THE BOUNDARY VALUE PROBLEMS FOR NON-LINEAR
ELLiptic EQUATIONS II.

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THE BOUNDARY VALUE PROBLEMS FOR NON-LINEAR
ELLIPTIC EQUATIONS II. *)

Estimates of solutions of the Dirichlet problem
for elliptic Euler-Lagrange equations and
related topics in differential geometry of
convex functions.

By

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In this preprint we investigate the two-sided estimates of solutions
\( u(x) \in W^{(n)}_2(\bar{B}) \cap C(\bar{B}) \) of the Dirichlet problem for elliptic Euler-Lagrange equations
related to the minimizer for multiple integrals

\[
I(u) = \int_B F(x,u(x),Du(x))dx
\]  

(1)

under the Assumption that the values of \( Du(x) \) belong to the given closed (open)
set \( G \subset \mathbb{R}^n \). We also assume that the origin \( 0 \) of \( \mathbb{R}^n \) is an interior point of \( G \).

For example such problems are considered in the relativity theory and in continuous
mechanics. The most traditional case \( G = \mathbb{R}^n \) which considered in our preprint [1]
is a simple particular case of the results established in the present paper. Moreover
it can be obtained as limiting case from the sets \( G \) with non-empty boundary \( \partial G \).

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The contents of this preprint is as follows.

§1. The main geometric constructions and Assumptions.

1.1. The main notations. 1.2. Assumptions with respect to Euler-Lagrange equations.


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The estimates of gradients and existence Theorems for elliptic Euler-Lagrange equations will be considered in the forthcoming preprint.

§ 1. The main geometric constructions and Assumptions.

1.1. The main notations. Let \( E^n \), \( P^n \), \( Q^n \) be three \( n \)-dimensional spaces with corresponding systems of Cartesian coordinates \( x_1, x_2, \ldots, x_n \); \( p_1, p_2, \ldots, p_n \) and \( q_1, q_2, \ldots, q_n \). Let \( dx_1^2 + dx_2^2 + \ldots + dx_n^2 \); \( dp_1^2 + dp_2^2 + \ldots + dp_n^2 \) and \( dq_1^2 + dq_2^2 + \ldots + dq_n^2 \) be the metrics of \( E^n, P^n \).
and \( Q^n \) correspondingly. We will use the notations

\[
\begin{align*}
x &= (x_1, x_2, \ldots, x_n) \text{ for points of } E^n ; \\
p &= (p_1, p_2, \ldots, p_n) \text{ for points of } P^n ; \\
q &= (q_1, q_2, \ldots, q_n) \text{ for points of } Q^n 
\end{align*}
\]

and

\[
\begin{align*}
(x, z) &= (x_1, x_2, \ldots, x_n, z) \text{ for points of } E^n \times \mathbb{R} ; \\
(p, w) &= (p_1, p_2, \ldots, p_n, w) \text{ for points of } P^n \times \mathbb{R} .
\end{align*}
\]

Let \( 0', 0'', 0''' \) be origins in \( E^n, P^n, Q^n \). We denote by \( \text{mes } e' \), \( \text{mes } e'' \) and \( \text{mes } e''' \) the Lebesgue measures in \( E^n, P^n, Q^n \) for the sets \( e', e'', e''' \).

Let \( B \) be a bounded domain in \( E^n \) and let \( \partial B \) be a closed hypersurface in \( E^n \). We denote by \( J_z : B \to E^n \) the tangential mapping, generated by every function \( z(x) \in C^1(B) \), i.e.

\[
p_i = \frac{\partial z}{\partial x_i} , \quad i = 1, 2, \ldots, n \tag{1.1}
\]

Let \( G \) be a closed star-shaped n-domain in \( P^n \) with respect to the origin \( 0'' \). There are two possibilities \( \partial G \neq \emptyset \) and \( \partial G = \emptyset \). In the first case \( G \) can be either bounded or unbounded closed domain in \( P^n \), but in the second case \( G = P^n \).

We denote by \( U(\mathcal{G}) \) the closed n-ball in \( P^n \) with the center \( 0'' \) and the radius \( \mathcal{G} \). Let \( r(G) = \sup \mathcal{G} \), where \( \sup \) is taken under the condition \( U(\mathcal{G}) \subset G \). If \( \partial G \neq \emptyset \), then \( r(G) \) is a positive finite number and

\[
U(r(G)) \subset G , \quad \tag{1.2}
\]

If \( \partial G = \emptyset \), i.e. \( G = P^n \), then \( r(G) = +\infty \).

