KILLED DIFFUSIONS AND ITS CONDITIONING

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IMA Preprint Series # 213

January 1986

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Abstract

Let $X = (X_t)_{t \geq 0}$ be a diffusion determined by an elliptic differential operator $L$ in $\mathbb{R}^n$ $(n > 1)$. For any bounded $C^{1,1}$ domain $D$, we define the conditional killed diffusion $X^\phi$ on $D$ by the semigroup:

$$T^\phi_t(x) = \phi_0(x)^{-1}E_x[f(X_t)\phi_0(X_t), \tau_D > t]e^{\lambda_0 t} \quad (t > 0)$$

where $\lambda_0$ and $\phi_0$ is the principle eigenvalue and eigenfunction of $L$ on $D$. In this paper, we prove that $X^\phi$ is a strong Feller process on $D$ and $\{T^\phi_t\}$ has the strong continuity on $C(\bar{D})$. For any $T > 0$ we consider the conditioned process $X^T$, i.e. the process $X$ in $D$ conditioned on $\{\tau_D > T\}$, and prove that $X^T$ weakly converges to $X^\phi$ as $T \to \infty$ without any additional hypotheses.

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1. INTRODUCTION

We consider the second order differential operator on $\mathbb{R}^n$:

$$Lu = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}(x) \frac{\partial u}{\partial x_i} + \sum b_j(x) \frac{\partial u}{\partial x_j}$$

where $a_{ij}(x), b_j(x) \in C_{1,\alpha}^1(\mathbb{R}^n), i,j = 1, \ldots, n, \alpha > 0$, and for all $N > 0$, there exist $\beta_{i,N} > 0 (i = 1,2)$ such that $a = (a_{ij}(x))$ satisfies

$$0 < \beta_{1,N} |x| < a(x) < \beta_{2,N} |x| (|x| < N)$$  \hspace{1cm} (1)

$L$ generates a diffusion process $X = (X_t)_{t \geq 0}$ on $\mathbb{R}^n$ with $L$ as the infinitesimal generator of the corresponding Markov semigroup. Let us deal with $L$ and $X$ locally, i.e. we confine $\{X_s: 0 < s < T\}$ on the pre-exit event $\{\tau_D > T\}$ (where $T > 0$, $\tau_D = \inf\{t > 0: X_t \in \mathbb{D}\}$), and then obtain the killed diffusion $X^D$. The corresponding killed semigroup is

$$T^D_t f(x) = E^x[f(X^D_t): t < \tau_D].$$

In this paper, it is given that for a bounded open $C^1,1$ domain $D$ and all $t_0 > 0$, there exists a constant $C = C(L,D,t_0) > 0$, such that

$$\frac{1}{C} d(x) e^{-\lambda_0 t} < P_x(\tau_D > t) < C d(x) e^{-\lambda_0 t}$$  \hspace{1cm} (3)

for all $(t,x) \in [t_0,\infty) \times D$, where $\lambda_0$ is the principle eigenvalue of $L$,

$$d(x) \triangleq \text{dis}(x,\partial D),$$

And a uniform estimate is given

$$\sup_{x \in D} \frac{e^{\lambda_0 t} P_x(\tau_D > t)}{\phi_0(x)} - \int_\mathbb{D} \psi_0(x) dx | < 0$$  \hspace{1cm} (4)

for time-space parameters, where $\phi_0, \psi_0$ are principle eigenfunctions of $L$ and its formal adjoint $L^\ast$ respectively.
However, the killed diffusion disappears gradually as $t \to \infty$. It is natural to consider the diffusion $\{X_t: 0 < t < T\}$ under the conditional probability measure $P^T_{s,x} = P_x(\cdot | \tau_D > T - s)$ (for each fixed $T > 0$) and its weak limit as $T \to \infty$. In the case of Markov chain, this was done by [2], [14]. For diffusions Ross Pinsky proved [12] that the conditioned diffusion $x^T$ up to time $T$ is inhomogeneous with the generator $L^T$ depending on $T$, and under three hypotheses (which seem hard to check) the coefficients of $L^T$ converge locally uniformly to those of the operator $L + a(V \ln \phi) V$ (therefore the weak convergence of the processes follows). But even in the case of $L$ being symmetric those three hypotheses have not been justified rigorously there. In this paper we approach this conditioning problem in a different way. We define directly a Markov process $X^\phi$ in $D$ given by the semigroup

$$T^\phi_t f(x) = \phi_0(x)^{-1} E_x[f(X_t)\phi_0(X_t): \tau_D > t] e^{\lambda_0 t} \tag{5}$$

Since $\phi_0$ is a positive "harmonic" function on $D$ with respect to $-(L + \lambda_0)$, this definition is a natural generalization of the concept of conditional Brownian motion given by Doob [3]. We call $X^\phi$ the conditional killed diffusion, which has been considered by Zhao [18] in a different situation. Moreover, we prove that $T^\phi_t$ has the strong Feller property. A more delicate property is the strong continuity of $T^\phi_t$ on $C_U(D)$ (the Banach space of uniformly continuous functions on $D$), or equivalently, for any given $\delta > 0$

$$\sup_{x \in D} |\phi_0(x)^{-1} E_x(\phi(X_t), |X_t - x| > \delta, \tau_D > t)| \to 0 \quad (t \to 0) \tag{6}$$

The strong continuity of $T^\phi_t$ is not only an analytical property of the semigroup, but also is the key step to get the tightness of the measures $P^T_{s,x}$. On the other hand, with the help of (5) and (4) we have the convergence of the transition functions of $x^T$. Thus, it turns out that $x^T$ weakly converges to $X^\phi$ without any additional assumptions.
2. SOME RESULTS IN ANALYSIS

Let \( a = (a_{ij}(x)), b = (b_1(x), ..., b_n(x)) \) be \( C^{1, \alpha} \) functions on \( D \), where \( i, j = 1, 2, ..., n \), and

\[
0 < \gamma_1 I < a < \gamma_2 I \quad (x \in D)
\]

(7)

Denote

\[
L = \frac{1}{2} \nabla a \nabla + b \nabla - c(x) \quad (c(x) > 0, c(x) \in C^{0, \alpha}(D))
\]

(8)

\[
L_a = \frac{1}{2} \nabla a
\]

and \( G_L, G_{L_a}, G_\Delta \) are Green functions of \( L, L_a, \frac{1}{2} \Delta \) respectively. Some analytic facts which we need are listed below.

