned} |r^{-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b| &\leq C|r^{-b}(x\partial_x)^i\partial_y^j r^b| \\ &\leq C|r^{-b}(x\partial_x)^i r^b| = C|b^i|. \end{aligned}$$

As for  $\Omega \setminus P$ , the complement of  $P = P_x \cup P_r$ , since  $r$  is smooth and bounded away from 0, the function  $r^{-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b$  is bounded. Thus, the proof is completed.  $\square$

Consequently, Lemma 2.4 shows

**Lemma 2.5.** *For the spaces  $K_a^m$ ,  $r^b K_a^m = K_{a+b}^m$ , where  $r^b K_a^m = \{r^b v, \forall v \in K_a^m\}$ . Therefore, the multiplication by  $r^b$  defines an isomorphism  $K_a^m \rightarrow K_{a+b}^m$ .*

*Proof.* Let  $v \in K_a^m$  and  $w = r^b v$ . Then  $|r^{-a}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j v| \in L^2$ , for  $i+j \leq m$ . Moreover, we have

$$\begin{aligned} |r^{-a-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j w| &= |r^{-a-b}(r_1r_2)^{i+j}(x\partial_x)^i\partial_y^j r^b v| \\ &= r^{-a-b}(r_1r_2)^{i+j} \sum_{k\leq i, h\leq j} (x\partial_x)^k \partial_y^h r^b (x\partial_x)^{i-k} \partial_y^{j-h} v \\ &\leq C \sum_{k\leq i, h\leq j} |r^{-a}(r_1r_2)^{i+j-k-h}(x\partial_x)^{i-k} \partial_y^{j-h} v| \in L^2. \end{aligned}$$

The last inequality is the consequence of Lemma 2.4. Thus,  $r^b K_a^m$  is continuously embedded in  $K_{a+b}^m$ . On the other hand, because this embedding holds for any real number  $b$ , we have the opposite

$$K_{a+b}^m = r^b r^{-b} K_{a+b}^m \subset r^b K_a^m,$$

which completes the proof.  $\square$

From a direct verification based on the definitions of  $H^m$  and  $K_a^m$ , we can also derive the following lemma.

**Lemma 2.6.** *We have  $K_0^0 = L^2$  and for  $m' \leq m$ ,  $a' \leq a$ ,*

1.  $K_a^m \subset K_{a'}^{m'}$
2.  $\|u\|_{K_{a'}^{m'}(\Omega_\xi)} \leq \xi^{a-a'} \|u\|_{K_a^m(\Omega_\xi)}, \forall u \in K_a^m, \xi < \frac{1}{2}$ .

*Proof.* The first argument is the result of the inequality below. For  $\forall v \in K_a^m$  and  $m' \leq m$ ,  $a' \leq a$

$$\sum_{i+j \leq m'} \|r^{-a'}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2}^2 \leq C \sum_{i+j \leq m} \|r^{-a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2}^2.$$

On  $\Omega_\xi = (0, \xi) \times (0, l)$ ,  $\xi < \frac{1}{2}$ , we have  $r = x$ . Then the second argument in this lemma follows

$$\begin{aligned} \|u\|_{K_{a'}^{m'}(\Omega_\xi)}^2 &= \sum_{i+j \leq m'} \|r^{-a'}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2(\Omega_\xi)}^2 \\ &\leq \sum_{i+j \leq m} \|r^{-a'}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2(\Omega_\xi)}^2 \\ &= \xi^{2(a-a')} \sum_{i+j \leq m} \|\xi^{a'-a} r^{-a'}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2(\Omega_\xi)}^2 \\ &\leq \xi^{2(a-a')} \sum_{i+j \leq m} \|r^{-a}(r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u\|_{L^2(\Omega_\xi)}^2 \\ &= \xi^{2(a-a')} \|u\|_{K_a^m(\Omega_\xi)}^2. \end{aligned}$$

□

We note that the parameters in  $K_a^m$  only depend on the distances to certain parts of the boundary. From Lemma 2.3, we claim that the  $H^m$ - and  $K_a^m$ -norm are equivalent on any compact subset  $A \subset \Omega$ .

**Lemma 2.7.** *Let  $\xi$  be a positive number, and let  $G \subset \Omega$  be an open subset such that  $r \geq \xi$  on  $G$ . Then  $\|u\|_{H^m(G)} \leq M_1 \|u\|_{K_a^m(G)}$  and  $\|u\|_{K_a^m(G)} \leq M_2 \|u\|_{H^m(G)}$ , for any  $u \in H^m(G)$ , where the constants  $M_1$  and  $M_2$  only depend on  $\xi$  and  $m$ .*

*Proof.* As stated in Lemma 2.3, the parameters  $r, r_1, r_2$  and  $x$  in the expression of  $K_a^m$ -norm are all bounded from 0 on  $G$ , since the distance from  $G$  to the set  $S$  has a lower bound  $\xi$ . Then, the proof follows the the definitions of the spaces and the norms involved. □

The following lemma states the relation between  $H^m$  and  $K_a^m$  on  $\Omega$ .

**Lemma 2.8.** *On  $\Omega$ ,  $\|u\|_{H^m} \leq M_1 \|u\|_{K_a^m}$ , and  $\|u\|_{K_a^m} \leq M_2 \|u\|_{H^m}$  for  $a \leq 0$ , where  $M_1$  and  $M_2$  depend on  $m$  and  $a$ .*

*Proof.* This lemma is basically a consequence of the definitions of those norms. Recall the definitions of  $P, P_x$  and  $P_r$ . Then, based on  $i+j \leq m$  and  $r = x$  on  $P_x$ ,

$$\begin{aligned} \|u\|_{K_a^m(P_x)}^2 &\geq C_1 \sum_{i+j \leq m} \|r^{-m} (x \partial_x)^i \partial_y^j u\|_{L^2(P_x)}^2 \\ &\geq C \sum_{i+j \leq m} \|\partial_x^i \partial_y^j u\|_{L^2(P_x)}^2 = C \|u\|_{H^m(P_x)}^2. \end{aligned}$$



On the other hand, we have

$$\begin{aligned} \|u\|_{K_m^m(P_r)}^2 &\geq C \sum_{i+j \leq m} \|r^{-m} r^{i+j} \partial_x^i \partial_y^j u\|_{L^2(P_r)}^2 \\ &\geq C \sum_{i+j \leq m} \|\partial_x^i \partial_y^j u\|_{L^2(P_r)}^2 = C \|u\|_{H^m(P_r)}^2, \end{aligned}$$

based on  $i + j \leq m$  and  $r = r_1$  or  $r_2$  on  $P_r$ .

For the region  $\Omega \setminus P$ , Lemma 2.7 shows that  $\|u\|_{H^m(\Omega \setminus P)} \leq M_1 \|u\|_{K_m^m(\Omega \setminus P)}$ , which completes the proof for the first argument in the lemma.

The second inequality can be proved in a similar way by comparing different norms on  $P$  and  $\Omega \setminus P$ , which will be shown in Lemma 2.10.  $\square$

From Lemma 2.5 and 2.8, we have the following corollary.

**Corollary 2.9.** *We have  $K_{m+a}^m \subset r^a H^m \subset K_a^m$ .*

The proof is based on the isomorphism arising from the multiplication in Lemma 2.5 and the inequalities in Lemma 2.8.

The following lemma will compare  $K_a^m$  and  $H^m$  near the y-axis and the vertices.

**Lemma 2.10.** *Let  $\xi$  be a positive number and let  $G'$  be an open subset of  $\Omega$  either close to  $x = 0$ , such that  $r = x$  and  $r < \xi$  on  $G'$ , or close to the vertices  $(1, 0), (1, l)$ , such that  $r < \xi$  on  $G'$ . Then  $\|u\|_{H^m(G')} \leq C_1 \xi^{a-m} \|u\|_{K_a^m(G')}$  if  $a \geq m$ , and  $\|u\|_{K_a^m(G')} \leq C_2 \xi^{-a} \|u\|_{H^m(G')}$  if  $a \leq 0$ , where  $C_1$  and  $C_2$  are generic constants depending on  $m$ .*

*Proof.* For  $a \geq m$ , we first have

$$\|u\|_{H^m(G')} \leq C_1 \|u\|_{K_m^m(G')}$$

from Lemma 2.8. Then, on the subregion of  $G'$  that is close to  $x = 0$ , Lemma 2.6 shows

$$\|u\|_{K_m^m} \leq \xi^{a-m} \|u\|_{K_a^m}.$$

For the subregion that is near one of the vertices, we have

$$\begin{aligned} \|u\|_{K_a^m}^2 &\geq C \sum_{i+j \leq m} \|r^{-a} r^{i+j} \partial_x^i \partial_y^j u\|_{L^2}^2 \\ &\geq \xi^{2(m-a)} \|\partial_x^i \partial_y^j u\|_{L^2}^2 = \xi^{2(m-a)} \|u\|_{H^m}^2. \end{aligned}$$

The last inequality is based on  $r < \xi$  and  $i + j \leq m$  on this subregion. Therefore,  $\|u\|_{H^m(G')} \leq C_1 \xi^{a-m} \|u\|_{K_a^m(G')}$  for  $a \geq m$  by combining the estimates on different subregions of  $G'$ .