In many cases it is sufficient to consider only convex closed n-domains \( G \). If \( G \) is the ball \( |p| \leq a \), then \( r(G) = a \) and \( U(r(G)) = G \). In some problems
it will be convenient to investigate our problems in the open n-domain \( \text{int.} G \).

Now let \( \Phi(p) \) be a convex (concave) function defined in \( \text{int.} G \). The cases when \( \Phi(p) \) can be extended to \( G \) or to a larger set \( G' \supset G \) (for example to \( \mathbb{R}^n \)) are not excluded. In this paper we assume that \( \Phi(p) \in C^2(\text{int.} G) \). We introduce the set function

\[
\omega(\Phi, e'') = \max_{Q^n} \Phi(e'')
\]

(1.3)

where \( \Phi: G \rightarrow Q^n \) is the tangential mapping generated by the convex function \( \Phi(p) \) and \( e'' \) is any measurable subset of \( G \). The number \( \omega(\Phi, G) \) is called the total area of the tangential mapping \( \Phi \).

1.2. Assumptions with respect to Euler-Lagrange equations. Let \( F(x, u, p) \) be a \( C^2 \) function in \( \overline{B} \times \mathbb{R} \times G \). It is well known that any \( C^2 \) function, minimizing (maximizing) the multiple integral

\[
I(u) = \int_B F(x, u(x), \nabla u(x)) dx
\]

(1.4)

for which \( \nabla u(x) \in G \) for any \( x \in B \), necessary satisfies the Euler-Lagrange equation

\[
\sum_{i=1}^n \frac{d}{dx_i} \left( \frac{\partial F(x, u(x), \nabla u(x))}{\partial p_i} \right) = 0
\]

(1.5)

We rewrite the equation (1.5) in the form

\[
\sum_{i,k=1}^n \frac{\partial^2 F(x, u, \nabla u)}{\partial p_i \partial p_k} u_{ik} = D(x, u, \nabla u)
\]

(1.6)

where

\[
D(x, u, p) = \frac{\partial F}{\partial u} - \sum_{i=1}^n \frac{\partial^2 F}{\partial p_i \partial u} p_i - \sum_{i=1}^n \frac{\partial^2 F}{\partial p_i \partial x_i}
\]

(1.7)
The equation (1.5) is elliptic if and only if the quadratic form
\[ \sum_{i,k=1}^{n} \frac{\partial^2 F}{\partial p_i \partial p_k} s_i s_k \]  
(1.8)
is positive (negative) definite in $B' \times R \times G$. If $G$ is different from $P^n$, then we have the variational problem with limitations on the gradient of solutions.

Now we formulate the main Assumption A consisting of a few special statements concerning the integrand $F(x,u,p)$ and its derivatives up to the second order.

**Assumption A.** Let $F(x,u,p)$ be a $C^2$-function in $\overline{B} \times R \times G$ and let the quadratic form (1.8) be positive definite in $\overline{B} \times R \times G$. Then there exist functions $g_1(x,u), g_2(x,u), R_1(p), R_2(p), l_1(x,u), l_2(x,u), \Phi_1(p), \Phi_2(p)$ such that the following conditions hold:

A.1.) The functions $R_1(p), R_2(p)$ are positive and locally summable in $G$ with degree $n$; the functions $g_1(x,u), g_2(x,u)$ are non-negative in $\overline{B} \times R$ and the inequalities
\[ D(x;u,p) \leq \frac{g_1(x,u)}{R_1(p)} \]  
(1.9)
and
\[ -D(x,u,p) \leq \frac{g_2(x,u)}{R_2(p)} \]  
(1.9)
correspondingly hold in every point $(x,u,p) \in B \times R \times G$ if $D(x,u,p) \geq 0$ or $D(x,u,p) \leq 0$.

A.2.) $\Phi_1(p), \Phi_2(p)$ are strictly convex $C^2$-functions in $G$; $l_1(x,u), l_2(x,u)$ are positive in $\overline{B} \times R$ and the inequalities
\[ l_k(x,u) \det \left( \frac{\partial^2 \Phi_k(p)}{\partial p_i \partial p_j} \right) \leq R_k(p) \det \left( \frac{\partial^2 F(x,u,p)}{\partial p_i \partial p_j} \right) \]  
(1.10)
(k=1,2) hold at every point \((x,u,p) \in \bar{B} \times R \times G\).