Theorem (A) (Widman [17])

\[
G_L(x, y) \leq C \frac{1}{|x - y|^{n-1}} (|x - y| \wedge d(x))
\]

\[
\left| \nabla G_L(x, y) \right| \leq K \frac{1}{|x - y|^{n-1}} \quad (n > 3)
\]

Theorem (B) (Hueber and Sieveking [7])

\[
\frac{1}{c} G_\Delta < G_L < cG_\Delta \quad (c > 0)
\]

Now we have

**Lemma 1**

\[
\sup_{x \in D} \int_{D} \frac{dy}{|x - y|^{n-1}} < \infty
\]

and measures \( \int \frac{dy}{A |x - y|^{n-1}} \) are absolutely continuous with respect to \( dy \) and uniformly for \( x \in D \).
Proof: Taking the $n$-dimensional spherical polar coordinate centered at $x$, we get

$$\sup_{x \in D} \int_{A \mid |x - y| < \delta} \frac{dy}{|x - y|^{n-1}} \leq \sup_{x \in D} \left( \int_{|x - y| < \delta} \frac{dy}{|x - y|^{n-1}} + \int_{|x - y| > \delta} \frac{dy}{|x - y|^{n-1}} \right)$$

$$\leq \frac{\delta}{\int_0^\infty \frac{r^{n-1}drd\alpha}{r^{n-1}}} + \frac{1}{\delta^{n-1}} m(A) = \delta \sigma + \frac{1}{\delta^{n-1}} m(A)$$

where $m(\cdot)$ is the Lebesgue measure and $\sigma$ is the area of the unit sphere in $\mathbb{R}^n$. Then the proof is complete.

**Lemma 2** If $f \in M_b(D)$ (bounded measurable function on $D$) $f > 0$ and $f > 0$ at least on a small ball, then there are $c_1 > 0$, $c_2$ such that

$$c_1 \leq \int_D \frac{G_L(x,y)f(y)dy}{d(x)} \leq c_2 \quad (x \in D).$$

Proof. Since Theorem (B), (A) and Lemma 1, the upper bound follows and for the lower bound we only need to investigate when $L = \Delta$ and $x$ satisfies $d(X) < \delta$ with small $\delta > 0$. If, on the contrary, this lemma fails, then there are $x_m \in D$ such that

$$\int_D G(x_m,y)f(y)dy = 0 \quad (m \to \infty)$$

Without loss of generality, we can assume $x_m + x_0 \in \partial D$. Denote the inward normal of $\partial D$ at $x_m$ to be $n_m$ and pick up $x^*_m \in \partial D$ such that $|x_m - x^*_m| = d(x_m)$. Because $\nabla x \Delta(x,y)$ can be extended continuously to $\overline{D \setminus \{y\}}$ [5],[6], with the help of the mean value theorem we obtain

$$G_\Delta(x_m,y) = G_\Delta(x_m,y) - G_\Delta(x^*_m,y)$$

$$= \frac{\partial G_\Delta}{\partial n_m} (\xi_m(y),y)d(x_m) \quad (\xi_m(y) \in x_m^{*m}).$$
Thus

$$\int_{D} \frac{G(x_m, y)}{d(x_m)} f(y) dy = \int_{D} \frac{a G(x_m, y)}{an_m} (\xi_m(y), y) f(y) dy$$

Now the estimate in Theorem (A) and Lemma 1 allows us to pass the limit under the integral and then get

$$\int_{D} \frac{G(x_m, y)}{d(x_m)} f(y) dy + \int_{D} \frac{a G(x_m, y)}{an} (x_0, y) f(y) dy \quad (m = \infty)$$

with $\frac{a G}{an} \mid_{\partial D} \geq \text{const} > 0$ [5], [6]. Hence it should be positive. This contradiction fulfills the validity of our lemma.

From now on, we assume $a, b$ satisfying conditions in Section 1 and $c(x) \in C^{0, \alpha}_{loc}(\mathbb{R}^n)$.

**Proposition 1.** Let

$$L(\lambda) \triangleq (L - \lambda), \quad G(\lambda) \triangleq G(\lambda) \quad (\lambda > 0)$$

Then there are $\alpha > 0$ and $C > 0$ such that

$$G(x, y) < C e^{-\alpha \sqrt{\lambda}} \frac{|x - y|}{|x - y|^{n-2}} \quad (n > 3) \quad (9)$$

where

$$G(x, y) = \int_{0}^{\infty} e^{-\lambda t} p^D(t, x, y) dt \quad (10)$$

and $p^D(t, x, y)$ is the transition density of the killed diffusion $X^D$ of $X$ at $\partial D$.

