# ON THE LARGE DEVIATION FUNCTIONS OF MARKOV CHAINS

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# ON THE LARGE DEVIATION FUNCTIONS OF MARKOV CHAINS

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## §1. Introduction

In this paper, we consider that how much information the large deviation rate functions contain. Namely, let us observe two processes $X^{(1)}$ and $X^{(2)}$ with the same large deviation function $I(\mu)$, for all $\mu$ with compact support, and ask what the relation should have between $X^{(1)}$ and $X^{(2)}$. In [1], Donsker and Varadhan gave the answer for the case of diffusion as the following Theorem.

<u>Theorem 0</u>. If $X^{(1)}$ and $X^{(2)}$ are two diffusion processes on $R^d$ with $C^\infty$ diffusion coefficients and drifts. Their diffusion matrices are positive definite for each $x \in R^d$. They have the same large deviation functions for all $\mu$ with compact supports and densities in $C^\infty$, iff either there exists a positive harmonic fucntion $u$ for $L^{(1)}$ such that $L^{(2)}f = \frac{L^{(1)}(uf)}{u}$ or there exists a positive invariant density $\rho$ for $L^{(1)}$ such that $L^{(2)}f = \frac{L^{(1)^*}(\rho f)}{\rho}$, where $L^{(i)}$ is the generator of $X^{(i)}$ $(i = 1,2)$, and $L^{(1)^*}$ is the formal adjoint of $L^{(1)}$.

Actually, in many cases this means that $X^{(2)}$ could only be $X^{(1)}$ or its time reversal with respect to $\rho$ in the sense of the law.

But this is not true for Markov processes other than diffusions, even for Markov chains with finite states. Let us look at an example as follows.

<u>Example</u> Let

$$Q^{(1)} = \begin{pmatrix} -1 & 1 & 0 & 0 & \\ 0 & -1 & 1 & & \\ 1 & 0 & -2 & 1 & 0 \\ & & 0 & -1 & 1 \\ & & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Q^{(2)} = \begin{pmatrix} -1 & 1 & 0 & & \\ 0 & -1 & 1 & & \\ 1 & 0 & -2 & 0 & 1 \\ & & 1 & -1 & 0 \\ & & 0 & 1 & -1 \end{pmatrix}$$

be the transition density matrices of the Markov chains $X^{(1)}$ and $X^{(2)}$ respectively. Then their invariant measures are both $\rho = (\rho_i)$, where $\rho_i = \frac{1}{5}$ $(i = 1,2,\ldots,5)$. The time reversal of $X^{(1)}$ with respect to $\rho$ just has the transition density matrix $(Q^{(1)})^T$, which is not equal to $Q^{(2)}$. Now look at the large deviation functions

$$I^{(\ell)}(\mu) = -\inf_{u_i>0} \sum_{i,j=1}^{5} \frac{\mu_i q_{ij}^{(\ell)} u_j}{u_i} \qquad (\ell = 1,2)$$

and set

$$v_1 = u_1,\ v_2 = u_2,\ v_3 = u_3,\ v_4 = \frac{u_4}{u_5} u_3,\ v_5 = u_4.$$

Then we have

$$I^{(1)}(\mu) = -\inf_{u_i>0} \sum_{i,j=1}^{5} \frac{\mu_i q_{ij}^{(1)} u_j}{u_i} = -\inf_{u_i>0} \sum_{i,j=1} \frac{\mu_i q_{ij}^{(2)} v_j}{v_i} = I^{(2)}(\mu)$$

Noticing that $Q^{(1)}$, $Q^{(2)}$ are the same if we restrict them in $\{1,2,3\}$ and $Q^{(2)}$, $(Q^{(1)})^T$ are the same if we restrict them in $\{3,4,5\}$, we can think about $Q^{(2)}$ is a kind of "combination" of $Q^{(1)}$ and $(Q^{(1)})^T$. Therefore, we can imagine that, generally more complicate things can happen for Markov processes. Even though Theorem 1 does not work in general, we can have some revision for Markov chains. In the case of Markov chains, it is substantial to study those with finite states.

Assume

$$Q^{(\ell)} = (q_{ij}^{(\ell)}) \qquad (\ell = 1,2;\ 1 \leq i,j \leq n)$$

are two Q-matrices, i.e.

$$q_{ii}^{(\ell)} \leq 0,\quad q_{ij}^{(\ell)} \geq 0\ (i \neq j),\quad \sum_{j=1}^{n} q_{ij}^{(\ell)} = 0.$$

The large deviation fucntions of $Q^{(\ell)}$ is

$$I^{(\ell)}(\mu) = -\inf_{u_i>0} \sum_{i,j=1}^{n} \frac{\mu_i q_{ij}^{(\ell)} u_j}{u_i} \qquad (\mu \in M)$$

where $M$ is the space of all probability measures on $\{1,2,\ldots,n\}$.