A.3. The functions

\[
\Psi_k(x,u) = \left[ \frac{g_k(x,u)}{\ell_k(x,u)} \right]^n,
\]

(1.11)

(k=1,2) are summable in \(B\) for every fixed \(u \in R\) and are non-decreasing with respect to \(u\) for every fixed \(x \in B\).

**Example.** Let

\[
F(x,u,p) = \Phi(p) + f(x,u)
\]

for every \((x,u,p) \in \bar{B} \times R \times G\) and \(\Phi(p) \in C^2(G)\). Then equation (1.5) becomes

\[
\sum_{i,k=1}^{n} \frac{\partial^2 \Phi(v_u)}{\partial p_i \partial p_k} u_{ik} = f_u(x,u).
\]

(1.12)

Let

\[
f_u(x,u) = f_u^+(x,u) - f_u^-(x,u)
\]

(1.13)

where \(f_u^+(x,u) \geq 0\), \(f_u^-(x,u) \geq 0\) are correspondingly the positive and negative parts of \(f_u(x,u)\). Thus

\[
g_1(x,u) = f_u^+(x,u) , \quad g_2(x,y) = f_u^-(x,u)
\]

(1.14)

and

\[
R_1(p) = R_2(p) = 1.
\]

(1.15)

Clearly

\[
\Phi_1(p) = \Phi_2(p) = \Phi(p)
\]

(1.16)

and

\[
\ell_1(x,u) = \ell_2(x,u) = 1
\]

(1.17)

From (1.14-17) it follows that the Assumption A consists only of one non-trivial statement A.3, which can be formulated in the following way. The functions \(f_u^+(x,u)\) and \(f_u^-(x,u)\) belong to \(L^\infty(B)\) for every fixed \(u \in R\) and these...
functions are non-decreasing with respect to $u$ for every fixed $x \in \overline{B}$.

Other important examples and applications to problems of global differential geometry and continuous mechanics will be considered below.


2.1. Solutions with limitations on their gradient. Let $W^{(n)}_2(B)$ be the well-known Sobolev space consisting of all functions $u(x) \in L^n(B)$, which have all second generalized Sobolev derivatives belonging to $L^n(B)$.

We assume that $r(G) = \text{dist}(0^n, \partial G)$ is positive and finite. Then the following Lemma is correct.

**Lemma 1.** Let $u(x) \in W^{(n)}_2(B) \cap \text{C}(\overline{B})$ and let $\nabla u(x) \in G$. Then the inequalities

$$m(u) - r(G)\text{diam } B \leq u(x) \leq M(u) + r(G)\text{diam } B \quad (2.1)$$

hold for all $x \in \overline{B}$, where

$$m(u) = \inf_{\partial B} u(x), \quad (2.2)$$

$$M(u) = \sup_{\partial B} u(x). \quad (2.3)$$

**Proof.** Clearly

$$-\infty < m(u) \leq M(u) < +\infty \quad (2.4)$$

for all $u(x) \in W^{(n)}_2(B) \cap \text{C}(\overline{B})$. Below we will write simply $m$ and $M$ instead of $m(u)$ and $M(u)$. We establish only the inequality

$$m - r(G)\text{diam } B \leq u(x) \quad (2.1a)$$

for all $x \in \overline{B}$, because the second inequality in (2.1) can be obtained in the similar way. Since $u(x) \in \text{C}(\overline{B})$, then only the case

$$u_0 = \inf_{\overline{B}} u(x) < m \quad (2.5)$$

is interesting. Clearly
where \( x_o \) is an interior point of \( B \). Let \( S_m \) be the part of the graph of \( u(x) \) located under the hyperplane \( \gamma_m : z = m \) in the space \( E^n \times \mathbb{R} \).

We introduce the following notations

\[ \bar{B}_m \] is the set in \( \gamma_m \) whose projection on \( E^n \) coincides with \( \bar{B} \);

\[ \Gamma_m \] and \( \Gamma \) are correspondingly the boundaries of closed convex hulls \( H_m \) and \( H \) of \( \bar{B}_m \) and \( B \);

\[ C_m \] is the closed convex hull of the set \( \bar{B}_m \cup S_m \).