**Proof.** It is well known that $p^D(t, x, y)$ is the Green function of $\frac{\partial u}{\partial t} = Lu$ on $D$ and $p^D(t, x, y) |_{x \in \partial D} = 0$. Meanwhile estimation

$$p(t, x, y) < C_{\varepsilon, T} \frac{1}{\varepsilon^{n/2}} e^{-\varepsilon \frac{(\gamma_1 - \varepsilon)|x - y|^2}{4t}} \quad (11)$$

$$0 < t < T, \quad x, y \in \mathbb{R}^n, \quad 0 < \varepsilon < \gamma_1$$
holds for the transition density $p(t,x,y)$ of $X$ with a constant $C_{\varepsilon,T}$ ($\gamma_1$ is defined in (7)). [10] And the existence of $p^D(t,x,y)$ and its property can be established in a purely probabilistic way without the help of PDE just like what is done in the killed brownian motion case [13]. Now the estimate (11) reduces our case to the classical brownian one. Combinind (11) and

$$p^D(t,x,y) \leq p(t,x,y),$$

we have for small $\eta > 0$ that

$$\int_0^\delta e^{-\lambda t} p^D(t,x,y) dy \leq C_{\varepsilon,\delta,1} \frac{e^{-\sqrt{\gamma_1-\varepsilon}/(1+\eta)\sqrt{\lambda}|x-y|}}{\frac{\delta \cdot d(D)^2}{(1+\eta)^{n/2}} \cdot \frac{1}{u^{n/2}}} du$$

($d(D) =$ diameter of $D$)

$$\leq C_{\varepsilon,\delta,\eta} \frac{e^{-\sqrt{\gamma_1-\varepsilon}/(1+\eta)\sqrt{\lambda}|x-y|}}{|x-y|^{n-2}}$$

On the other hand

$$\int_\delta^\infty e^{-\lambda t} p^D(t,x,y) dt$$

$$\leq e^{1+\eta} \frac{d(D)}{4\delta} e^{-\sqrt{\gamma_1-\varepsilon}/(1+\eta)\sqrt{\lambda}|x-y|} \int_\delta^\infty p^D(y,x,y) dt,$$

And by Theorem (B) and (A)

$$\int_\delta^\infty p^D(t,x,y) dt \leq \int_0^\infty p^D(t,x,y) dt = G_L(x,y)$$

$$< cG_\Delta(x,y) < \frac{\tilde{C}}{|x-y|^{n-2}}$$
Thus (9) holds for $\alpha = \sqrt{\frac{\gamma_1 - \varepsilon}{1 + \eta}}, C = C_{\varepsilon, \delta, \eta} V(Ce^{1+\eta}) \frac{d(D)}{4\delta}$.

Remark Let

$$\hat{L} = \frac{1}{2} \nabla \nabla - \nabla(b \cdot)$$

to be the formal adjoint of $L$, and $\lambda > - \inf \{(\div b(x))\}$, then proposition 1 keeps valid for $G_{\lambda} = G_{L(\lambda)}$ where $L(\lambda) = \hat{L} - \lambda$.

Proposition 2 There exists $\beta > 0$ such that

$$G_{\lambda}(x,y) \leq \frac{C_{d(x)} e^{-\beta\sqrt{\lambda}|x-y|}}{|x-y|^{n-1}} \quad (n > 3) \quad (12)$$

Proof Inequality (12) follows by modifying Widman's arguments in [17]. For the specified annular domain $A$ (which is denoted as $D$ in [17]), $G_{\lambda}(z,y)$ is majorized by $\frac{C_{e^{-B\sqrt{\lambda}|x-y|}}}{|x-y|^{n-2}}$ and zero on $\partial A \cap D$ and $\partial D$ respectively by Proposition 1. Then by the maximum principle and Theorem (B) $G_{\lambda}(z,y)$ ($\lambda$ sufficiently large) in $A$ can be majorized by the harmonic function with the boundary value $\frac{C_{e^{-B\sqrt{\lambda}|x-y|}}}{|x-y|^{n-2}}$ and zero on $\partial A \cap D$ and $\partial D$ respectively. Thus the same arguments as [17] lead to the inequality (12).

From now on we always assume $c(x) \equiv 0$.

We need a version of Frobenius theorem of the elliptic PDE. A familiar fact is that this kind of counterpart can be proven by Krein-Rutman's theorem [9] and the strong maximum principle [5]. We state it as follows

Theorem (C)

$-L$ and $-\hat{L}$ with the Dirichlet boundary condition on $\partial D$ have a common simple positive eigenvalue $\lambda_0$ at the left of their spectrum. The corresponding eigenfunctions, $\phi_0$ and $\psi_0$, can be chosen positive on $D$ and vanish on $\partial D$. 
For this we have to do a little bit explanation both for the sketch of its proof and convenience of its application.

Let us introduce two Banach spaces:

\[ B(D) \triangleq \{ f: f(x) = d(x)\tilde{f}(x), \tilde{f} \in C_b(D), \| f \|_{B(D)} \triangleq \| \tilde{f} \|_{C_b(D)} \} \]

\[ B_0(D) \triangleq \{ f: f(x) = d(x)\tilde{f}(x), \tilde{f} \in C_U(D), \| f \|_{B_0(D)} \triangleq \| \tilde{f} \|_{C_U(D)} \} \]

where \( C_b(D) \) = space of bounded continuous functions and \( C_U(D) \) is equivalent to \( c(\overline{D}) \). \( B_0(D) \) is a closed subspace of \( B(D) \). \( B_0(D) \) is included in \( C_0(D) \) (\( C_b(D) \) functions vanishing at \( \partial D \)) but equipped with a stronger topology.

**Lemma 3**

1° Let \( G_\lambda \) be the operator with the kernel \( G_\lambda(x,y) \). Then we have

\[ G_\lambda M_b(D) \subset B_0(D). \]

And \( G_\lambda \) is compact on \( B_0(D) \).