We identify cycle $R$ of k-order with an ordered set $i_1 \to i_2 \to \cdots \to i_k$, and agree that two such sets correspond to the same cycle, if they differ only by a cyclic permutation. Let us denote it as $R = \langle i_1, i_2, \ldots, i_k \rangle$

Definition 1. For a cycle $R$ of k-order

$$R = \langle i_1, i_2, \ldots, i_k \rangle,$$

the flow along $R$ for a matrix $Q = (q_{ij})$ is defined as

$$Q_R \triangleq \prod_{r=1}^{k} q_{i_r i_{r+1}} \qquad (\text{where } k+1 \triangleq 1).$$

The reversed cycle of $R$ is

$$R^- \triangleq \langle i_k, i_{k-1}, \ldots, i_1 \rangle.$$

Notations. If $Q$ is a matrix, then the matrix from $Q$ by setting the diagonal zero is denoted by $\tilde{Q}$, and the submatrix on $i_1, i_2, \ldots, i_s$ columns and rows from $Q$ is denoted by $Q(i_1, i_2, \ldots, i_s)$.

Definition 2. A $k \times k$ matrix $D$ is said to be single for cycles, if there exists at most one pair of k-order cycle $R$ and $R^-$ such that $D_R + D_{R^-} > 0$, where $D_R$ is the flow for the matrix $D$ along $R$.

Definition 3. A pair of transition density matrices $Q^{(i)}$ $(i = 1,2)$ is called regular, if for $\forall k \geqslant 4$, there is no $R = \langle i_1, \ldots, i_k \rangle$ such that $Q^{(i)}(i_1, \ldots, i_k)$ are not single and

$$Q_R^{(1)} = Q_{R^-}^{(1)} = Q_R^{(2)} = Q_{R^-}^{(2)}$$

(they could all vanish).

<u>From now on we assume that $Q^{(i)}$ (i = 1,2) is a regular pair, unless specified.</u>

<u>Definition 4</u>. Cycle R is called of type I (of type II), if

$$Q_R^{(1)} = Q_R^{(2)} > 0 \quad (Q_R^{(1)} = Q_{R^-}^{(2)} > 0).$$

R is called typed, if R is either of type I or of type II.

<u>Remark 1</u>. If R is of type I (or type II), then

$$Q_{R^-}^{(1)} = Q_{R^-}^{(2)} \quad (Q_{R^-}^{(1)} = Q_R^{(2)})$$

and if $Q_{R^-}^{(1)} > 0$, then $R^-$ has the same type with R.

<u>Theorem 1</u>. For Markov chains with finite states, the following conditions are equivalent:

(i) $\quad I^{(1)}(\mu) = I^{(2)}(\mu) \qquad$ (for $\forall\mu \in M$);

(ii) For any cycle R, we have

$$Q_R^{(1)} + Q_{R^-}^{(1)} = Q_R^{(2)} + Q_{R^-}^{(2)};$$

(iii) For any cycle R, we have

$$\text{either} \begin{cases} Q_R^{(1)} = Q_R^{(2)} \\ Q_{R^-}^{(1)} = Q_{R^-}^{(2)} \end{cases} \text{or} \begin{cases} Q_R^{(1)} = Q_{R^-}^{(2)} \\ Q_{R^-}^{(1)} = Q_R^{(2)} \end{cases}.$$

When $X^{(\ell)}$ ($\ell$ = 1,2) is ergodic, Theorem 2 gives a probabilistic explanation for Theorem 1.

Let $J_R^{(\ell)}$ be the constant such that

$$J_R^{(\ell)} = \lim_{T\to\infty} \frac{1}{T}\, {}^{(\ell)}W_T^R(\omega) \qquad (\text{a.e.}, \ \ell = 1,2),$$

where

$${}^{(\ell)}W_T^R(\omega) = \text{the times of the cycle } R \text{ formed by } X^{(\ell)} \text{ up to time } T$$

(see [2])

Now Theorem 1 can be restated as

<u>Theorem 2</u> $I^{(1)}(\mu) = I^{(2)}(\mu)$ (for $\forall\mu \in M$) iff for any cycle $R$ either

$$\begin{cases} J_R^{(1)} = J_R^{(2)} \\ \\ J_{R^-}^{(1)} = J_{R^-}^{(2)} \end{cases} \quad \text{or} \quad \begin{cases} J_R^{(1)} = J_{R^-}^{(2)} \\ \\ J_{R^-}^{(1)} = J_R^{(2)} \end{cases}$$

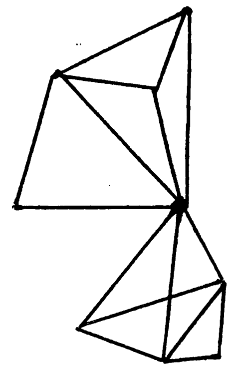
holds.

Theorem 2 tells that along any cycle the process $X^{(2)}$ could only behave like $X^{(1)}$ or its time reversal.

<u>Definition 5</u>. For a Q-matrix $Q = (q_{ij})$, we say $\overline{ij}$ is a chord, if $q_{ij} + q_{ji} > 0$.