For  $a \leq 0$ , similarly, on the subregion of  $G'$ , which is close to  $x = 0$ , because  $r = x, a \leq 0$  and  $r < \xi$ , we have the following inequalities,

$$\begin{aligned} \|u\|_{K_a^m}^2 &\leq C_1 \sum_{i+j \leq m} \|r^{-a} (x \partial_x)^i \partial_y^j u\|_{L^2}^2 \\ &\leq C \xi^{-2a} \sum_{i+j \leq m} \|\partial_x^i \partial_y^j u\|_{L^2}^2 = C \xi^{-2a} \|u\|_{H^m}^2. \end{aligned}$$

On the subregion close to one of the vertices, the inequalities are

$$\begin{aligned} \|u\|_{K_a^m}^2 &\leq C \sum_{i+j \leq m} \|r^{-a} r^{i+j} \partial_x^i \partial_y^j u\|_{L^2}^2 \\ &\leq C \xi^{-2a} \sum_{i+j \leq m} \|\partial_x^i \partial_y^j u\|_{L^2}^2 = C \xi^{-2a} \|u\|_{H^m}^2. \end{aligned}$$

Therefore, for  $a \leq 0$ ,  $\|u\|_{K_a^m(G')} \leq C \xi^{-a} \|u\|_{H^m(G')}$ . This also provides the proof of the second inequality in Lemma 2.8.  $\square$

We have derived several lemmas to reveal the relations between weighted Sobolev spaces  $K_a^m$  and regular Sobolev spaces  $H^m$ . They are the preliminaries for our main results in the next section. Now, we shall give an important lemma for the homogeneity of the norms of weighted Sobolev spaces  $K_a^m(\Omega_\xi)$ . This is one of the main reasons that we use weighted Sobolev spaces for the analysis.

We define the dilation of a function on  $\Omega_\xi$  first. For  $0 < \lambda < 1$ , let  $G \subset \Omega_\xi$  be an open subset of  $\Omega$ . Also, let  $v$  be a function on  $G$ . Then, we define the dilation function  $v_\lambda(x, y) := v(\lambda x, y)$  for any point  $(x, y) \in G' \subset \Omega_{\xi/\lambda}$ , such that  $(\lambda x, y) \in G$ . The relation of the norms of  $v$  and its dilation  $v_\lambda$  is given by the following lemma.

**Lemma 2.11.** *Let  $G \subset \Omega_\xi \setminus \Omega_{\lambda\xi}$  be an open subset and  $u(x, y)$  be a function on  $G$ ,  $0 < \lambda < 1$ ,  $\xi/\lambda < \frac{1}{2}$ . Then  $\|u_\lambda\|_{K_a^m(G')}^2 = \lambda^{2a-1} \|u\|_{K_a^m(G)}^2$  for any  $u \in K_a^m(G)$ . This relation also holds for  $G \subset \Omega_{\lambda\xi}$ ,  $\xi < \frac{1}{2}$ .*

*Proof.* The proof of the lemma follows the change of variables and the fact that  $r = x$  on  $G'$  with a direct calculation. Let  $w = \lambda x$ , then

$$\begin{aligned} \|u_\lambda(x, y)\|_{K_a^m(G')}^2 &= \sum_{i+j \leq m} \int_{G'} (x^{-a} (r_1 r_2)^{i+j} (x \partial_x)^i \partial_y^j u_\lambda)^2 dx dy \\ &= \sum_{i+j \leq m} \int_G (\lambda^a w^{-a} (r_1 r_2)^{i+j} (w \partial_w)^i \partial_y^j u(w, y))^2 \frac{1}{\lambda} dw dy \\ &= \sum_{i+j \leq m} \int_G \lambda^{2a-1} (w^{-a} (r_1 r_2)^{i+j} (w \partial_w)^i \partial_y^j u(w, y))^2 dw dy \\ &= \lambda^{2a-1} \sum_{i+j \leq m} \int_G (w^{-a} (r_1 r_2)^{i+j} (w \partial_w)^i \partial_y^j u(w, y))^2 dw dy \\ &= \lambda^{2a-1} \sum_{i+j \leq m} \|w^{-a} (r_1 r_2)^{i+j} (w \partial_w)^i \partial_y^j u\|_{L^2(G)}^2 \\ &= \lambda^{2a-1} \|u\|_{K_a^m(G)}^2. \end{aligned}$$

We note that the proof above can be carried out without any restriction on  $G \subset \Omega_{\lambda\xi}$ ,  $\xi < \frac{1}{2}$  as well. Therefore, this relation for  $u$  and  $u_\lambda$  also holds on this region.  $\square$

The significance of Lemma 2.11 is that it shows the possibility to estimate the weighted norms on another region under dilation.

Let us explain why we considered Lemma 2.11 only on  $\Omega_\xi$ . Recall the notation  $P_x$  and  $P_r$  defined in Lemma 2.3. Let  $P = P_x \cup P_r$ . Then, the strong ellipticity of the operator  $\mathcal{L}_\delta$  on  $\Omega \setminus P$  admits a unique solution  $u \in H^{m+1}(\Omega \setminus P) \cap \{u|_{\partial\Omega} = 0\}$  for  $f \in H^{m-1}$ . Therefore,  $u$  has no singularity on  $\Omega \setminus P$ , and no extra study is needed

in general on this region. In fact, Lemma 2.11 is particular for the analysis of the solution near  $x = 0$ , since one can refer to [2, 13, 28] for the solution around the vertices  $(1, 0), (1, l)$ .

We have listed several lemmas on  $K_a^m$  for the degenerate problem and we shall conclude this section by the following result.

**Lemma 2.12.** *The operator  $\mathcal{L}_\delta$  defines a continuous map  $\mathcal{L}_\delta: K_{a+1}^{m+1}(\Omega) \rightarrow K_{a-1}^{m-1}(\Omega)$ .*

*Proof.* We need to show that for  $\forall u \in K_{a+1}^{m+1}(\Omega)$ ,  $\|\mathcal{L}_\delta u\|_{K_{a-1}^{m-1}} \leq C\|u\|_{K_{a+1}^{m+1}}$  on  $P_r, P_x$  and  $\Omega \setminus P$ , which are defined in Lemma 2.3.

On  $\Omega \setminus P$ ,  $\mathcal{L}_\delta = -\partial_x^2 - \frac{\delta}{x^2} \partial_y^2$  is strongly elliptic. Therefore, it is a bounded operator  $H^{m+1}(\Omega \setminus P) \rightarrow H^{m-1}(\Omega \setminus P)$  [24]. Then, the argument for this lemma follows the equivalence of the spaces  $H^{m+1}(\Omega \setminus P)$  and  $K_a^{m+1}(\Omega \setminus P)$ .

On  $P_r$ , let  $g = r_1 r_2$ ,  $H^m = H^m(P_r)$  and  $K_a^m = K_a^m(P_r)$  in the proof for simplicity. Then, based on  $g \simeq r$ , the following inequalities hold with the coefficient frozen in  $x$  at 1,

$$\begin{aligned} \|\mathcal{L}_\delta u\|_{K_{a-1}^{m-1}} &\leq C_0 \sum_{i+j \leq m-1} \|r^{1-a} g^{i+j} (x \partial_x)^i \partial_y^j (-\partial_x^2 u - \frac{\delta}{x^2} \partial_y^2 u)\|_{L^2} \\ &= C_0 \sum_{i+j \leq m-1} \|r^{1-a} g^{i+j} \partial_x^i \partial_y^j (-\partial_x^2 u - \delta \partial_y^2 u)\|_{L^2} \\ &\leq C_1 \sum_{i+j \leq m-1} (\|r^{1-a} g^{i+j} \partial_x^{i+2} \partial_y^j u\|_{L^2} + \|r^{1-a} g^{i+j} \partial_x^i \partial_y^{j+2} u\|_{L^2}) \\ &\leq C_2 \sum_{i+j \leq m-1} (\|r^{-a-1} g^{i+j+2} \partial_x^{i+2} \partial_y^j u\|_{L^2} + \|r^{-a-1} g^{i+j+2} \partial_x^i \partial_y^{j+2} u\|_{L^2}) \\ &\leq C \|u\|_{K_{a+1}^{m+1}}. \end{aligned}$$