The semigroup \( T_t^D \) \((t \geq 0)\) in (2) maps \( M_b(D) \) into \( B_0(D) \). Besides, \( T_t^D \)
is compact on \( B_0(D) \) and on \( C_0(D) \) and strongly continuous on \( C_0(D) \).

**Proof:**

1°. We have \( G_\lambda f \in C_b(D) \) and

\[
\left| \frac{G_\lambda f(x')}{d(x')} - \frac{G_\lambda f(x)}{d(x)} \right| < \int_D \left| \frac{G_\lambda(x,y)}{d(x)} - \frac{G_\lambda(x',y)}{d(x')} \right| |f(y)|dy
\]

for any \( f \in M_b(D) \). The left hand side goes to zero when \( x' + x \in D \). In the case of \( x' + x \in \partial D \), \( \nabla G(x,y) \) can be extended continuously to \( \overline{D} \setminus \{ y \} \) [5], [6] which ensures that the left hand side turns also to zero. That means \( \frac{G_\lambda f}{d(x)} \) can be taken as a function in \( C(\overline{D}) \), i.e. \( G_\lambda f \in B_0(D) \). A very similar
argument shows that $\frac{G_x f}{d(x)}$ is equivalently continuous on $\overline{D}$ for all $f = \tilde{f}_{t}$ with $\tilde{f}_{t} < 1$ which leads to the compactness of $G_x$ on $B_0(D)$.

2°. First we point out that for any $f \in M_b(D)$

$$\frac{1}{d(x)} \int_D p^D(t,x,y) f(y) dy \in C^b(D)$$

holds, which says $T^D_{t} \subseteq B(D)$. In fact, $\int_D p^D(t,x,y) f(y) dy$ is continuous [10]. Combining Theorem (B), (A) and Lemma 1, we get

$$\sup_{x \in D} \frac{1}{d(x)} \left| \int_D p^D(t,x,y) f(y) dy \right|$$

$$< \sup_{x \in D} \frac{1}{d(x)} \int_D E^D(t) \int_D \frac{G_L(x,y)}{d(x)} dy \leq C \sup_{x \in D} \int_D \frac{G_A(x,y)}{d(x)} dy$$

$$< C \sup_{x \in D} \int_D \frac{dy}{|x-y|^{n-1}} < \infty.$$ 

Next we show that $\frac{1}{d(x)} \int_D p^D(t,x,y) f(y) dy$ is uniformly continuous on $D$. This can be done as Lemma 2 with the well-known fact that $\int_D p^D(t,x,y)$ can be extended continuously to $D$ with respect to $(x,y)$ ([10], Th 16.3). From these two steps we get $T^D_{t} f \in B_0(D)$. The compactness of $T^D_{t} (t > 0)$ on $B_0(D)$ or $C_0(D)$ can be proven just as we did in 1° with $G_x$. Finally, let us prove that $T^D_{t} f$ is strongly continuous on $C_0(D)$. For this we pick up

$$\bar{a}_{ij}(x), \bar{b}_i(x) \in C_b(R^n) \cap C^1_{loc}(R^n)$$

such that

$$\bar{a}_{ij}(x) = a_{ij}(x), \quad \bar{b}_i(x) = b_i(x) \quad (x \in D, i,j = 1,\ldots,n)$$
The Markov process $\mathcal{X}$ generated by $\mathcal{L}$ with coefficients $a_{i,j}$, $b_i$ is a strong Feller process with strongly continuous semigroup on $C_0(D)$. However $\mathcal{X}^D$ and $\chi^D$ have the same (killed) semigroup on $D$. By a fact in [1], $T^D_t$ is strongly continuous on $C_0(D)$. That accomplishes the lemma.

Late on, let $G_\lambda$ be the operator with the kernel $G_\lambda(x,y)$: the Green function of $-\mathcal{L} - \lambda$.

We have $\int_D \frac{G_\lambda(x,y)}{d(x)} f(y) dy > \epsilon_\lambda > 0$ (for some $\epsilon_\lambda$ and $\forall f \in B^+_0(D)$ i.e. $f \in B_0(D)$ and $f > 0$) as in Lemma 2 (Similar inequality holds for $G_\lambda$ with $\lambda$ large enough). It implies that $G_\lambda$ is strictly positive on $B^+_0(D)$ in the sense of Krein-Rutman [9]. Now their famous theorem works: The first eigenvalue $\gamma(\lambda)$ of $G_\lambda$ is positive and simple with a positive eigenfunction $\phi_0^\lambda(x)$. And the resolvent identity $G_\mu - G_\lambda = (\lambda - u)G_\mu G_\lambda$ assures that $\phi_0(x)$ is independent of $\lambda$, i.e. $\phi_0^\lambda(x) = \phi_0(x)$, and $\gamma(\lambda) = \frac{1}{\lambda + \lambda_0}$. Similarly $G_\lambda$ has a positive eigenfunction $\psi_0(x)$ with positive simple first eigenvalue $\frac{1}{\lambda + \lambda_0}$ when $\lambda$ is large enough. Since we have $\langle G_\lambda f, g \rangle_{L^2(D)} = \langle f, G_\lambda g \rangle_{L^2(D)}$ for $f, g \in L^2(D)$ and $G_\lambda$, $\hat{G}_\lambda$ are compact both on $B_0(D)$ and $L^2(D)$, we should have $\lambda_0 = \lambda_0$, and the other generalized eigenspace $M_j$, $\hat{M}_j$ of $G_\lambda$, $\hat{G}_\lambda$ are finite dimensional and can be chosen that the corresponding point spectrum are $\frac{1}{\lambda + \lambda_j}$ and $\frac{1}{\lambda + \overline{\lambda}_j}$, and $M_j \perp \hat{M}_k$ in $L^2(D)$ when $j \neq k$. Meanwhile $\phi_0$, $\psi_0$ are determined uniquely by

$$
\begin{align*}
-L\phi &= \lambda_0\phi \\
\phi|_{\partial D} &= 0 \\
-L\psi &= \lambda_0\psi \\
\psi|_{\partial D} &= 0 \\
\phi &> 0 \text{ on } D \\
\psi &> 0 \text{ on } D.
\end{align*}
$$