Subsets $J_1$ and $J_2$ of the state space are called being chord joint, if there exists a chord $\overline{ij}$ such that $i,j \in J_1 \cap J_2$, otherwise - chord disjoint. $Q$ is said to be quasi-reducible, if the state space can be the union of its two chord disjoint subsets.

Obviously, quasi-reducible $Q$ only can happen when $Q$ is reducible or the graph of $Q$ is the union of two part connected with only one state as the graph at right side shows.




Theorem 3. For a quasi-irreducible $Q^{(1)}$.

$$I^{(1)}(\mu) = I^{(2)}(\mu) \quad (\forall\mu \in M)$$

iff

$$Q^{(1)} = Q^{(2)}$$

or

$$Q^{(2)} = \left( q_{ij}^{(1)} \frac{\mu_j}{\mu_i} \right)$$

where $(\mu_i)$ is the invariant measure on $Q^{(1)}$.

i.e. the Markov chain $X^{(i)}$ generated by $Q^{(i)}$ $(i = 1,2)$ are either the same or one is the time reversal of the other one with respect to the invariant measure.

For general (i.e. irregular case), Theorems 1-3 are not true, the things are more complicated.

Example. (Varadhan) Set $a > b > 0$, and

$$Q_1 = \begin{pmatrix} -1 & \frac{a}{3+a} & \frac{1}{3+a} & \frac{2}{3+a} \\ \frac{b}{3+b} & -1 & \frac{1}{3+b} & \frac{2}{3+b} \\ 3+a & 3+b & -(6+a+b+c) & c \\ \frac{3+a}{2} & \frac{3+b}{2} & d & -(3+\frac{a+b}{2}+d) \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} -1 & \frac{b}{3+b} & \frac{1}{3+b} & \frac{2}{3+b} \\ \frac{a}{3+a} & -1 & \frac{1}{3+a} & \frac{2}{3+a} \\ 3+b & 3+a & -(6+a+b+c) & c \\ \frac{3+b}{2} & \frac{3+a}{2} & d & -(3+\frac{a+b}{2}+d) \end{pmatrix}$$

Since $\tilde{Q}_1$, $\tilde{Q}_2$ have same minors and diagonal elements, by Proposition 1 (in §2) they have the same large deviation function. But we have

$$Q^{(1)}_{\langle 1,4,2,3\rangle} = Q^{(2)}_{\langle 1,4,2,3\rangle} = Q^{(1)}_{\langle 3,2,4,1\rangle} = Q^{(2)}_{\langle 3,2,4,1\rangle} = 1$$

and

$$Q^{(1)}_{\langle 1,2,3,4\rangle} = \frac{ac}{2(3+b)} = Q^{(2)}_{\langle 3,4,2,1\rangle}$$

$$Q^{(1)}_{\langle 3,4,2,1\rangle} = \frac{bc}{2(3+a)} = Q^{(2)}_{\langle 1,2,3,4\rangle}$$

$$Q^{(1)}_{\langle 4,3,2,1\rangle} = \frac{2bd}{3+a} = Q^{(2)}_{\langle 1,2,4,3\rangle}$$

$$Q^{(1)}_{\langle 1,2,4,3\rangle} = \frac{2ad}{3+b} = Q^{(2)}_{\langle 4,3,2,1\rangle}$$

However the sufficiency of Theorem 2 is always valid.

We claim that, in general, $I^{(1)}(\mu) = I^{(2)}(\mu)$ $(\forall\mu \in M)$ iff there exists a 1-1 mapping $f:R \to f(R)$ such that $f(R)$ is a cycle with the same states as $R$, and

$$\begin{cases} Q^{(1)}_R = Q^{(2)}_{f(R)} \\ Q^{(1)}_{R^-} = Q^{(2)}_{f(R)^-} . \end{cases}$$

The proof of the neccesity is very cumbersome, we are not go through that here.

## §2. Lemmas and the proof of Theorems

If $\mu$ only charges on subset $\{i_1,\dots,i_k\}$, obviously we have

$$I^{(\ell)}(\mu) = - \inf_{\substack{u_{j_r}>0 \\ 1\leq r\leq k}} \sum_{r,m=1}^{k} \frac{u_{i_r} q^{(\ell)}_{i_r i_m} u_{i_m}}{u_{i_r}} \qquad (\ell = 1,2)$$

Now, assume we have

$$I^{(1)}(\mu) = I^{(2)}(\mu) \qquad (\text{for } \forall\mu \in M).$$

Setting $\mu = (\delta_i) = (0,\dots,0,1,0,\dots,0)$, it follows that

$$q^{(1)}_{ii} = q^{(2)}_{ii} \qquad (i = 1,2,\dots,n).$$

Taking $\mu = (\lambda,1-\lambda,0,\dots,0)$, we get

$$I^{(\ell)}(\mu) = -\lambda q^{(\ell)}_{11} - (1-\lambda)q^{(\ell)}_{22} - 2\sqrt{\lambda q^{(\ell)}_{12}(1-\lambda)q^{(\ell)}_{21}}\ ,$$

since Markov chains with two states are symmetric.