On  $P_x$ , similarly, let  $g = r_1 r_2$ ,  $H^m = H^m(P_x)$  and  $K_a^m = K_a^m(P_x)$  in the proof for simplicity. Then, based on  $r = x$ , we have the estimates below,

$$\begin{aligned} \|\mathcal{L}_\delta u\|_{K_{a-1}^{m-1}} &\leq C_1 \sum_{i+j \leq m-1} \|r^{1-a} g^{i+j} (x \partial_x)^i \partial_y^j (-\partial_x^2 u - \frac{\delta}{x^2} \partial_y^2 u)\|_{L^2} \\ &= C_1 \sum_{i+j \leq m-1} \|r^{1-a} g^{i+j} (r \partial_r)^i \partial_y^j (-\partial_r^2 u - \frac{\delta}{r^2} \partial_y^2 u)\|_{L^2} \\ &\leq C_1 \sum_{i+j \leq m-1} (\|r^{1-a} g^{i+j} (r \partial_r)^i \partial_r^2 \partial_y^j u\|_{L^2} + \|r^{1-a} g^{i+j} (r \partial_r)^i \partial_y^{j+2} \frac{\delta}{r^2} u\|_{L^2}) \\ &\leq C_2 \sum_{i+j \leq m-1} (\|r^{1-a} g^{i+j} ((r \partial_r)^{i+2} - (r \partial_r)^{i+1}) r^{-2} \partial_y^j u\|_{L^2} + \\ &\quad \|r^{-a-1} g^{i+j} (-2 + r \partial_r)^i \partial_y^{j+2} u\|_{L^2}) \\ &\leq C_2 \sum_{i+j \leq m-1} (\|r^{-a-1} g^{i+j} (-2 + r \partial_r)^{i+2} \partial_y^j u\|_{L^2} + \\ &\quad \|r^{-a-1} g^{i+j} (-2 + r \partial_r)^{i+1} \partial_y^j u\|_{L^2} + \|r^{-a-1} g^{i+j} (-2 + r \partial_r)^i \partial_y^{j+2} u\|_{L^2}) \\ &\leq C \|u\|_{K_{a+1}^{m+1}}. \end{aligned}$$

We here use  $r^{-a} (r \partial_r)^i r^a u = (a + r \partial_r)^i u$  to simplify the expression.  $\square$

As a direct result from Lemma 2.12, the operator  $\mathcal{L}_\delta$  defines an injective map  $\mathcal{L}_\delta : K_{a+1}^{m+1} \rightarrow K_{a-1}^{m-1}$ . Moreover, we will show that this map  $\mathcal{L}_\delta : K_{a+1}^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{a-1}^{m-1}$  is also surjective if the index  $a$  satisfies some condition, which makes the map a bijection.

### 3. THE WELL-POSEDNESS AND REGULARITY OF THE SOLUTION

First, we need the following estimates on the solution of Equation (1). Recall that we defined the set  $S = \{(1, 0), (1, l), [0, y], 0 \leq y \leq l\}$  in Section 2. It has been shown in [13, 30] that the trace or restriction of  $u \in K_a^m$  on the boundary follows

$$u|_{\partial\Omega \setminus S} \in K_{a-\frac{1}{2}}^{m-\frac{1}{2}}(\partial\Omega \setminus S).$$

However, the estimate of the trace on  $x = 0$  is needed to derive the corresponding bilinear form  $a(\cdot, \cdot)$  for the boundary value problem (1), although the trace on this part of boundary is generally undefined in the weighted Sobolev spaces  $K_a^m$  [8, 13, 30]. We note that for  $m \geq 1$ ,  $K_m^m \subset H^m$  by Lemma 2.8. Denote each segment of  $\partial\Omega$  by  $\bar{D}_i$ ,  $i = 1, 2, 3, 4$ , where  $D_i$  is open. In particular, let  $D_1 := (0, y)$ ,  $0 \leq y \leq l$ , be the corresponding open set for the degenerate boundary  $x = 0$ . Then, the trace

$$u|_{D_i} \in H^{m-\frac{1}{2}}(D_i), \quad \forall u \in K_m^m(\Omega),$$

is defined [28]. Consequently, for  $u \in K_1^1$ , the trace of  $u$  is well defined in  $L^2$  on every  $D_i$  of the boundary  $\partial\Omega$ . Furthermore, we can even show that  $u|_{D_1} = 0$  for  $u \in K_1^1$ .

**Lemma 3.1.** *For any function  $u \in K_1^1$ , its trace on  $D_i$  is well defined in  $L^2$  and moreover, we have  $u|_{D_1} = 0$ . Consequently, the corresponding bilinear form for Equation (1) is  $a(u, v) = \int_\Omega \partial_x u \partial_x v + \frac{\sqrt{\delta}}{x} \partial_y u \frac{\sqrt{\delta}}{x} \partial_y v$ ,  $\forall v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega} = 0\}$ .*

*Proof.* For  $u \in K_1^1$ , the trace  $u|_{D_i}$  belongs to  $H^{\frac{1}{2}}(D_i)$  by the arguments above, hence in  $L^2(D_i)$ .

Moreover, on  $\Omega_\xi = (0, \xi) \times (0, l)$ ,  $\xi < 1/2$ , since  $r = x$ , we have

$$\frac{1}{\xi^2} \int_{\Omega_\xi} u^2 \leq \int_{\Omega_\xi} \frac{1}{x^2} u^2 \leq C \|u\|_{K_1^1(\Omega_\xi)}^2 < \infty.$$

Therefore,  $\int_{\Omega_\xi} u^2 \rightarrow 0$  as  $\xi \rightarrow 0$ . Hence, the trace  $u|_{D_1} = 0$  in  $L^2$  by continuity.

Thus, the following bilinear form is obtained by integration by parts,

$$a(u, v) = \int_\Omega \partial_x u \partial_x v + \frac{\sqrt{\delta}}{x} \partial_y u \frac{\sqrt{\delta}}{x} \partial_y v, \quad \forall v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega} = 0\}.$$

□

Now, we shall prove the existence and uniqueness of the solution of Equation (1) in weighted Sobolev spaces.

**Theorem 3.2.** *On  $\Omega$ , the map  $\mathcal{L}_\delta : K_1^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1}^{m-1}(\Omega)$  is an isomorphism, for  $\delta > 0$ ,  $m \geq 0$ . Namely, there is a unique solution  $u \in K_1^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$  for Equation (1) if  $f \in K_{-1}^{m-1}(\Omega)$ .*

*Proof.* We shall first prove it for  $m = 0$ . From Lemma 3.1, we have the following weak formulation for Equation (1),

$$a(u, v) = \int_{\Omega} \partial_x u \partial_x v + \frac{\sqrt{\delta}}{x} \partial_y u \frac{\sqrt{\delta}}{x} \partial_y v = \int_{\Omega} f v, \quad \forall v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega} = 0\}.$$

Then, we will show the equivalence between the energy norm induced by  $a(\cdot, \cdot)$  and the  $K_1^1$ -norm  $\|\cdot\|_{K_1^1(\Omega)}$  to complete the proof.

Based on the definitions of  $a(\cdot, \cdot)$  and the  $K_1^1$ -norm on  $\Omega$ , the continuity of  $a(\cdot, \cdot)$  can be verified as follows. From the Holder inequality, there exists a constant  $C$ , not depending on  $u$  and  $v$ , such that  $a(u, v) \leq C \|u\|_{K_1^1} \|v\|_{K_1^1}$ . Therefore,  $a(\cdot, \cdot)$  is a continuous (bounded) bilinear form on  $K_1^1$ .