Surely, we can assume that $\int \phi_0(x)\psi_0(x) dx = 1$. Moreover $\{\phi_0, \phi_j^\lambda : j, \lambda \text{ varies}\} \cup \{\psi_j^\lambda : j, \lambda \text{ varies}\}$ generates $L^2(D)$, where $\phi_j^\lambda \in M_j$, $\psi_j^\lambda \in \hat{M}_j$ are given by
\[(\lambda_j + L)^{m_{jL}} \phi_{jL} = 0 \quad \text{and} \quad (\lambda_j + L)^{\hat{m}_{jL}} \psi_{jL} = 0\]
\[\phi_{jL} \mid_{\partial D} = 0 \quad \text{and} \quad \psi_{jL} \mid_{\partial D} = 0\]

with \(m_{jL} < \dim M_j\), \(\hat{m}_{jL} < \dim \mathfrak{A}_j\) respectively.

**Proposition 3**

\[\phi_0, \psi_0 \in B_0(D).\]

And there are \(\beta_2 > \beta_1 > 0\) such that

\[\beta_1 d(x) < \phi_0(x) < \beta_2 d(x),\]
\[\beta_1 d(x) < \psi_0(x) < \beta_2 d(x).\]

**Proof:** Since we can take a sufficiently large \(\lambda\) such that

\[\psi_0(x) = \frac{(\lambda + \lambda_0) \int_D G_\lambda(x,y)\psi_0(y)dy}{d(x)},\]

the proposition follows immediately from Theorem (B) and Lemma 2.

**Corollary** \(\phi_0(x)\) or \(\psi_0(x)\) can be used instead of \(d(x)\) in the definition of \(B_0(D)\).

Let us denote the Green function of \(\frac{\partial^2 u}{\partial t^2} = Lu\) on \(D\) by \(\hat{p}^D(t,x,y)\) \((= p^D(t,y,x))\) and set

\[\hat{r}^D_t f(x) = \int_D \hat{p}^D(t,x,y)f(y)dy.\]

**Proposition 4**

Semigroups \(T^D_t\) and \(T^D_t\) on \(B_0(D)\) have a common first eigenvalue \(e^{-\lambda_0 t}\)

which is simple with eigenfunctions \(\phi_0\) and \(\psi_0\) respectively.
Proof: By Lemma 3, $T^D_t$ only has non-zero pure point spectrum. Then it follows from the spectral mapping theorem ([8] Th. 16.7.2) that $T^D_t$ and $T^D_t$ have the first eigenvalue $e^{-\lambda_0 t}$. Since $G_\lambda$ and $G_{\lambda}$ have no generalized eigenfunction with the eigenvalue $e^{-\lambda_0 t}$ either. That shows $e^{-\lambda_0 t}$ is a simple eigenvalue of $T^D_t$ and $T^D_t$.

**Theorem 1**

$T^D_t$, $T^D_t$ as semigroups on $B_0(D)$ are strongly continuous.

Proof: By Hille-Yosida Theorem, it suffices to prove that

$$\|\lambda G_\lambda - f\|_{B_0(D)} \to 0, \|\lambda G_\lambda f - f\|_{B_0(D)} \to 0 \quad (\lambda \to \infty) \quad (13)$$

It follows from Proposition 2 that

$$\frac{1}{d(x)} \int_D \lambda G_\lambda(x, y)I_{\{|x - y| > \delta\}} dy$$

$$\leq \frac{c}{\delta^{n-1}} \lambda e^{-B\sqrt{\lambda} \delta} m(D) \to 0 \quad (\lambda \to \infty)$$

(uniformly with respect to $x$).

Thus Proposition 3 gives us

$$\sup_{x \in D} \frac{\lambda G_\lambda(x, y)}{\phi_0(x)} \to 0 \quad (\lambda \to \infty) \quad (14)$$

Now for any $f \in B_0(D)$, by the corollary of Proposition 3 there is a $\tilde{f} \in C_0(D)$ such that $f = \tilde{f} \phi_0$. Then we have
\[ \| \lambda G f - f \|_{B_0(D)} = \| \frac{\lambda G}{\phi_0} f \|_C(D) \]

\[ \leq \sup_{x \in D} \int_D \frac{\lambda G(x,y) \phi_0(y)}{\phi_0(x)} |\tilde{f}(y) - \tilde{f}(x)| \, dy + \]

\[ + \sup_{x \in D} |\tilde{f}(x)| \int_D \frac{\lambda G(x,y) \phi_0(y)}{\phi_0(x)} \, dy - 1 | \]

\[ \leq C_1 \sup_{x \in D} \int_{|x-y| > \delta} \frac{\lambda G(x,y)}{\phi_0(x)} \, dy + \max_{x \in D} \sup_{|y-x| < \delta} |\tilde{f}(y) - \tilde{f}(x)| \frac{\lambda}{\lambda + \lambda_0} + \]

\[ + \| \tilde{f} \|_{C(D)} \left( \frac{\lambda}{\lambda + \lambda_0} - 1 \right) . \]

Hence (13) follows from (14).