Then

\[ \partial C_m = H_m \cup S_v, \tag{2.6} \]

where \( S_v \) is the graph of a convex function \( v(x) \in C(H) \) such that

\[ v \big|_{\partial B} = m, \tag{2.7} \]

\[ v(x) \leq u(x) \tag{2.8} \]

for all \( x \in \bar{B} \). Clearly

\[ u_o = \inf_{\bar{B}} u(x) = u(x_o) = v(x_o) = \inf_{H} v(x) \tag{2.9} \]

where the point \( x_o \in \text{int} \ B \) was introduced above. The point \( M_o(x_o,u_o) \) belongs to the set \( S_v \cap S_m \).

Now consider two convex cones \( K_o \) and \( K_1 \), which have the common vertex at the point \( M_o \) and \( \Gamma_m \) is the base of \( K_o \) and the \((n-1)\)-sphere \( \Sigma_m \) is the base of \( K_1 \), where \( \Sigma_m \) has the center at the point \( M_o \) and its radius equal to \( \text{diam} B \) (note that \( \text{diam} H_m = \text{diam} \bar{B}_m = \text{diam} B \)).

For general convex hypersurfaces, which are graphs of continuous convex (concave) functions there is the natural generalization of the concept of the tangential mapping. This mapping (see [1], [2]) is constructed in the terms of supporting hyperplanes of the graphs of these functions and is called the normal mapping. It associates the gradient of the explicit equation of every supporting hyperplane with itself. We shall use the same notation for the normal
mapping as for the tangential mapping. We denote by \( T_m \) the open ball whose boundary is \( \sum_m \). Let \( T \) be the projection of \( T_m \) on the hyperplane \( \mathbb{E}^n \), then clearly

\[
\mathcal{K}_{K_1}(T) \subset \mathcal{K}(\text{int } H) \subset \mathcal{K}_v(\text{int } H) .
\]  
(2.10)

Since \( K_1 \) is the cone of revolution, then \( \mathcal{K}_{K_1}(T) \) is the closed \( n \)-ball in \( \mathbb{P}^n \). The explicit inequality of \( \mathcal{K}_{K_1}(T) \) is as follows

\[
|p| \leq \frac{h}{\text{diam } B} \]
(2.11)

where \( h = m - u_o \geq 0 \). Let

\[
P_m = (S_v \cap S_m) \setminus \Gamma_m
\]
(2.12)

and let \( P \) be the projection of \( P_m \) on the hyperplane \( \mathbb{E}^n \). Clearly \( P \) is a Borel subset of \( B \). From the definition of convex hull it follows that every supporting hyperplane \( \alpha \) of \( S_v \) is also a supporting hyperplane of \( S_m \) and \( \alpha \cap P_m \neq \emptyset \). Therefore

\[
\mathcal{K}_v(S_v) \subset \mathcal{K}_u(\text{int } H) \subset \mathcal{G}
\]
(2.13)

Thus from (2.10) and (2.13) it follows that

\[
\mathcal{K}_{K_1}(T) \subset \mathcal{G}
\]
(2.14)

Hence the \( n \)-ball \( \mathcal{K}_{K_1}(T) : |p| \leq \frac{h}{\text{diam } B} \) is contained in \( U(r(\mathcal{G})) \).

Thus we obtain

\[
\frac{h}{\text{diam } B} \leq \frac{m - u}{\text{diam } B} \leq r(\mathcal{G}),
\]
(2.15)

because \( h = m - u_o \geq 0 \). Finally we obtain

\[
m - r(\mathcal{G})\text{diam } B \leq u_o \leq u(x)
\]

for all \( x \in \overline{B} \).

The proof of Lemma 1 is completed.
The inequalities (2.1) have meaning only in the case \( r(G) < +\infty \).
They don't have meaning if \( G = \mathbb{P}^n \). Therefore we slightly amplify the
conditions of Lemma 1 and obtain more delicate estimates, which can be used
for both cases \( r(G) < +\infty \) and \( r(G) = +\infty \).