To prove the coercivity, we adopt the following notations. Let  $\Omega_{\xi} = (0, \xi) \times (0, l)$  be the rectangular domain near the boundary  $x = 0$ . Denote by  $B(v, r)$  the open ball of radius  $r$  centered at  $v$ . For any of the vertices  $v_1 = (1, 0)$ ,  $v_2 = (1, l)$ , let  $\Omega_{r\xi_i} = \Omega \cap B(v_i, \xi)$ ,  $i = 1, 2$ , be the corresponding conical domain, such that  $\Omega_{r\xi_i}$  can be characterized in polar coordinates by

$$\Omega_{r\xi_i} = \{(r, \theta) | 0 < r < \xi, 0 < \theta < \frac{\pi}{2}\}.$$

Note that  $a(u, u)$  is equivalent to  $|u|_{K_1^1}^2 = \sum_{i+j=1} \|r(r_1 r_2)(x \partial_x)^i (\partial_y^j u)\|_{L^2}^2$  by their definitions. Then, we shall first prove the following weaker inequalities for  $\xi$  small,

$$\begin{aligned} \int_{\Omega_{\xi}} \frac{u^2}{x^2} &\leq C \int_{\Omega_{\xi}} (\partial_x u)^2 + \frac{\delta}{x^2} (\partial_y u)^2 \\ \int_{\Omega_{r\xi_i}} \frac{u^2}{r^2} &\leq C \int_{\Omega_{r\xi_i}} (\partial_x u)^2 + \delta (\partial_y u)^2. \end{aligned}$$

On the domain  $\Omega_{\xi}$ , we first have the one-dimensional Poincaré inequality for  $y$ ,

$$\int_0^l u^2 dy \leq C_1 \int_0^l (\partial_y u)^2 dy.$$

By integrating with respect to  $x$ , we obtain

$$\int_{\Omega_{\xi}} \frac{u^2}{x^2} dy dx \leq C_1 \int_{\Omega_{\xi}} \frac{(\partial_y u)^2}{x^2} dy dx \leq C \int_{\Omega_{\xi}} (\partial_x u)^2 + \frac{\delta}{x^2} (\partial_y u)^2 dy dx,$$

where  $C$  is independent of  $u$ .

Similarly, we have the one-dimensional Poincaré inequality for  $\theta$  on  $\Omega_{r\xi_i}$ ,

$$\int_0^{\frac{\pi}{2}} u^2 d\theta \leq C_1 \int_0^{\frac{\pi}{2}} (\partial_{\theta} u)^2 d\theta.$$

By integrating in polar coordinates, we have

$$\int_{\Omega_{r\xi_i}} \frac{u^2}{r^2} dx dy = \int_{\Omega_{r\xi_i}} \frac{u^2}{r} dr d\theta \leq C_1 \int_{\Omega_{r\xi_i}} \frac{(\partial_{\theta} u)^2}{r} dr d\theta.$$

Since  $\int_{\Omega_{r\xi_i}} (\partial_x u)^2 + (\partial_y u)^2 dx dy = \int_{\Omega_{r\xi_i}} r (\partial_r u)^2 + \frac{(\partial_{\theta} u)^2}{r} dr d\theta$ , we now have

$$\int_{\Omega_{r\xi_i}} \frac{u^2}{r^2} dx dy \leq C \int_{\Omega_{r\xi_i}} (\partial_x u)^2 + \delta (\partial_y u)^2 dx dy,$$

with  $C$  independent of  $u$ .

Let  $\Omega_{r_\xi} := \Omega_{r_{\xi_1}} \cup \Omega_{r_{\xi_2}}$ . Thus, based on the usual Poincaré inequality in  $\Omega \setminus (\Omega_\xi \cup \Omega_{r_\xi})$  and the inequalities above, we complete the proof for the coercivity  $a(u, u) \geq C \|u\|_{K_1^1(\Omega)}^2$ . Then, the existence of the unique solution  $u \in K_1^1(\Omega) \cap \{u|_{\partial\Omega} = 0\}$  follows the Lax-Milgram Theorem [17].  $\square$

For  $m \geq 1$ , the proof follows [13, 29], which is based on the regularity of the solution derived by the Mellin transform on an infinite domain.

As an extension from this theorem, one has the following corollary.

**Corollary 3.3.** *There exists a constant  $\eta > 0$ , depending on  $\Omega$ , such that*

$$\mathcal{L}_\delta : K_{1+\epsilon}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1+\epsilon}^{m-1}(\Omega)$$

is an isomorphism for  $0 < |\epsilon| < \eta$ .

*Proof.* Denote by  $\mathcal{L}_{\delta\epsilon}$  the operator defined by  $\mathcal{L}_\delta$  but on the space  $K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\}$ . Then, from Theorem 3.2 and Lemma 2.5 (see the diagram below), the operator  $\mathcal{L}_{\delta\epsilon}$  is an isomorphism if, and only if

$$\mathcal{A}_{\delta\epsilon} := r^{-\epsilon} \mathcal{L}_{\delta\epsilon} r^\epsilon : K_1^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1}^{m-1}$$

is an isomorphism.

$$\begin{array}{ccc} K_{1+\epsilon}^{m+1} & \xrightarrow{\mathcal{L}_{\delta\epsilon}} & K_{-1+\epsilon}^{m-1} \\ r^\epsilon \uparrow & & \downarrow r^{-\epsilon} \\ K_1^{m+1} & \longrightarrow & K_{-1}^{m-1} \end{array}$$

Then the proof follows the fact that  $\mathcal{A}_{\delta\epsilon}$  is a continuous bijection as  $\epsilon = 0$  and the operator  $\mathcal{A}_{\delta\epsilon}$  depends continuously in norm on the parameter  $\epsilon$ .  $\square$

*Remark 3.4.* For a brief summary, we have taken the advantage of weighed spaces  $K_a^m$  to prove the well-posedness of the solution  $u \in K_1^{m+1} \cap \{u|_{\partial\Omega} = 0\}$  of Equation (1), for  $\forall f \in K_{-1}^{m-1}$ . Furthermore, there exists some constant  $\eta$ , such that  $\mathcal{L}_\delta$  is still invertible on weighted spaces that depend on  $\eta$ . In fact, one will find that it is important to know the exact upper bound of  $\epsilon$  in the FEM. To be more precise,  $\eta$  is determined by the local behavior of the solution for Equation (1) near the set  $S = \{(1, 0), (1, l), [0, y], 0 \leq y \leq l\}$ . Recall  $\Omega_\xi = (0, \xi) \times (0, l)$  and  $\Omega_{r_\xi} = \Omega_{r_{\xi_1}} \cup \Omega_{r_{\xi_2}}$  in Theorem 3.2. Then, it is possible to show that  $\eta_1 = \frac{\sqrt{l^2 + 4\delta\pi^2}}{2l}$  on  $\Omega_\xi$  for  $\xi < 1/2$ . Namely,  $\mathcal{A}_{\delta\epsilon} : K_1^{m+1}(\Omega_\xi) \cap \{u|_{y=0,l} = 0\} \rightarrow K_{-1}^{m-1}(\Omega_\xi)$  is invertible for  $|\epsilon| < \eta_1 = \frac{\sqrt{l^2 + 4\delta\pi^2}}{2l}$ . Here, we include some arguments on operator  $\mathcal{A}_{\delta\epsilon}$  and the constant  $\eta$ .

We focus on the region  $\Omega_\xi = (0, \xi) \times (0, l)$  for  $\xi < 1/2$  first. The indicial family of  $\mathcal{A}_{\delta\epsilon} = r^{-\epsilon} \mathcal{L}_{\delta\epsilon} r^\epsilon$  for  $K_1^{m+1}(\Omega_\xi) \cap \{u|_{y=0,l} = 0\}$  is  $(i\tau + \epsilon + \frac{1}{2})(i\tau + \epsilon - \frac{1}{2}) + \delta\partial_y^2$  acting on  $H^2([0, l]) \cap \{u|_{y=0,l} = 0\}$ . The eigenvalues of  $\partial_y^2$  on  $H^2([0, l]) \cap \{u|_{y=0,l} = 0\}$  are  $-(\frac{k\pi}{l})^2$  for  $k \in \{1, 2, 3, \dots\}$ . On  $\Omega_{r_{\xi_i}}$ , the indicial family of  $\mathcal{A}_{\delta\epsilon}$  for the vertex can be derived in a similar way as in [14] by the Mellin transform [29]. Then, we have the corresponding eigenvalues that can be calculated numerically. Based on Kondratiev's results [30], the operator  $\mathcal{A}_{\delta\epsilon} : K_1^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1}^{m-1}$  is Fredholm if, and only if its indicial family for the degenerate boundary  $x = 0$

and the indicial families for the two vertices are invertible for all  $\tau \in \mathbb{R}$ . This is seen to be the case for  $(i\tau + \epsilon + \frac{1}{2})(i\tau + \epsilon - \frac{1}{2}) + \delta\partial_y^2$ , unless  $\epsilon = \pm \frac{\sqrt{l^2 + 4k^2\delta\pi^2}}{2l}$ ,  $k \in \mathbb{N}$ . Let  $\eta_1 = \frac{\sqrt{l^2 + 4\delta\pi^2}}{2l}$ . Denote by  $\eta_2$  the smallest positive value, such that the indicial families of  $\mathcal{A}_{\delta\epsilon}$  for the vertices is not invertible as  $\epsilon = \eta_2$ . Then,  $\mathcal{A}_{\delta\epsilon} = r^{-\epsilon}\mathcal{L}_{\delta\epsilon}r^\epsilon : K_1^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1}^{m-1}$  is Fredholm of index 0 when  $|\epsilon| < \eta = \min(\eta_1, \eta_2)$ , since  $\mathcal{A}_{\delta\epsilon}$  is Fredholm of index 0 as  $\epsilon = 0$ . Moreover, we note that the kernel of  $\mathcal{A}_{\delta\epsilon} = r^{-\epsilon}\mathcal{L}_{\delta\epsilon}r^\epsilon$  with Dirichlet boundary condition is always zero when  $|\epsilon| < \eta$ . Then, the cokernel has to be zero for  $|\epsilon| < \eta$ , since the operator has a zero index in this case. Consequently,  $\mathcal{A}_{\delta\epsilon} : K_1^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1}^{m-1}$  is surjective as well, hence, invertible when  $|\epsilon| < \eta$ . As the conclusion of the arguments above, we can take  $\eta = \min(\eta_1, \eta_2)$ , such that for  $\forall |\epsilon| < \eta$ , the operator  $L_\delta : K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1+\epsilon}^{m-1}$  is still an isomorphism.