3. KILLED DIFFUSIONS

Let us assume that \( X = (\Omega, F, F_t, P, \theta_t, X) \) is the Markov process generated by \( L \) on \( \mathbb{R}^n \).

Denote

\[ T_t^\psi f(x) = e^{\lambda_0 t} \frac{T_t^D(\phi_0 f)(x)}{\phi_0(x)} = \int_D p^\psi(t, x, y) f(y) \, dy \]

and

\[ \hat{T}_t^\psi f(x) = e^{\lambda_0 t} \frac{\hat{T}_t^D(\psi_0 f)(x)}{\psi_0(x)} = \int_D \hat{p}^\psi(t, x, y) f(y) \, dy \]

where \( f \in C_0(D) \) and
\[ p_{\phi}(t,x,y) \triangleq \frac{\lambda_0 t D(t,x,y) \phi_0(y)}{\phi_0(x)} \]

\[ \hat{p}_{\psi}(t,x,y) \triangleq \frac{\lambda_0 t D(t,x,y) \psi_0(y)}{\psi_0(x)} = \frac{\lambda_0 t D(t,y,x) \psi_0(y)}{\psi_0(x)} \]

It is easy to see that \( T_{t1}^\phi = 1, T_{t1}^\psi = 1. \)

**Theorem 2** There exist \( C_2 \) (depending on \( t_0 \)) \( > C_1 > 0 \), such that for \( t > t_0 \) and \( x \in D \)

\[ C_1 e^{-t \lambda_0 D(x)} < p_x(\tau_D > t) < C_2 e^{-t \lambda_0 D(x)} \] \( (15) \)

and

\[ C_1 e^{-t \lambda_0 \phi_0(x)} < p_x(\tau_D > t) < C_2 e^{-t \lambda_0 \phi_0(x)} \] \( (16) \)

hold.

**Proof:** Since Proposition 3 we only need to check (16). For the lower bound, we have

\[ p_x(\tau_D > t) = \int_D \frac{p_D(t,x,y) \phi_0(y)}{\phi_0(x)} e^{\lambda_0 t} \frac{1}{\phi_0(y)} \ dy \]

\[ > \sup_{x \in D} \frac{1}{\phi_0(y)} \triangleq C_1 > 0. \]

For the upper bound, we recall from Lemma 3 and Proposition 3 that there exists a constant \( C_{t2} \) such that \( g_t(x) \triangleq \int_D \frac{p_D(t,x,y)}{\phi_0(x)} \ dy \leq C_{t2}. \) Hence for \( t > t_0 \)

\[ p_x(\tau_D > t) = T_{t-t_0}^\phi g_{t_0}(x) \leq C_{t0} T_{t-t_0}^\phi 1 \leq C_{t2} \]

since \( T_{t1}^\phi = 1. \)
Theorem 3

There exists a constant \( c \) only depending on \( D \), such that

\[
\sup_{x \in D} \left| e^{\lambda_0 t} \frac{P_x(\tau_D > t)}{\phi_0(x)} - c \right| \to 0 \quad (t \to \infty)
\]  

(17)

Moreover

\[
\sup_{x \in D} \left| e^{\lambda_0 t} \frac{\int_D P(t,x,y)f(y)dy}{\phi_0(x)} - \int_D f(y)\psi_0(y)dy \right| \to 0 \quad (t \to \infty)
\]  

(18)

\[
\sup_{x \in D} \left| e^{\lambda_0 t} \frac{\int_D \rho(t,x,y)f(y)dy}{\psi_0(x)} - \int_D f(y)\phi_0(y)dy \right| \to 0
\]  

(18')

where \( f \in M_b(D) \).

Proof:

First we assume \( f \in B_0(D) \).

The compactness of \( T^D_t \) on \( B_0(D) \) implies that there exists a decomposition of \( B_0(D) \) into the direct sum of its subspaces \( \mathcal{M}_{\lambda_0} \) and \( \mathcal{N}_{\lambda_0} \) which are invariant under \( e^{-\lambda_0 t} - T^D_t \), and

\[
B_0(D) = \mathcal{M}_{\lambda_0} + \mathcal{N}_{\lambda_0}, \quad \mathcal{M}_{\lambda_0} = \{ a\phi_0 : \alpha \in \mathbb{R}^\prime \}
\]

\[
T^D_t \mathcal{N}_{\lambda_0} \subset \mathcal{N}_{\lambda_0}
\]

and \( T^D_t \) is compact on \( \mathcal{N}_{\lambda_0} \), while \( e^{-\lambda_0 t} \) is no more in the spectrum of \( T_t \) on \( \mathcal{N}_{\lambda_0} \) ([11] Section 6.2, Th. 6).

Now \( f \) can be written into

\[
f = c_f \phi_0 + g \quad g \in \mathcal{N}_{\lambda_0}.
\]

And then
\[ T^D_t f = c_f e^{-\lambda_0 t} \phi_0 + T^D_t g, \]

Let \( \lambda_1 \) be in the point spectrum of \( T^D_t \) with the smallest real part bigger than \( \lambda_0 \). The spectrum radius theorem tells us that

\[
\lim_{t \to \infty} \|T^D_t\|^{1/t} = e^{-\text{Re} \lambda_1} < e^{-\mu} \quad (\lambda_0 < \mu < \text{Re} \lambda_1).
\]

Thus we have

\[
\|T^D_t\|^{1/t} < e^{-\mu} \quad (t \text{ large enough}).
\]

Hence

\[
\sup_{x \in D} \left| \frac{e^{\lambda_0 t} T^D_t f}{\phi_0(x)} - C_f \right| = \sup_{x \in D} \left| \frac{e^{\lambda_0 t} T^D_t g}{\phi_0(x)} \right| < \text{const} e^{\lambda_0 t} = \text{const} e^{\lambda_0 t} \|T^D_t g\| \quad (t \to \infty).
\]