We shall use the following definitions related to functions \( u(x) \in W_{2}^{(n)}(B) \cap C(\overline{B}) \).
The convex function \( v(x) \) (see the proof of Lemma 1) is called the convex
support of the function \( u(x) \). In the similar way one defines the concave
support \( w(x) \) of the same function \( u(x) \). We use the part \( S_M \) of the graph
of \( u(x) \) disposed over the hyperplane \( z = M \) for this purpose. We denote by
\( S_v \) and \( S_w \) the graphs of \( v(x) \) and \( w(x) \). Clearly \( S_v \) and \( S_w \) are convex
hypersurfaces. Let \( H \) be the closed convex hull of a bounded domain \( B \). Then

\[
\bigstar_{v}(\text{int} \ H) = \bigstar_{v}(P_1) = \bigstar_{u}(P_1),
\]

\[
\bigstar_{w}(\text{int} \ H) = \bigstar_{w}(P_2) = \bigstar_{u}(P_2)
\]

where \( P_1 \) and \( P_2 \) are correspondingly the projections of the sets
\[
(S_v \cap S_M) \setminus \Gamma_M \quad \text{and} \quad (S_w \cap S_M) \setminus \Gamma_M
\]
on \( \mathbb{P}^n \) (see notations in the proof of Lemma 1). Clearly \( P_1 \subset \subset B \subset \subset \text{int} \ H \)
and \( P_2 \subset \subset B \subset \subset \text{int} \ H \). Since \( P_1 \) and \( P_2 \) are Borel subsets of \( B \), then
\( \bigstar_{v}(\text{int} \ H) \), \( \bigstar_{w}(\text{int} \ H) \) are measurable sets in \( \mathbb{P}^n \).

Below we call the function \( u(x) \) admissible if \( u(x) \in W_{2}^{(n)}(B) \cap C(\overline{B}) \)
and \( \text{grad} \ u(x) \in G \) for all \( x \in B \). According to Assumption A we suppose that \( G \)
is a closed star-shaped domain with respect to its interior point \( 0'' \) and
\( r(G) = \text{dist} \ (0'', \partial G) > 0 \), where \( 0'' \) is the origin of \( \mathbb{P}^n \).

Let \( \Phi_i(p) \), \( i = 1, 2 \) be \( C^2 \) - convex functions defined in \( G \) and
let \( \tilde{\chi}_{\Phi_i} : G \to \mathbb{P}^n \) be tangential mappings generated by \( \Phi_i \). Clearly the sets
\[ \mathcal{H}_1^*(u) = \mathcal{J}_{\Phi_1} \left( \mathcal{F}_y(\text{int } H) \right) \] 

and

\[ \mathcal{H}_2^*(u) = \mathcal{J}_{\Phi_2} \left( \mathcal{F}_y(\text{int } H) \right) \] 

are measurable in \( Q^n \). Now consider the family of the concentric balls

\[ U(\rho) : |p| \leq \rho \]

in the space \( P^\mathbb{N} \), where \( 0 \leq \rho \leq r(G) \). Clearly

\[ U(\rho) \subseteq G \] 

for all \( \rho \in [0, r(G)] \). We introduce two functions

\[ c_1(\rho) = \operatorname{mes}_{Q^n} (\mathcal{J}_{\Phi_1}(U(\rho))) \] 

and

\[ c_2(\rho) = \operatorname{mes}_{Q^n} (\mathcal{J}_{\Phi_2}(U(\rho))) \] 

of the variable \( \rho \in [0, r(G)] \). The cases \( c_1(r(G)) = +\infty \) , \( c_2(r(G)) = +\infty \)

are not excluded.

The functions \( c_1(\rho), c_2(\rho) \) are continuous, strictly increasing and take non-negative values. Therefore there exist inverses \( b_1(t), b_2(t) \), which are non-negative, continuous and strictly increasing on \([0, \mathcal{C}(\Phi_1)], [0, \mathcal{C}(\Phi_2)]\)

where

\[ \mathcal{C}(\Phi_i) = \operatorname{mes}_{Q^n} (\mathcal{J}_{\Phi_i}(U(r(G)))) , \ (i = 1, 2). \] 

**Lemma 2.** Let \( u(x) \) be an admissible function in \( \Omega \) and let the inequalities

\[ \operatorname{mes}_{Q^n} \mathcal{H}_1^*(u) < \mathcal{C}(\Phi_1) \] 

and

\[ \operatorname{mes}_{Q^n} \mathcal{H}_2^*(u) < \mathcal{C}(\Phi_2) \]
hold. Then

\[ m(u) - b_1(\text{mes}_{n} H_1^*(u)) \text{diam } B \leq u(x) \leq M(u) + b_2(\text{mes}_{n} H_2^*(u)) \text{diam } B \]  \tag{2.24}

hold for all \( x \in \overline{B} \), where \( m(u), M(u), H_1^*(u), H_2^*(u) \) and the functions \( b_1(t), b_2(t) \) were correspondingly defined by \( (2.2), (2.3), (2.16), (2.17), (2.21) \).