For values of  $\epsilon$  outside the range above, the operator  $\mathcal{A}_{\delta\epsilon}$  will no longer be invertible. In fact, it will have a non-zero index that can be computed using the results of [34]. To compute  $\eta_2$  for  $\Omega_{r_\epsilon}$ , one can follow the method in [5, 14, 28], and similar examples will be presented in a forthcoming paper.

#### 4. ANALYSIS FOR THE FINITE ELEMENT METHOD

In this section, we will consider the numerical approximation for the solution  $u$  of Equation (1) by using the FEM. Denote by  $\Gamma = \{T_i\}$  the triangulation of  $\Omega$  with triangles. Let  $V_n = V_n(\Gamma, m+1)$  be the finite element space associated to the degree  $m$  Lagrange triangle such that  $V_n$  consists of polynomials of degree  $\leq m$  on each triangle  $T_i \in \Gamma$ . Then, we shall try to find the optimal finite element solution  $u_n \in V_n$  by an appropriate construction of the finite space. By optimal finite element solution, we here mean the finite element solution that satisfies

$$\|u - u_n\|_{H^1(\Omega)} \leq C \dim(V_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)}, \quad \forall f \in H^{m-1}(\Omega) \cap K_{-1+\epsilon}^{m-1}(\Omega),$$

which recovers the asymptotic rate of convergence for elliptic equations on smooth domains [17].

To look for the extensions of the classical approximation theory, we have the following tasks to complete here: 1. the error estimate for the interpolation of the solution; 2. the construction of finite element spaces  $V_n$ .

**4.1. Estimate for the Interpolation.** Let us recall the following well-known approximation theorem [17, 19] in order to carry out further analysis.

**Theorem 4.1.** *Suppose the bilinear form  $a(\cdot, \cdot)$  for an equation is both continuous and coercive on  $H^1$  for a star-shaped two-dimensional domain  $D$ . Let  $V = V(\Gamma, m+1)$ . Assume that all triangles  $T_j$  of the triangulation  $\Gamma$  of domain  $D$  have angles  $\geq \alpha$  and edges of length  $\leq h$  and  $\geq ah$ . Namely, the triangulation is quasi-uniform and satisfies the minimum-angle condition. Let  $u_n \in V$  and  $u_I \in V$  be the finite element solution and the interpolation function respectively. Then, there exist constants  $c$  and  $C_1 = C_1(\alpha, m)$  such that*

$$c\|u - u_n\|_{H^1(D)} \leq \|u - u_I\|_{H^1(D)} \leq C_1 h^m \|u\|_{H^{m+1}(D)}$$

for  $\forall u \in H^{m+1}(D)$ ,  $m \geq 1$ . Also, there exists a constant  $C_2 = C_2(\alpha, a, m)$ , such that

$$c\|u - u_n\|_{H^1(D)} \leq \|u - u_I\|_{H^1(D)} \leq C_2 \dim(V)^{-m/2} \|u\|_{H^{m+1}(D)}.$$

The constant  $c$  depends only on the bilinear form  $a(\cdot, \cdot)$  by Céa's Lemma, while  $C_1$  and  $C_2$  are independent of the solution.

Let  $M := C_1(\alpha)M_1M_2$ , where  $C_1(\alpha)$  is as in Theorem 4.1, and  $M_1$  and  $M_2$  are from Lemma 2.7. Then, we have the following estimate for the error  $\|u - u_I\|_{K_1^1}$  on a subset of  $\Omega$ .

**Theorem 4.2.** *Fix  $\alpha > 0$  and  $0 < \xi < 1/4$ . Let  $P \subset \Omega$  be a star-shaped polygonal domain, such that  $r > \xi$  on  $P$ . Let  $\Gamma = (T_j)$  be a triangulation of  $P$  with angles  $\geq \alpha$  and sides  $\leq h$ . Then*

$$\|u - u_I\|_{K_{1+\epsilon}^1(P)} \leq Mh^m \|u\|_{K_{1+\epsilon}^{m+1}(P)}$$

for  $\forall u \in K_{1+\epsilon}^{m+1}(P)$ .

*Proof.* The proof for the error estimate follows the arguments in Theorem 4.1 and the equivalence of the  $H^m$ - and the  $K_a^m$ -norms on  $P$  immediately.  $\square$

We need to point out here that the estimate in Theorem 4.2, for the solution  $u$  of Equation (1) and its interpolation  $u_I$ , can not extend to the entire domain  $\Omega$  in general. It is reasonable to expect singularities for  $u$  as it is approaching the degenerate boundary  $x = 0$ , since it is just sitting in the weighted Sobolev spaces instead of the usual Sobolev spaces. Also, It is possible that we have corner singularities near any vertex away from  $x = 0$ , depending on the parameter  $\delta$  and the interior angle corresponding to the corner. In either of the cases, the constant  $M$  can not be uniformly bounded, which will destroy the optimal rate of convergence [3, 4]. The FEM for elliptic equations with Dirichlet boundary condition near the corners has been widely studied [3, 5, 14] in different principles. The basic idea is to reflect the singularities in finite elements by using anisotropic meshes near the corners. In those graded meshes, the size and shape of the triangles are carefully designed without increasing the number of degrees of freedom. Then, the optimal rate of convergence can be recovered in terms of the dimension of the finite space  $V_n$ . From this point, we shall concentrate on the estimate near the degenerate boundary  $x = 0$ , since we have been quite clear for the corners. Recall  $\Omega_\xi = (0, \xi) \times (0, l)$ . Let  $0 < \kappa < 1$ . The extension of Theorem 4.2 on the thin rectangular region  $\Omega_\xi \setminus \Omega_{\kappa\xi}$  is as follows.

**Theorem 4.3.** *Let  $\Omega_\xi = (0, \xi) \times (0, l)$  be a subset sitting in  $\Omega$  with  $\xi < 1/4$ . Given  $0 < \kappa < 1$ , let  $\Gamma = (T_j)$  be a triangulation for the rectangular region  $U := \Omega_\xi \setminus \Omega_{\kappa\xi}$  with triangles of sides  $\leq h_x$  and  $\leq h_y$  in the  $x$ -,  $y$ -direction respectively. Denote by  $u_I$  the interpolant as the polynomial of degree  $\leq m$  on each triangle. Then the interpolation error on  $U$  is*

$$\|u - u_I\|_{K_1^1(U)} \leq C(\kappa)\xi^\epsilon (\max(h_y, \frac{h_x}{2\xi}))^m \|u\|_{K_{1+\epsilon}^{m+1}(U)}$$

for  $\forall u \in K_{1+\epsilon}^{m+1}$ ,  $m \geq 1$ ,  $\epsilon > 0$ .



*Proof.* Recall the dilation function  $u_\lambda(x, y) = u(\lambda x, y)$  from Section 2 and note that  $u_{I\lambda} = u_{\lambda I}$ . Namely, dilation commutes with interpolation. Then, if we let  $\lambda = 2\xi$  and  $U' = (\frac{\kappa}{2}, 1/2) \times (0, l)$ , by Lemma 2.11 and Theorem 4.2, we have

$$\begin{aligned}
\|u - u_I\|_{K_1^1(U)} &= \lambda^{-\frac{1}{2}} \|u_\lambda - u_{I\lambda}\|_{K_1^1(U')} \\
&= \lambda^{-\frac{1}{2}} \|u_\lambda - u_{\lambda I}\|_{K_1^1(U')} \\
&\leq \lambda^{-\frac{1}{2}} M (\max(\frac{h_x}{\lambda}, h_y))^m \|u_\lambda\|_{K_1^{m+1}(U')} \\
&= M (\max(\frac{h_x}{\lambda}, h_y))^m \|u\|_{K_1^{m+1}(U)} \\
&\leq M \xi^\epsilon (\max(h_y, \frac{h_x}{2\xi}))^m \|u\|_{K_{1+\epsilon}^{m+1}(U)}.
\end{aligned}$$

The last inequality is based on Lemma 2.6 in Section 2.  $\square$

This theorem provides us the interpolation error for  $u \in K_{1+\epsilon}^{m+1}$  on a rectangular strip  $U$  near  $x = 0$ . Based on our observation above, it is possible to construct a class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  of partitions  $\Gamma$  of  $\Omega$ , such that the optimal convergence rate is obtained.