Let us specify \( c_f \):

\[
0 = \lim_{t \to \infty} \int_D \frac{e^{\lambda_0 t} T^D_t f}{\phi_0(x)} \psi_0(x) \phi_0(x) dx = \lim_{t \to \infty} \int_D \psi_0(y) f(y) dy - c_f
\]

Secondly, for \( f \in M_0(D) \), we have \( T^D_t f \in B_0(D) \) from Lemma 3. Applying the formula just got, we obtain

\[
\sup_{x \in D} \left| \frac{e^{\lambda_0 t} T^D_t f}{\phi_0(x)} \right| - \int_D \psi_0(y) T^D_t F^D_t f(y) dy + 0 = 0 \quad (t \to \infty)
\]

Then (18) follows.
4. CONDITIONAL KILLED DIFFUSIONS

Theorem 4

$T^\psi_t$ and $\hat{T}^\psi_t$ are strongly continuous semigroups on $C_u(D)$. If $f \in C^\infty_K(D)$ (infinitely differential functions with compact support), then

$$A^\phi f = \left( \frac{1}{Z} \nabla \psi + \nabla \psi + a(\nabla \log \phi_0) \nabla \right) f,$$

$$A^\psi f = \left( \frac{1}{Z} \nabla \psi - \nabla \psi + a(\nabla \log \psi_0) \nabla \right) f,$$

(19)

where $A^\phi$, $A^\psi$ are the generator of $T^\phi_t$ and $\hat{T}^\psi_t$ on $C_u(D)$ respectively.

Proof: The first assertion is just a restatement of Theorem 1. For the second one, we have $f \in D(A^\phi)$ iff $\phi f \in D(A)$ ($A$: generator of $T_t$) and under this condition $A^\phi f = A(\phi_0 f)/\phi_0$. Then (19) follows from the direct calculation.

As usual $\{p^\phi(t,x,y)\}$ and $\{p^\psi(t,x,y)\}$ generate families of Markov measures $\{p^\phi_x\}$ and $\{p^\psi_x\}$ with state space $D$ respectively.

Theorem 5

$X$ is a continuous homogeneous conservative Markov process with strong Feller (and $C_u(D)$ strongly continuous semigroup) under both $\{p^\phi_x\}$ and $\{p^\psi_x\}$ with invariant measure $\nu_0(\Gamma) = \int_{\Gamma} \phi_0(x) \psi_0(x) dx$ ($\Gamma \in B(D)$).

$\{p^\psi_x\}$ is the time reversal of $\{p^\phi_x\}$ with respect to the invariant measure $\nu_0$ which is mixing.

And we have for any initial measure $\mu(\Gamma)$ ($\Gamma \in B(D)$):

$$\int_D \int_D p^\phi(t,x,y) f(y) \mu(dx) dy + \int f(y) \mu_0(dy) \quad (f \in M_b(D))$$

(20)

$$\sup_{x \in D} \int_D \left| p^\psi(t,x,y) - \psi_0(y) \psi_0(y) \right| dy + 0 \quad (t + \infty)$$

(21)

(the same for $\hat{p}^\psi$!)
Proof:

First, we show the path continuity of $X$ under \( \{p_t^x\} \) (or \( \{p_t^x\} \)). For any $\xi \in \sigma(X_t: s < t)$, we have

$$E_{X_t}^\xi = \frac{e^{\lambda_0 t}}{\phi_0(x)} E_x(\xi \phi_0(X_t)).$$

It follows that

$$p_{X_t}^x(w: X_s(w) \text{ discontinuous on } [0,n]) = \frac{e^{\lambda_0 t}}{\phi_0(x)} E_x (I(X_s \text{ discontinuous on } [0,n]) \phi_0(X_n)) = 0$$

Next, we set

$$q_{t_0}(y) = \int_D \frac{p^D(t_0,x,y)}{\phi_0(x)} \mu(dx).$$

Then Theorem 2 implies

$$\int_D p^\phi(t,x,y)f(y)\mu(dx) = \int_{D \times D} \int_D p^D(t - t_0,x,y,z)\phi_0(z)f(z)q_{t_0}(y)dydz$$

$$= \int_D e^{\lambda_0(t - t_0)} \int_{t_0}^{t} (\phi_0(y))e^{-\lambda_0 t_0}q_{t_0}(y)dy$$

$$+ \int_{L^2(\Omega)} \phi_0(y)e^{\lambda_0 t_0}q_{t_0}(y)dy (t + \infty)$$

$$= \langle \phi_0^t, \psi_0^t \rangle_{L^2(\Omega)} = \int_D f(y)\mu_0(dy).$$

To see the mixing of $\mu_0$, we calculate, for instance, for bounded measurable functions $f$ and $g$

$$E_{\mu}^\phi[f(C_{t},X_{t+s_1},X_{t+s_2})g(X_{u_1},X_{u_2})]$$

$$= \int \phi_0(x)\psi_0(x)p^\phi(u_1,x,y)p^\phi(u_2-u_1,y,z)g(y,z)p^\phi(t - u_2,z,a)$$

$$p^\phi(s_1,a,b)p^\phi(s_2 - s_1,b,c)f(a,b,c)dxdydzdadbdc.$$
The right hand side goes to $E^\Phi f(X_0, X_{s_1}, X_{s_2})E^\Phi g(x_{U_1}, x_{U_2})$ as $t \to \infty$, because we have

$$
\int \phi_0(x)\psi_0(x)p^\phi(u_1, x, y)dx = \phi_0(y)\psi_0(y)
$$

$$
\int \phi_0(y)\psi_0(y)p^\phi(u_2 - u_1, y, z)g(y, z)dy =
\quad = e^{\lambda_0(u_2 - u_1)}\phi_0(z)\int p^D(u_2 - u_1, y, z)\psi_0(y)g(y, z)dz
$$

$$
\hat{\phi}_0(z)g(z)
$$

and

$$
\int \phi_0(z)g(z)p^\phi(t - u_2, z, a)dz = \phi_0(a)\int e^{\lambda_0(t - u_2)}p^D(t - u_2, a, z)g(z)dz
$$

$$
+ \phi_0(a)\psi_0(a)\int g(z)\phi_0(z)dz \quad (t \to \infty,\text{ uniformly})
$$