Hence

$$q^{(1)}_{12}q^{(1)}_{21} = q^{(2)}_{12}q^{(2)}_{21}$$

and in general we have

$$q^{(1)}_{ij}q^{(1)}_{ji} = q^{(2)}_{ij}q^{(2)}_{ji} \tag{1}$$

i.e.

$$Q^{(1)}_{\langle i,j\rangle} = Q^{(2)}_{\langle i,j\rangle} \tag{2}$$

The following well known lemma about the convex duality we need later.

Lemma 1. If $a = (a_{ij})_{k\times k}$ satisfies

$$a_{ij} \geqslant 0 \qquad (i \neq j)$$
$$a_{ii} = 0$$

then

$$-\inf_{\substack{u_i>0\\ 1\leqslant i\leqslant K}} \sum_{i,j=1}^{k} \frac{\mu_i a_{ij} u_j}{u_i} = \sup_{c\varepsilon R^k} \left(\sum_{i=1}^{K} c_k \mu_k - \lambda_0(a,c)\right)$$

where $c = (c_1,\dots,c_k)$ and

$$\lambda_0(a,c) = \lim_{t\to\infty} \frac{1}{t} \log \| e^{Dt} \|_{op}$$

($\|\cdot\|_{op}$ is the operator norm of $e^{Dt}$ on $R^k$, see [3]), which is also equal to the eigenvalue of $D = (A + \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_k \end{pmatrix})$ with the largest real part.

<u>Lemma 2</u> For $1 \leqslant i,j \leqslant k$ and fixed $r(=1,2)$ set

$$a_{ij} = \begin{cases} q^{(r)}_{\ell_i \ell_j} & \text{when} \quad i \neq j \\ \beta_i & \text{when} \quad i = j \quad (i,j = 1,2,\dots,k) \end{cases}$$

and

$$D^{(r)}(\ell_1,\dots,\ell_k;\beta_1,\dots,\beta_k) = \det(a_{ij})_{k\times k}.$$

Then

$$D^{(r)}(\ell_1,\dots,\ell_k;\beta_1,\dots,\beta_k) = \sum_{\{i_1,\dots,i_k\}\subset\{1,2,\dots,k\}} \alpha(i_1,\dots,i_k)\beta_{i_1}\cdots\beta_{i_k}$$

where $\alpha(i_1,\dots,i_k)$ ($i_r \neq i_s$ when $r \neq s$) is a linear combination of the minors $\{D^{(r)}(\ell_{i_1},\dots,\ell_{i_k};0,\dots,0)\}$ with $\pm 1$ as their coefficients only depending on $(i_1,i_2,\dots,i_k)$.

Proof. Noticing

$$D(\ell_1,\dots,\ell_k;\beta_1,\dots,\beta_k) =$$

$$\beta_k D(\ell_1,\dots,\ell_{k-1};\beta_1,\dots,\beta_{k-1}) + D(\ell_1,\dots,\ell_k;\beta_1,\dots,\beta_{k-1},0)$$

the lemma is easy to prove by induction.

Proposition 1. The following statements are equivalent:

(i) $I^{(1)}(\mu) = I^{(2)}(\mu)$ for $(\forall\mu \in M)$;

(ii) $\lambda_0(Q^{(1)},c) = \lambda_0(Q^{(2)},c)$ (for $\forall c \in R^n$);

(iii) For any $\{\ell_1,\dots,\ell_k\} \subset \{1,2,\dots,n\}$, $k \leq n$

$$\lambda_0 \triangleq \lambda_0\left(\begin{pmatrix} 0 & q_{\ell_1\ell_2} & \cdots & q_{\ell_1\ell_k} \\ q_{\ell_2\ell_1} & 0 & \cdots & q_{\ell_2\ell_k} \\ \cdots & \cdots & \cdots & \cdots \\ q_{\ell_k\ell_1} & q_{\ell_k\ell_2} & \cdots & 0 \end{pmatrix}, c\right) = \lambda_0\left(\begin{pmatrix} 0 & q_{\ell_1\ell_2} & \cdots & q_{\ell_1\ell_k} \\ q_{\ell_2\ell_1} & 0 & \cdots & q_{\ell_2\ell_k} \\ \cdots & \cdots & \cdots & \cdots \\ q_{\ell_k\ell_1} & q_{\ell_k\ell_2} & \cdots & 0 \end{pmatrix}, c\right)$$

(for $\forall c \in R^k$);

(iv) $q_{ii}^{(1)} = q_{ii}^{(2)}$ $(1 \leq i \leq n)$ and all the corresponding minors of $\tilde{Q}^{(1)}$ and $\tilde{Q}^{(2)}$ are equal respectively.

(v) All the corresponding submatrices on the diagonal have the same characteristic polynomials.

Proof. From the convex dual theorem it follows that (1), (2), (3) are equivalent.