Before we define the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$ , we assume that, near the vertices  $(1, 0)$  and  $(1, l)$ , a proper graded mesh has already been chosen to recover the optimal convergence rate, which is reasonable based on our previous discussions. For this reason, we will not consider the graded mesh near the vertices in the definition below. In fact, we will use a uniform mesh near the corners to demonstrate our method to generate triangles. However, it is only for the purpose to simplify the expressions. One needs to keep in mind that the uniform mesh near the corners in the following definition will be replaced by an appropriate graded mesh generally in practical computation. Some similar examples on the graded mesh near corners for elliptic operators will be presented in [35].

**4.2. Construction of the Mesh.** We shall introduce the construction of a class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  of triangulations and the finite element spaces  $V_n$  associated to it. In the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$ , we try to even-distribute the interpolation error, and keep the same number of triangles as in the usual mid-point triangulation.

In the notation  $\mathcal{C}(l, h, \kappa, m, \epsilon)$ , we denote by  $h$  the size of triangles in the triangulation for  $(\frac{1}{2}, \frac{3}{4}) \times (0, l) \subset \Omega$ ;  $\kappa$  is the parameter to control the decay of triangles near  $x = 0$ . We focus only on  $\Omega_\xi = (0, \xi) \times (0, l)$ , for  $\xi < \frac{1}{2}$ . (Graded mesh for both of the vertices  $(1, 0)$ ,  $(1, l)$  is needed in general, but it will have the same number of triangles as in the uniform mesh.) We here define the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  for our problem.

**Definition 4.4.** For a fixed  $m = \{1, 2, \dots\}$ ,  $\epsilon \in (0, 1]$ ,  $l, h > 0$ , we define  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  to be the set of triangulations as follows. For a positive integer  $N$ , choose  $\kappa$ , such that

$$\kappa^{N\epsilon} \leq C(l)h^m.$$

Then, starting with three initial triangles sitting in  $\Omega \setminus \Omega_{\frac{1}{2}}$  (Fig. 4.1), we decompose  $\Omega_{\frac{1}{2}}$  into rectangular subdomains  $\Omega^0 := \Omega_{\frac{1}{2}} \setminus \Omega_{\frac{\kappa}{2}}$ ,  $\Omega^1 := \Omega_{\frac{\kappa}{2}} \setminus \Omega_{\frac{\kappa^2}{2}}$ ,  $\dots$ ,  $\Omega^j := \Omega_{\frac{\kappa^j}{2}} \setminus \Omega_{\frac{\kappa^{j+1}}{2}}$ , for  $j = 0, 1, 2, \dots, N-1$  and  $\Omega^N := \Omega_{\frac{\kappa^N}{2}} = (0, \frac{\kappa^N}{2}) \times (0, l)$ .

Note  $h = l$  in the initial mesh. Denote by  $h_{x_j}$  and  $h_{y_j}$  the lengths of sides of triangles in the  $x$ - and  $y$ -direction respectively on  $\Omega^j$ . In the  $j$ th refinement,  $1 \leq j < N$ , we triangulate  $\Omega^{j-1}$  with three triangles, such that they satisfy

$$h_{x_{j-1}} = \kappa^{j-1}/2 - \kappa^j/2, \quad \text{and} \quad h_{y_{j-1}} = l.$$

Meanwhile, new triangles are generated in  $\Omega \setminus \Omega_{\frac{\kappa^{j-1}}{2}}$ , by connecting the mid-points of the old triangles. Note that there is no triangle in  $\Omega^i, i \geq j$ , yet. We simply repeat this process for  $\Omega^j$  and  $\Omega \setminus \Omega_{\frac{\kappa^j}{2}}$ , in the next step. In the  $N$ th refinement, besides generating triangles for  $\Omega^{N-1}$  and  $\Omega \setminus \Omega_{\frac{\kappa^{N-1}}{2}}$ , we divide  $\Omega^N$  into two triangles by connecting the diagonal of the rectangle. Then, after  $N$  refinements, the triangulation  $\Gamma$  is the union of the triangles in the triangulations of all the subdomains.

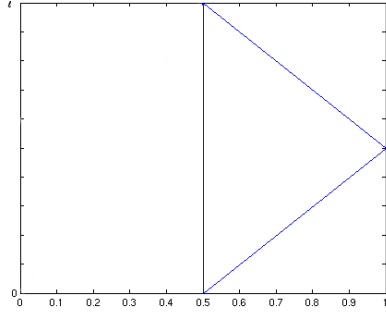


Fig. 4.1. Initial mesh with three triangles

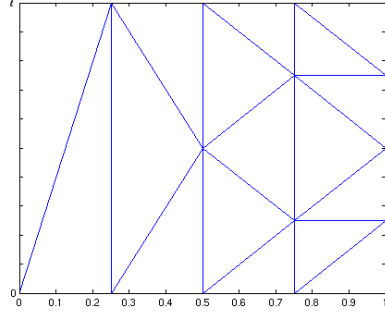


Fig. 4.2. Triangulation after one refinement

$$N = 1, \kappa = 0.5$$

We now state the following property for the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  we defined above.

**Theorem 4.5.** *There exists a constant  $C$ , such that*

$$\|u - u_I\|_{K_1^1(\Omega)} \leq C(2h)^m \|u\|_{K_{1+\epsilon}^{m+1}(\Omega)}$$

for any triangulation  $\Gamma$  in  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  and  $u \in K_{1+\epsilon}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$ ,  $\epsilon > 0$ ,  $m \geq 1$ .

The proof of this theorem needs the estimate on every  $\Omega^j$ ,  $j < N$  and  $\Omega^N$ , since it holds for  $\Omega \setminus \Omega_{\frac{1}{2}}$  by our assumption. Due to the construction of the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$ , we present the following lemma for the last region  $\Omega^N = (0, \frac{\kappa^N}{2}) \times (0, l)$  first.

**Lemma 4.6.** *On  $\Omega^N = (0, \frac{\kappa^N}{2}) \times (0, l)$ , from the construction of  $\mathcal{C}(l, h, \kappa, m, \epsilon)$ , the estimate on the error gives*

$$\|u - u_I\|_{K_1^1(\Omega^N)} \leq Ch^m \|u\|_{K_{1+\epsilon}^{m+1}(\Omega^N)},$$

for  $\forall u \in K_{1+\epsilon}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\}$ ,  $\epsilon > 0$ ,  $m \geq 1$ .

*Proof.* The proof follows the dilation of  $u$  and the introduction of an auxiliary function  $v$ . We define the dilation  $u_\lambda(x, y) = u(\lambda x, y)$  for  $(\lambda x, y) \in \Omega^N$ . Let  $\lambda = \kappa^N$ . Then,  $u_\lambda(x, y) \in K_{1+\epsilon}^{1+m}(\Omega_{\frac{1}{2}})$  by Lemma 2.11. Meanwhile, let  $\chi : \Omega_{\frac{1}{2}} \rightarrow [0, 1]$  be a non-decreasing smooth function of  $x$ , which is equal to 0 in a neighborhood of

$x = 0$ , but is equal to 1 at all the nodal points that do not lie on  $x = 0$ . Then we introduce the auxiliary function  $v = \chi u_\lambda$  on  $\Omega_{\frac{1}{2}}$ . Consequently, for a fixed  $m$  and the corresponding nodal points in the triangulation, we have

$$\begin{aligned} \|v\|_{K_1^{m+1}(\Omega_{\frac{1}{2}})}^2 &= \|\chi u_\lambda\|_{K_1^{m+1}(\Omega_{\frac{1}{2}})}^2 \\ &= \sum_{i+j \leq m+1} \left\| \sum_{k \leq i} x^{-1} (r_1 r_2)^{i+j} (x \partial_x)^{i-k} \partial_y^j u_\lambda (x \partial_x)^k \chi \right\|_{L^2(\Omega_{\frac{1}{2}})}^2 \\ &\leq C \|u_\lambda\|_{K_1^{m+1}(\Omega_{\frac{1}{2}})}^2, \end{aligned}$$

where  $C$  depends on  $m$  and the function  $\chi$ . Moreover, one notes that the interpolation  $v_I = u_{\lambda I}$  on  $\Omega_{\frac{1}{2}}$  by the definition of  $v$ .