What we get above deduces that for any $\sigma(X_t, 0 < t < \infty)$ cylinder sets $A$ and $B$, it is valid that

$$
p^\phi_{\mu_0}(\theta_t A \cap B) + p^\phi_{\mu_0}(A)p^\phi_{\mu_0}(B) \quad (t \to \infty).
$$

Thus the standard way of approximating $\sigma(X_t, 0 < t < \infty)$ measurable sets $A, B$ by cylinder sets $A_n, B_n$ ensures that

$$
|p^\phi_{\mu_0}(\theta_t A \cap B) - p^\phi_{\mu_0}(\theta_t A \cap B_n)| < p^\phi_{\mu_0}(\theta_t A \Delta \theta_t A_n) + p^\phi_{\mu_0}(B \Delta B_n)
$$

$$
= p^\phi_{\mu_0}(A \Delta A_n) + p^\phi_{\mu_0}(B \Delta B_n) \to 0 \quad (n \to \infty,\text{ uniformly for } t)
$$

which implies

$$
p^\phi_{\mu_0}(\theta_t A \cap B) + p^\phi_{\mu_0}(A)p^\phi_{\mu_0}(B) \quad (t \to \infty)
$$
Finally, we estimate
\[
\sup_{x \in \mathbb{D}} \int_D |p^\phi(t, x, y) - \phi_0(y)\psi_0(y)| \, dy
\]
\[
= \sup_{x \in \mathbb{D}} \int_D (p^\phi T, x, y) - p^\phi(t, x', y)\phi_0(x')\psi_0(x') \, dx' \, dy
\]
\[
\leq \sup_{x \in \mathbb{D}} \int_D |p^\phi(t, x, y) - p^\phi(t, x', y)\phi_0(x')\psi_0(x')| \, dx'
\]
\[
= \sup_{x \in \mathbb{D}} \sup_{\|M\|_D < 1} |T_t f(x) - T_t f(x')| \phi_0(x')\psi_0(x') \, dx'
\]
\[
+ 0 \quad (t + \infty)
\]
since (18) holds uniformly for \( f : \mathbb{N} \leq 1 \).

The other conclusions of the Theorem are easy to obtain.

**Corollary** If \( L \) is symmetric, then \( \psi_0(x) = \text{const.} \phi_0(x)e^{2/\lambda}(a^{-1}b) \).

The next theorem says that \( X \) under \( \{P'_x\} \) (we denote it as \( X^\phi \)) can be regarded as original process \( X \) (under \( \{P_x\} \)) conditioned on \( D \) in the following sense: \( \{P'_x\} \) is the weak limit (when \( T + \infty \)) of \( X^T \) the killed diffusion conditioned up to time \( T \). Actually \( X^T \) is determined by the nonhomogeneous Markov measures \( \{P'_{s,x}\} [12] \) with the density:

\[
p^T(s, t, x, y) = p^D(t - s, x, y) \frac{p_y(y_D > T - t)}{p_x(y_D > T - s)}.
\]

(22)

Hence we call \( \{P'_x\} \) conditional killed diffusion on \( D \).

**Theorem 6** For any \( T_0 > 0 \), \( X^T \) converges weakly to \( X^\phi \) in \( D[0, T_0] \) as \( T + \infty \).
Proof.

Two things have to be done for the proof. First, the convergence of the finite distributions of $p^T_{s,x}$ on $[0,T_0]$. Second, the uniform tightness of the family $p^T_{s,x}$ with $T_0 < T$. In the light of Theorem 1 and 2, we have

$$p^T(s,t,x,y) < \text{Const.} \ p^\phi(t - s,x,y) \quad (23)$$

and

$$p^T(s,t,x,y) + p^\phi(t - s,x,y) \quad (T \to \infty).$$

It implies that the finite distributions of $p^T_{s,x}$ converge to those of $p^\phi_x$. On the other hand, since Theorem 3 $T^\phi_t$ is strongly continuous on $C_0(D)$, then $p^\phi(t,x,y)$ satisfies that

$$\sup_{x \in D} \int_{|x-y| > \delta} p^\phi(t,x,y)dy \to 0 \quad (t \to 0)$$

by an argument similar to that in [4].

It follows from (23) that

$$\sup_{x \in D} \int_{|x-y| > \delta} p^T(s,t,x,y)dy \to 0 \quad (t \to 0)$$

uniformly with respect to $s$ and $T$. Now a theorem [16] about the tightness of the Markov families of $D[0,T_0]$ provides us that $p^T_{s,x}$ is compact in $D[0,T_0]$. Thus it implies that $p^T_{s,x} \Rightarrow p^\phi_x$ in $D[0,T_0]$. \qed

We see that this Theorem gives us a comprehensive understanding of the Markov process $X^\phi$.

All these Propositions and Theorems above remain true with a usual modifi-
cation in the case of $n = 2$ or $n = 1$. 
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