Taking $\beta_1 = \dots = \beta_k = -\lambda_0$ (as in (iii)) in Lemma 2, from

$$D^{(1)}(\ell_1,\ldots,\ell_k;-\lambda_0,\ldots,-\lambda_0) = 0 = D^{(2)}(\ell_1,\ldots,\ell_k;-\lambda_0,\ldots,-\lambda_0),$$

we have that (iii) implies (iv) by induction.

On the other hand, let

$$\beta_i = c_i \qquad (i = 1,2,\ldots,k),$$

and (iv) is immediate from Lemma 2.

It is obvious that (iv) and (v) are equivalent.

Lemma 3. Assume $I^{(1)}(\mu) = I^{(2)}(\mu)$ for $\forall\ \mu \in \mathbf{M}$, and $Q^{(1)}(i_1,\ldots,i_k)$ is single of cycles. Then for an arbitrary k-order cycle $R = \langle i_1,\ldots,i_k\rangle$, either

$$\begin{cases} Q_R^{(1)} = Q_R^{(2)} \\ Q_{R^-}^{(1)} = Q_{R^-}^{(2)} \end{cases}$$

or

$$\begin{cases} Q_R^{(1)} = Q_{R^-}^{(2)} \\ Q_{R^-}^{(1)} = Q_R^{(2)} \end{cases}$$

holds.

Proof. From Proposition 1, it is easy to show the following equalities by the definition of determinants and $Q^{(1)}$ being k-single:

$$Q_R^{(1)} - Q_{R^-}^{(1)} = \det\begin{pmatrix} 0 & q_{\ell_1\ell_2}^{(1)} & \cdots & q_{\ell_1\ell_k}^{(1)} \\ q_{\ell_2\ell_1}^{(1)} & 0 & \cdots & q_{\ell_2\ell_k}^{(1)} \\ q_{\ell_k\ell_1}^{(1)} & q_{\ell_k\ell_2}^{(1)} & \cdots & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & q_{\ell_1\ell_2}^{(2)} & \cdots & q_{\ell_1\ell_k}^{(2)} \\ q_{\ell_2\ell_1}^{(2)} & 0 & \cdots & q_{\ell_2\ell_k}^{(2)} \\ q_{\ell_k\ell_1}^{(2)} & q_{\ell_k\ell_2}^{(2)} & \cdots & 0 \end{pmatrix}$$

$$= Q_R^{(2)} - Q_{R^-}^{(2)}.$$

On the other hand, from (1) we have

$$Q_R^{(1)} Q_{R^-}^{(1)} = Q_R^{(2)} Q_{R^-}^{(2)}. \tag{3}$$

Therefore the Lemma is shown by the comparison of (2) and (3).

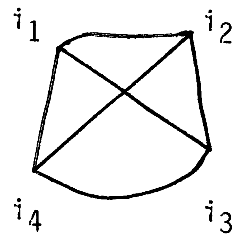
Notation. Let $R_1 \circ R_2$ be the product of orineted cycles of $R_1$ and $R_2$ as usual. Thus when $Q_{R_i} > 0$, $Q_{R_i^-} > 0$ $(i=1,2)$, we have

$$\frac{Q_{R_1 \circ R_2}^{(1)}}{Q_{R_1}^{(1)}\ Q_{R_2}^{(1)}} = \frac{Q_{R_1 \circ R_2}^{(1)}}{Q_{R_1^-}^{(1)}\ Q_{R_2^-}^{(1)}} = \frac{Q_{R_1 \circ R_2}^{(2)}}{Q_{R_1}^{(2)}\ Q_{R_2}^{(2)}} = \frac{Q_{R_1 \circ R_2}^{(2)}}{Q_{R_1^-}^{(2)}\ Q_{R_2^-}^{(2)}}$$

We see that if $R_1$, $R_2$ (or $R_1 \circ R_2$) are of the same type, then so is $R_1 \circ R_2$ (or $R_2$).

Lemma 4. Set

$$R_1 = \langle i_1,i_2,i_3\rangle, \quad R_2 = \langle i_2,i_3,i_4\rangle$$
$$R_3 = \langle i_1,i_3,i_4\rangle, \quad R_4 = \langle i_1,i_2,i_4\rangle$$
$$S_1 = \langle i_1,i_2,i_3,i_4\rangle, \quad S_2 = \langle i_1,i_2,i_4,i_3\rangle$$
$$S_3 = \langle i_1,i_3,i_2,i_4\rangle$$




If $I^{(1)}(\mu) = I^{(2)}(\mu)$ $(\forall\mu \in \mathbf{M})$

and $Q^{(1)}(i_1,i_2,i_3,i_4)$ is not single.

Then

$$(Q^{(1)}_{R_k} + Q^{(2)}_{R_k^-})(Q^{(2)}_{R_k} + Q^{(2)}_{R_k^-}) > 0 \quad (k = 1,2,3,4)$$

Proof. Without loss of generality, we assume

$$Q^{(1)}_{S_1} \cdot Q^{(2)}_{S_2} > 0.$$

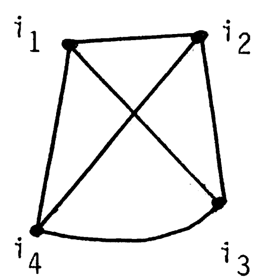
Since $Q^{(1)}(i_1,i_2,i_3,i_4)$ is nonsingle. We prove the lemma in two cases.