Therefore, the proof is completed by the following inequalities

$$\begin{aligned} \|u - u_I\|_{K_1^1(\Omega^N)} &= \lambda^{-1/2} \|u_\lambda - v + v - u_{\lambda I}\|_{K_1^1(\Omega_{\frac{1}{2}})} \\ &\leq \lambda^{-1/2} (\|u_\lambda - v\|_{K_1^1(\Omega_{\frac{1}{2}})} + \|v - u_{\lambda I}\|_{K_1^1(\Omega_{\frac{1}{2}})}) \\ &\leq \lambda^{-1/2} (C_1 \|u_\lambda\|_{K_1^1(\Omega_{\frac{1}{2}})} + C_2 h_{y_N}^m \|v\|_{K_1^{m+1}(\Omega_{\frac{1}{2}})}) \\ &\leq \lambda^{-1/2} (C_1 \|u_\lambda\|_{K_1^1(\Omega_{\frac{1}{2}})} + C_3 h_{y_N}^m \|u_\lambda\|_{K_1^{m+1}(\Omega_{\frac{1}{2}})}) \\ &= C_1 \|u\|_{K_1^1(\Omega^N)} + C_3 h_{y_N}^m \|u\|_{K_1^{m+1}(\Omega^N)} \\ &\leq C_4 \left(\frac{\kappa^N}{2}\right)^\epsilon \|u\|_{K_{1+\epsilon}^m(\Omega^N)} + C_5 h_{y_N}^m \left(\frac{\kappa^N}{2}\right)^\epsilon \|u\|_{K_{1+\epsilon}^{m+1}(\Omega^N)} \\ &\leq C h^m \|u\|_{K_{1+\epsilon}^{m+1}(\Omega^N)}. \end{aligned}$$

The first and the fifth are from Lemma 2.11; the third and the fourth are the results of Theorem 4.2 and the relation between  $v$  and  $u_\lambda$ ; the sixth and the seventh are based on the construction of the triangulation.  $\square$

We here provide the proof for Theorem 4.5 by summing up the estimates on every region  $\Omega^j$  for  $j \leq N$ .

*Proof.* Since we assume the estimate is valid on  $\Omega \setminus \Omega_{\frac{1}{2}}$  that contains the vertices, it is sufficient to show the estimate still holds on  $\Omega_{\frac{1}{2}}$  for completing the proof.

The basic idea is to establish the estimate  $\|u - u_I\|_{K_1^1}$  on every  $\Omega^j$  for  $j = 0, 1, 2, \dots, N-1$  and on  $\Omega^N$ .

On every  $\Omega^j$ , based on the construction of the triangulation,  $\xi = \frac{\kappa^j}{2}$  for Theorem 4.3. Then, we have

$$\frac{h_{x_j}}{2\xi} \simeq h_{y_j} = \frac{l}{2^{N-(j+1)}}.$$

Recall that  $h$  represents the size of triangles in the region  $(\frac{1}{2}, \frac{3}{4}) \times (0, l)$ . Then  $h = O(l/2^N)$  after  $N$  successive refinements. Since  $\kappa^{N\epsilon} \leq C h^m$ , by Theorem 4.3, we have

$$\|u - u_I\|_{K_1^1(\Omega^j)} \leq C (2h)^m \|u\|_{K_{1+\epsilon}^{m+1}(\Omega^j)}.$$

As for the last region  $\Omega^N$ , we have  $\|u - u_I\|_{K_1^1(\Omega^N)} \leq C h^m \|u\|_{K_{1+\epsilon}^{m+1}(\Omega^N)}$  by Lemma 4.6. The proof of Theorem 4.5 then follows by adding the squares of all these norms on  $\Omega^j$  and  $\Omega^N$ .  $\square$

From this point, we assume that  $0 < \epsilon < 1$  is chosen such that

$$\mathcal{L}_\delta : K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1+\epsilon}^{m-1}$$

is an isomorphism, which is possible due to Corollary 3.3. Denote by  $V$  the finite element space corresponding to the triangulation in Definition 4.4. Let  $u_V \in V$  be the finite element solution of Equation (1). Then, we have the following estimate on  $\|u - u_V\|_{H^1(\Omega)}$ .

**Theorem 4.7.** *Let  $u \in K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\}$  be the solution for Equation (1),  $0 < \epsilon < 1$ . There exists a constant  $C$  for every  $m \geq 1$ , such that*

$$\|u - u_V\|_{H^1(\Omega)} \leq C(2h)^m \|f\|_{H^{m-1}(\Omega)}$$

for any  $\Gamma \in \mathcal{C}(l, h, \kappa, m, \epsilon)$  and  $\forall f \in H^{m-1}$ .

*Proof.* We denote by  $\gamma_\delta$  the norm of the inverse operator  $\mathcal{L}_\delta^{-1} : K_{-1+\epsilon}^{m-1} \rightarrow K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\}$ . The theorem can be proved by the following inequalities,

$$\begin{aligned} \|u - u_V\|_{H^1} &\leq M \|u - u_V\|_{K_1^1} \\ &\leq C_1 M \|u - u_I\|_{K_1^1} \\ &\leq C_2 M (2h)^m \|u\|_{K_{1+\epsilon}^{m+1}} \\ &\leq C_2 M \gamma_\delta (2h)^m \|f\|_{K_{-1+\epsilon}^{m-1}} \\ &\leq C (2h)^m \|f\|_{H^{m-1}}. \end{aligned}$$

The first and fifth inequalities are from Lemma 2.8; the second inequality is based on Céa's Lemma and the third inequality is from Theorem 4.5; the fourth inequality is obtained by the invertibility of the operator  $\mathcal{L}_\delta : K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\} \rightarrow K_{-1+\epsilon}^{m-1}$ .  $\square$

We have proved our theorems based on an explicit construction of the class  $\mathcal{C}(l, h, \kappa, m, \epsilon)$  for  $\Omega$ . Our estimates on the error were expressed by  $h$ , the size of those triangles in  $(\frac{1}{2}, \frac{3}{4}) \times (0, l)$ , as in the usual quasi-uniform finite element spaces. However, since there is no uniform size for the triangles in the triangulation, it is better to formulate the estimate in terms of the dimension of the finite subspace  $V$ . Based on the structure of the mesh we developed above, we attain the rate of convergence for the finite element solution  $u_V \in V$  as follows.

**Theorem 4.8.** *Let  $u \in K_{1+\epsilon}^{m+1} \cap \{u|_{\partial\Omega} = 0\}$  be the solution for Equation (1),  $0 < \epsilon < 1$ . There exists a constant  $C = C(l, \kappa, h, m, \epsilon)$  for  $m \geq 1$ , such that*

$$\|u - u_V\|_{H^1(\Omega)} \leq C \dim(V)^{-m/2} \|f\|_{H^{m-1}(\Omega)}$$

for any partition  $\Gamma \in \mathcal{C}(l, h, \kappa, m, \epsilon)$  and  $\forall f \in H^{m-1}(\Omega)$ .

*Proof.* Let  $Q_j$  and  $Q_{j-1}$  be the numbers of the elements in the meshes after  $j$  and  $j - 1$  refinements from the initial mesh respectively. Then, based on our construction, we have  $Q_j = 4 \times Q_{j-1} + 3$ , hence,  $Q_N = O(4^N)$ . On the other hand, the size of the triangles in  $(\frac{1}{2}, \frac{3}{4}) \times (0, l)$  satisfies  $h = O(2^{-N})$ , after  $N$  levels of refinement. Thus,  $\dim(V) \approx Q_N \approx (2h)^{-2}$  for every  $m \geq 1$ . From Theorem 4.7, we have the following estimate in terms of the dimension of the finite space

$$\|u - u_V\|_{H^1(\Omega)} \leq C \dim(V)^{-m/2} \|f\|_{H^{m-1}(\Omega)}.$$

This is also the optimal convergence rate of the finite element solution expected for a smooth solution.  $\square$

Based on the estimates for the interpolation error in weighted Sobolev spaces, we have designed a class of triangulations, on which the optimal rate of convergence is attained. In the next section, our theoretical results will be verified by comparing the convergence rates numerically in triangulations with different parameters  $\kappa$ .