**Proof.** The following abbreviated notations

\[ m = m(u), \quad M=M(u), \]  \tag{2.25}

\[ t^*_i = \text{mes}_{n} H_i^*(u), i = 1,2 \]  \tag{2.26}

will be used below. As we saw in the proof of Lemma 1 it is sufficient to establish only the inequality

\[ m - b_1(t^*_1) \text{diam } B \leq u(x) \]

for all \( x \in \overline{B} \).

Let \( v(x) \) be the convex support of the function \( u(x) \). Then

\[ v(x) \leq u(x) \]

for all \( x \in \overline{B} \). The same considerations as in the proof of Lemma 1 lead to the equalities

\[ u_0 = \inf_{\overline{B}} u(x) = u(x_0) = v(x_0) = \inf_{H} v(x) \]

where \( H \) is the closed convex hull of \( \overline{B} \) and \( x_0 \) is the interior point of \( B \).

Let \( K_1 \) be the convex cone of revolution introduced in Lemma 1. From (2.10) and (2.13) it follows that

\[ \mathcal{J}_{K_1}^{(T)} \subset \mathcal{J}_{V}^{(\text{int } H)} \subset \mathcal{J}_{u}^{(\text{int } H)} \subset \mathcal{G}, \]

where \( T \) is the projection of \( K_1 \) on \( E^n \). Therefore

\[ \mathcal{J}_{\Phi}(\mathcal{J}_{K_1}^{(T)}) \subset \mathcal{J}_{\Phi}(\mathcal{J}_{V}^{(\text{int } H)}) = H_1^*(u). \]
Hence
\[
\text{mes} \rho n \mathcal{K} \left( \mathcal{K}_{K_1}(T) \right) \leq \text{mes} \rho n \mathcal{H}_1^*(u) = t_1^* < c(\Phi_1) = \\
= \text{mes} \rho n \mathcal{K} \left( U(r(G)) \right)
\]

On the other hand, \( \mathcal{K}_{K_1}(T) \) and \( U(r(G)) \) are concentric \( n \)-balls and the radius of \( \mathcal{K}_{K_1}(T) \) is equal to \( \frac{h}{\text{diam } B} \), where \( h = m - u_0 > 0 \).

Then from (2.27) it follows that
\[
\mathcal{K}_{K_1}(T) \subset U(b_1(t_1^*)) \subset U(r(G))
\]
and moreover
\[
\frac{m - u_0}{\text{diam } B} \leq b_1(t_1^*) < r(G).
\]

Thus we obtain the desired inequality
\[
m - b_1(t_1^*) \text{diam } B \leq u_0 \leq u(x)
\]
for all \( x \in \overline{B} \). The estimate of \( u(x) \) from above can be proved in the similar way.

The proof of Lemma 2 is completed.

**Theorem 1.** Let the admissible function \( u(x)^* \) be a solution of Euler-Lagrange equation (1.5). We assume that Assumption A (see Subsection 1.2) is fulfilled and
\[
\omega_1 < c(\Phi_1), \quad \omega_2 < c(\Phi_2), \quad \vdots
\]
where
\[
^* \text{ i.e. } u(x) \in W^{(n)}_2(B) \cap C(\overline{B}) \text{ and grad } u(x) \in G \text{ for all } x \in \overline{B}.
\]
\[ \omega_1 = \frac{1}{n} \int_B \psi_1(x, m) \, dx \quad ; \quad \omega_2 = \frac{1}{n} \int_B \psi_2(x, m) \, dx \quad ; \quad (2.30) \]

\[ m = \inf_{\bar{B}} u(x) \quad , \quad M = \sup_{\bar{B}} u(x) \quad (2.31) \]

and \( c(\Phi_1) \), \( i = 1, 2 \) are defined by \((2.21)\).