Case 1°. $Q^{(1)}_{s_3} + Q^{(1)}_{s_3^-} > 0$. Obviously the conclusion holds.

Case 2°. $Q^{(1)}_{s_3} = Q^{(1)}_{s_3^-} = 0$. From the assumption of this lemma we have

$$q_{i_1i_2}q_{i_2i_3}q_{i_3i_4}q_{i_4i_1}q_{i_2i_4}q_{i_4i_3}q_{i_3i_1} > 0$$

as in the graph on the right. Thus




$R_1$ and $R_4$ are typed and

$$q^{(2)}_{i_3i_4}q^{(2)}_{i_4i_3} > 0$$

If $R_1$, $R_4$ are of the same type (say type I), then

$$q^{(2)}_{i_1i_4} = q^{(2)}_{i_3i_2} = 0$$

implies $Q^{(2)}_{s_3} = Q^{(2)}_{s_{\bar{3}}} = 0$. This contradicts to the regularity.

If $R_1$, $R_4$ are of different type, then $q^{(2)}_{i_1 i_2} > 0$ and $q^{(2)}_{i_2 i_1} > 0$ implies $q^{(1)}_{i_2 i_1} > 0$. Thus $R_2$ and $R_3$ are typed for $Q^{(2)}$, and this leads for the same conclusion for $Q^{(1)}$. $\square$

<u>Lemma 5</u>. Let us have the same notations as in Lemma 4. Set

$$C \triangleq \{s_1, s_2, s_3, s_{\bar{1}}, s_{\bar{2}}, s_{\bar{3}}\}.$$

If $I^{(1)}(\mu) = I^{(2)}(\mu)$ (for$_\rho$ $\forall \mu \in M$), then for $\forall R \in C$, when $Q_R > 0$, $R$ is typed.

Proof. If $Q^{(1)}(i_1, i_2, i_3, i_4)$ is single for cycles, then the conclusion of Lemma 5 holds by Lemma 3. Otherwise we have two in $C$ (without loss of generality say $S_1$ and $S_2$) such that

$$Q_{s_1} \cdot Q_{s_2} > 0.$$

From Lemma 4, we have

$$Q^{(i)}_{R_k} + Q^{(i)}_{R_{\bar{k}}} > 0 \qquad (\forall k = 1,2,3,4;\ i = 1,2).$$

Now by symmetry, there are only three cases need to be considered as follows.

For simplicity, we say $R_k$, $R_\ell$ are of the same type, if two cycles in one of pairs $\{(R_k, R_\ell), (R_{\bar{k}}, R_\ell), (R_k, R_{\bar{\ell}}), (R_{\bar{k}}, R_{\bar{\ell}})\}$ are both of type I (or of type II).

Case 1°. $R_1$, $R_2$, $R_3$ and $R_4$ are all of the same type, say type II. Thus we have

$$Q^{(1)} = \frac{Q^{(1)}_{R_3} Q^{(1)}_{R_{\bar{2}}}}{Q^{(1)}_{\langle i_3, i_4 \rangle}} = \frac{Q^{(2)}_{R_{\bar{3}}} Q^{(2)}_{R_2}}{Q^{(2)}_{\langle i_3, i_4 \rangle}} = Q^{(2)}_{s_{\bar{3}}} \tag{4}$$

(and, the same,

$$Q^{(1)}_{s_3^-} = Q^{(2)}_{s_3} ).$$

Since $Q^{(1)}_{s_3} + Q^{(2)}_{s_3^-} > 0$ (regularity), we have

$$q^{(1)}_{i_1 i_3} q^{(1)}_{i_3 i_1} > 0 \quad \text{or} \quad q^{(1)}_{i_1 i_4} q^{(1)}_{i_4 i_1} > 0.$$

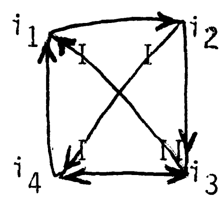
Therefore

$$Q^{(1)}_{s_i} = Q^{(2)}_{s_i^-} , \quad Q^{(1)}_{s_i^-} = Q^{(2)}_{s_i} \quad (i = 1,2)$$

holds as (4).