## 5. NUMERICAL RESULTS

We here present the numerical results to demonstrate our method to approximate the solution. The following model problem in the case  $\delta = 1$  and the domain  $\hat{\Omega} = (0, 1) \times (0, 10)$  for Equation (1) is considered,

$$\begin{cases} -\partial_x^2 u - \frac{1}{x^2} \partial_y^2 u = 1 & \text{in } \hat{\Omega} \\ u = 0 & \text{on } \partial\hat{\Omega}. \end{cases}$$

We also have chosen  $m = 1$ , namely, piecewise linear functions for the FEM, because the implementation is then simpler, while the results are still relevant. From the previous theorems, the solution is not automatically in  $H^2(\hat{\Omega})$  near the degenerate boundary. In fact, a more accurate a priori estimate in the usual Sobolev spaces  $H^m(\hat{\Omega})$  shows that  $u \in H^s(\hat{\Omega})$  for  $s < 1 + \frac{\sqrt{100+4\pi^2}}{20} \approx 1.59$  [30]. Note that the operator  $\mathcal{L}_1$  is actually the Laplace operator  $-\Delta$  near the vertices  $(0, 1), (0, l)$ . The solution near those two corners, therefore, behaves like

$$u(r, \theta) = r^{2k} \zeta(\theta), \quad k \in \mathbb{Z}_+,$$

in polar coordinates [28], where the function  $\zeta$  is smooth, only depending on  $\theta$ . For this reason, the solution has no singularity near those two vertices in  $H^2$ . The vertices  $(1, 0), (1, l)$  do not affect the regularity of the solution in this case. Consequently, it is not necessary to apply graded mesh there. Moreover, with a direct calculation based on our arguments in Section 3, we have  $\eta_1 = \frac{\sqrt{100+4\pi^2}}{20}$  and  $\eta_2 = 2$  on  $\hat{\Omega}$ . Then, one can set  $0 < \epsilon < \eta = \min(\eta_1, \eta_2) = \frac{\sqrt{100+4\pi^2}}{20} \approx 0.59$  and take  $0 < \kappa = 2^{-1/\epsilon}$  for the graded mesh near the degenerate boundary  $x = 0$ . Therefore, we have the range  $\kappa < 2^{-1/0.59} \approx 0.309$ , on which the optimal rate of convergence in Theorem 4.8 holds for the model problem.

To construct the mesh on which we have the convergence rate as in Theorem 4.8, we start with three initial triangles (Fig. 5.1). In every step of refinement, we pick two points  $(x_1, 0)$  and  $(x_2, 10)$  as two vertices of the new triangle and the third vertex of the new triangle is placed at the mid-point of the base of the old triangle. Denote by  $d$  the minimum distance from any point in the old triangles to  $x = 0$ . Then, the parameter  $\kappa$  controls the position of the new points, such that  $x_1 = x_2, \kappa = x_1/d$ . Meanwhile, other new triangles are generated on the region that is enclosed by old triangles, by the mid-points as described in the previous section (Fig. 5.2). Therefore, the new triangle that is approaching  $x = 0$  is specially designed to fulfill the requirement in Definition 4.4, while all the other new triangles are generated by connecting the mid-points of the old triangles. In the last step, the last region  $\hat{\Omega}_{\frac{\kappa N}{2}}$  is divided into two triangles by the diagonal of the rectangle, as described in Section 4.

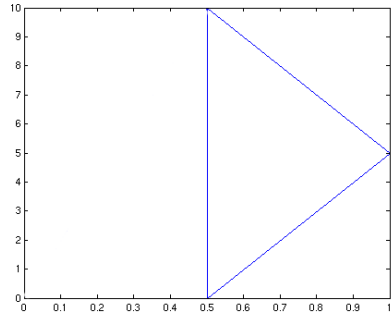


Fig. 5.1. Initial mesh with three triangles

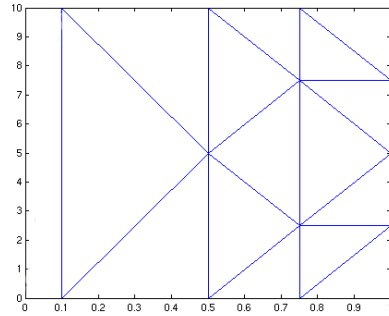


Fig. 5.2. Triangulation after one refinement  $\kappa = 0.2$

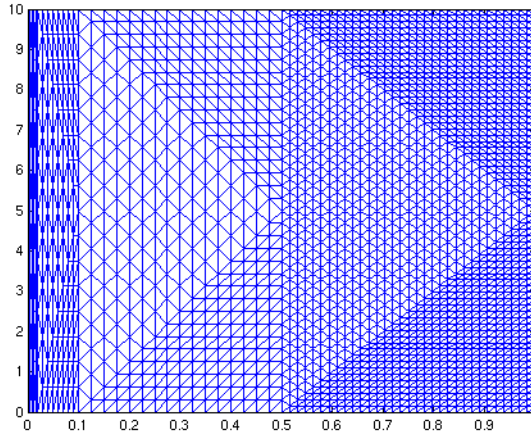


Fig. 5.3. The mesh after 5 levels of refinement for  $\kappa = 0.2$

One also notes that the triangles near the degenerate boundary are getting thinner and thinner in our construction, in which the maximum-angle condition is apparently violated. Nevertheless, the difficulty is already overcome by Theorem 4.3 and Theorem 4.5.

The finest mesh in our numerical experiments is obtained after 10 successive refinements of the coarsest mesh and has roughly  $2^{22} \approx 4 \times 10^6$  elements. The preconditioned conjugate gradient (PCG) method is used to solve the resulting system of algebraic equations.

$j$	$e : \kappa = 0.1$	$e : \kappa = 0.2$	$e : \kappa = 0.25$	$e : \kappa = 0.35$	$e : \kappa = 0.4$	$e : \kappa = 0.5$
2	0.45	0.46	0.43	0.35	0.31	0.22
3	0.66	0.63	0.59	0.48	0.42	0.31
4	0.82	0.78	0.73	0.61	0.54	0.40
5	0.92	0.88	0.84	0.71	0.63	0.47
6	0.97	0.94	0.89	0.76	0.68	0.52
7	0.99	0.96	0.93	0.79	0.71	0.55
8	1.00	0.98	0.94	0.81	0.73	0.56
9	1.00	0.99	0.96	0.82	0.74	0.57

Table 5.1. Convergence history with  $e = \log_2 \left( \frac{|u_j - u_{j-1}|_{K_1^1}}{|u_{j+1} - u_j|_{K_1^1}} \right)$ .

We have tested several values of the parameter  $\kappa$  for the model problem. The convergence rates are summarized in Table 5.1. These results convincingly show that the theoretical approximation order can be verified in practical calculations with  $\kappa < 0.3$ .

The left most column in Table 5.1 shows the number of the refinement level, and  $u_j$  denotes the numerical solution corresponding to the mesh after  $j$  levels of refinement. The quantity printed out in other columns in the table represents the convergence rate in the manner

$$e = \log_2 \left( \frac{|u_j - u_{j-1}|_{K_1^1}}{|u_{j+1} - u_j|_{K_1^1}} \right).$$

The quantity  $e$  is not an exact convergence rate, but it turns out to be a quite reasonable approximation to it. Recall  $h$  stands for the triangle size in  $(\frac{1}{2}, \frac{3}{4}) \times (0, l)$ . We have already seen that the correctly graded refinement improves the convergence rate with a factor of about  $h^{0.43}$ . In fact, the improvement may be better as our theory shows that the convergence rate in the case  $\kappa < 0.3$  is  $h^1$  by Theorem 4.5.

Our theoretical prediction for the convergence rate in the case  $\kappa = 0.5$  is about  $h^{0.59}$ , and the theoretical prediction for the convergence rate in the cases  $\kappa = 0.1$ ,  $\kappa = 0.2$  is  $h^1$ . Moreover, one can see a big jump in the rates between  $\kappa = 0.25$  and  $\kappa = 0.35$ , which strongly supports our theory for the critical number  $\kappa \approx 0.3$ . Thus, our numerical results completely agree with the theory we have presented in this paper. Based on the behavior of the sequence of the numbers in every column, it is also reasonable to expect the optimal rate by doing more refinements for all  $\kappa < 0.3$ . Therefore, we conclude from our numerical results, that, for a correct refinement, the difference between consecutive solutions is decreasing like  $\dim(V_n)^{-1/2}$ , which verifies Theorem 5.1.

One may also notice that those numbers in each column keep increasing as  $\kappa > 0.3$ . An explanation is that the solution consists of a singular and a regular part:  $u = u_s + u_r$ . The regular part  $u_r$  dominates the behavior of the solution until  $x$  is sufficiently close to the degenerate boundary, when the singular part can be taken into account [5]. Therefore, as shown in the table, the increasing rate slows down and will be fixed at some point.

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