Case 2°. Three of $R_1, R_2, R_3$ and $R_4$ are of the same type (say $R_1, R_3, R_4$ are of type I), and the other one is of different type (say $R_2$ is of type II). Therefore

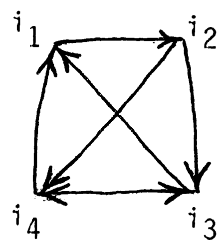
$$Q^{(2)}_{\langle i_2, i_4, i_1\rangle} = Q^{(1)}_{\langle i_2, i_4, i_1\rangle} > 0 \tag{5}$$




Since $R_2$ or $R_2^-$ is typed, we have

$$q^{(1)}_{i_4 i_2} + q^{(1)}_{i_3 i_2} > 0$$

In virtue of Lemma 3 we have $Q^{(1)}_{R_2} = Q^{(2)}_{R_2^-}$ , $Q^{(1)}_{R_2^-} = Q^{(2)}_{R_2}$, it follows




$$q^{(i)}_{i_4 i_2} \; q^{(i)}_{i_3 i_2} > \; (i = 1,2).$$

Then

$$Q^{(i)}_{\langle i_2, i_4\rangle} = Q^{(i)}_{\langle i_2, i_3\rangle} > 0 \quad (i = 1,2) \tag{6}$$

When $Q^{(1)}_{i_1, i_3} > 0$, it implies

$$Q^{(1)}_{s_1} = \frac{Q^{(1)}_{R_1} \; Q^{(1)}_{R_3}}{Q^{(1)}_{\langle i_1, i_3\rangle}} = \frac{Q^{(2)}_{R_1} \; Q^{(2)}_{R_3}}{Q^{(2)}_{\langle i_1, i_3\rangle}} = Q^{(2)}_{s_1}$$

Similarly

$$Q^{(1)}_{\bar{s}_1} = Q^{(2)}_{\bar{s}_1}.$$

Hence, it follows that $R_2$ must be of type I by Remark 2. Thus it is reduced to case 1°.

When $q^{(1)}_{i_1 i_3} = 0$, we get $q^{(1)}_{i_1 i_4} > 0$ and $s_2$ is of type I. On the other hand,

$$Q^{(1)}_{s_1} Q^{(1)}_{s_2} = Q^{(1)}_{R_1} Q^{(1)}_{R_4} Q^{(1)}_{\langle i_3, i_4 \rangle} =$$
$$= Q^{(2)}_{R_1} Q^{(2)}_{R_4} Q^{(2)}_{\langle i_3, i_4 \rangle} = Q^{(2)}_{s_1} Q^{(2)}_{s_2},$$

Then $s_1$ is of type I, and $R_2$ is of type I as well. And so is $s_3$.

Case 3°. Two of $R_1$, $R_2$, $R_3$ and $R_4$ are of type I (e.g. $R_1$ and $R_4$), the other two are of type II (e.g. $R_2$ and $R_3$). As well as (6), we have

$$Q^{(1)}_{\langle i_k, i_\ell \rangle} = Q^{(2)}_{\langle i_k, i_\ell \rangle} > 0$$

(for $\forall$ $\langle i_k, i_\ell \rangle \neq \langle i_3, i_4 \rangle$ or $\langle i_1, i_2 \rangle$)

When

$$Q^{(1)}_{\langle i_3, i_4 \rangle} + Q^{(1)}_{\langle i_1, i_2 \rangle} > 0,$$

e.g.

$$Q^{(1)}_{\langle i_3, i_4 \rangle} > 0,$$

we have

$$Q^{(1)}_{s_{\bar{3}}} Q^{(1)}_{s_1} = Q^{(1)}_{R_1} Q^{(1)}_{R_2} Q^{(1)}_{\langle i_1, i_4\rangle}$$

$$= Q^{(2)}_{R_1} Q^{(2)}_{R_{\bar{2}}} Q^{(2)}_{\langle i_1, i_4\rangle}$$

$$= Q^{(2)}_{R_4} Q^{(2)}_{R_{\bar{3}}} Q^{(2)}_{\langle i_2, i_3\rangle}$$

$$= Q^{(1)}_{R_4} Q^{(1)}_{R_3} Q^{(1)}_{\langle i_2, i_3\rangle}$$

$$= Q^{(1)}_{s_1} Q^{(1)}_{s_3}$$

and

$$Q^{(1)}_{s_3} = \frac{Q^{(1)}_{R_2} Q^{(1)}_{R_3}}{Q^{(1)}_{\langle i_3, i_4\rangle}} = \frac{Q^{(2)}_{R_{\bar{2}}} Q^{(2)}_{R_{\bar{3}}}}{Q^{(2)}_{\langle i_3, i_4\rangle}} = Q^{(2)}_{s_{\bar{3}}}$$

(and the same,

$$Q^{(1)}_{s_{\bar{3}}} = Q^{(2)}_{s_3}).$$

Thus we have

$$Q^{(1)}_{s_3} = Q^{(1)}_{s_{\bar{3}}} = Q^{(2)}_{s_3} = Q^{(2)}_{s_{\bar{3}}}$$

which contradicts to the regularity of $Q^{(i)}$ $(i = 1,2)$.

When $Q^{(1)}_{\langle i_3, i_4\rangle} + Q^{(1)}_{\langle i_1, i_2\rangle} = 0$, e.g.

$$Q^{(1)}_{i_2 i_1} = q^{(1)}_{i_3 i_4} = 0,$$

we have

$$Q^{(1)}_{s_2} Q^{(1)}_{s_3} = Q^{(2)}_{s_1} Q^{(2)}_{\langle i_4, i_2\rangle} Q^{(2)}_{\langle i_3, i_1\rangle} = Q^{(1)}_{s_2} Q^{(1)}_{s_{\bar{3}}}.$$

It follows from $Q_{s_2} > 0$ that

$$Q^{(1)}_{s_3} = Q^{(1)}_{s_3^-}.$$

Similar argument leads to

$$Q^{(2)}_{s_3} = Q^{(2)}_{s_3^-}.$$

Now, by $Q^{(1)}_{s_3} Q^{(1)}_{s_3^-} = Q^{(2)}_{s_3} Q^{(2)}_{s_3^-}$, we obtain

$$Q^{(1)}_{s_3} = Q^{(2)}_{s_3} = Q^{(1)}_{s_3^-} = Q^{(2)}_{s_3^-}$$

which cannot appear in the regular case. Q.E.D.

Remark. When there are states between $i_1$, $i_2$, $i_3$ and $i_4$, assuming that $Q^{(1)}(i_1,\ldots,i_2,\ldots,i_3,\ldots,i_4)$ is not single for cycles and any $R \in \{R_k, R_k^-, k=1,2,3,4\}$ is typed if $Q_R > 0$, Lemma 4 and Lemma 5 remain valid.

Moreover, all cycles which are typed, are of the same type.

Lemma 6 If $I^{(1)}(\mu) = I^{(2)}(\mu)$ $(\forall \mu \in M)$, then for any cycle $R$, we have either

$$\left\{\begin{array}{l} Q^{(1)}_R = Q^{(2)}_R \\ \\ Q^{(1)}_{R^-} = Q^{(2)}_{R^-} \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} Q^{(1)}_R = Q^{(2)}_{R^-} \\ \\ Q^{(1)}_{R^-} = Q^{(2)}_R \end{array}\right. \tag{7}$$

proof. Set

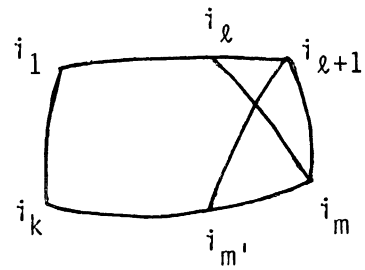
$$R = \langle i_1, i_2, \ldots, i_k \rangle,$$

when $K \leqslant 4$, (7) is true from Lemma 5.

Now, we are going to prove our assertion for $K \geqslant 5$ by induction. Assume that (7) is true for $\ell < K$ ($\ell$ is the order of the cycle), and try to prove (7) for $\ell = K$.

If $Q^{(1)}(i_1,i_2,\ldots,i_K)$ is single for cycles, (7) is true by lemma 3. Otherwise, there exists two chords $\overline{i_\ell i_m}$ and $\overline{i_{\ell+1} i_{m'}}$ such that $m > \ell + 1$. Set

$$R_1 = \langle i_1,\ldots,i_\ell,i_m,\ldots,i_K\rangle$$
$$R_2 = \langle i_{\ell+1},\ldots,i_m,\ldots,i_{m'}\rangle$$
$$R_3 = \langle i_1,\ldots,i_\ell,i_m,\ldots,i_k\rangle$$
$$R_4 = \langle i_1,\ldots,i_\ell,i_{\ell+1},i_{m'},\ldots,i_k\rangle$$




Then (7) follows from the hypothesis of the induction and Lemma.

Summing up Lemmas, we obtain the 'if' part of Theorem 1, and the 'only if' part is just an immediate conclusion from Proposition 1 and

$$\det\begin{pmatrix} 0 & q^{(r)}_{i_1 i_2} & \cdots & q^{(r)}_{i_1 i_k} \\ q^{(r)}_{i_2 i_1} & 0 & \cdots & q^{(r)}_{i_2 i_k} \\ & & & \\ q^{(r)}_{i_k i_1} & q^{(r)}_{i_k i_2} & \cdots & 0 \end{pmatrix} = \sum_{\substack{R \text{ is of } k\text{-order} \\ R=\langle i_{\ell_1},\ldots,i_{\ell_k}\rangle}} (-1)^{\rho(R)} \prod_{s=1}^{k} q^{(r)}_{i_{\ell_s \ell_{s+1}}} ,$$

where $\rho(R)$ only depends on $\langle \ell_1,\ldots,\ell_k\rangle$.

Theorem 2 follows immediately from

$$J^{(r)} = \prod_{\ell=1}^{k} q^{(r)} \quad \frac{\tilde{D}^{(r)}(\{i_1,\ldots,i_k\}^c)}{\sum_j \tilde{D}^{(r)}(\{j\}^c)} \qquad (r = 1,2)$$

(See [2], where $\tilde{D}^{(r)}(\{i_1,\ldots,i_k\}^c)$ is the determinant of $Q^{(r)}$ with rows and columns indexed by $i_1,i_s,\ldots,i_k$ deleted) and Proposition 1.

Theorem 3 follows from the definition of quasi-irriducible and the proof of Lemma 5 and the induction.

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