Then the inequalities

\[ m - b_1(\omega_1) \text{diam } B \leq u(x) \leq M + b_2(\omega_2) \text{diam } B \quad (2.32) \]

hold for all \( x \in \bar{B} \).

**Proof.** It is sufficient only to establish the inequalities

\[ \text{mes} \, Q^n H_1^*(u) \leq \omega_1 \quad (2.33) \]

\[ \text{mes} \, Q^n H_2^*(u) \leq \omega_2 \quad (2.34) \]

Since \( b_1(t), b_2(t) \) are strictly increasing functions, the direct application of Lemma 2 completes the proof of Theorem 1.

Now we establish the inequality \((2.33)\), first of all we obtain the important expression \((2.36)\) for \( \text{mes} \, Q^n H_1^*(u) \) by means of the second generalized derivatives of the function \( u(x) \). From the definition of convex hull and its properties, mentioned in the proof of Lemma 1 and page 10, it follows that

\[ \text{mes} \, Q^n H_1^*(u) = \text{mes} \, Q^n \mathcal{F}_1^* (\mathcal{F}_v (\text{int } H)) = \]

\[ = \text{mes} \, Q^n \mathcal{K}_1^* (\mathcal{K}_v (P_1)) = \text{mes} \, Q^n \mathcal{K}_1^* (\mathcal{K}_u (P_1)) = \quad (2.35) \]

\[ = \int \det \left( \frac{\nabla^2 \Phi_1(p)}{p_i \partial p_i} \right) \, dp. \quad \mathcal{F}_u (P_1) \]

Since \( u(x) \in W^{(n)}_{2}(B) \), then the measure of the tangential image (with respect
to the mapping $\mathcal{L}_u$ is absolutely continuous on the family of Borel subsets of $B$. Thus (2.35) becomes

$$\text{mes}_{H^1_0}^*(u) = \int_{P_1} \det(\frac{\nabla^2 \Phi_1(Du(x))}{\partial p_1 \partial p_j}) \det(u_{ij}(x)) \, dx \quad (2.36)$$

Since

$$u(x) = v(x)$$

for all $x \in P_1$, where $v(x)$ is the convex support of $u(x)$, then all points $(x, u(x))$, $x \in P_1$ are convex points of the graph $S_u$ of the function $u(x)$. Therefore the form

$$\sum_{i,j=1}^{n} u_{ij} \xi_i \xi_j$$

is non-negative in the set $P_1$. On the other hand, $\nabla u(x) \in G$ and according to Assumption A the form

$$\sum_{i,j=1}^{n} \frac{\nabla^2 F(x, u(x), Du(x))}{\partial p_i \partial p_j} \xi_i \xi_j$$

is positive definite for all $x \in B$.

Let

$$\sum_{i,k=1}^{n} \frac{\nabla^2 F(x, u, Du(x))}{\partial p_i \partial p_k} u_{ik} = D(x, u, Du(x)) \quad (1.5)$$

be the Euler-Lagrange equation mentioned in the statement of Theorem 1.

Using the inequality between traces of non-negative quadratic forms and inequalities (I. 9 - 11) we obtain the following inequalities

$$\frac{g_1(x, u(x))}{R_1(Du(x))} \geq D(x, u(x), Du(x)) = \sum_{i,k=1}^{n} \frac{\nabla^2 F(x, u(x), Du(x))}{\partial p_i \partial p_k} u_{ik} \geq$$
\[
\begin{align*}
\frac{n}{R_1(Du(x))} & \left( \det \frac{\partial^2 \Phi_1(Du(x))}{\partial p_1 \partial p_k} \right)^{1/n} \cdot \left( \det (u_{1k}(x)) \right)^{1/n} \cdot 1(x,u(x))
\end{align*}
\]

for all \( x \in p_1 \). Using the statement of Assumption A and the last inequality and the formula (2.36) we obtain the following inequality

\[
\text{mes}^*_{Q_1} (u) \leq \int_{\partial_1} \left[ \frac{g_1(x,u(x))}{n \cdot 1(x,u(x))} \right]^n \, dx = \frac{1}{n} \int_{\partial_1} \Psi_1(x,m) \, dx \leq
\]

\[
\leq \frac{1}{n} \int_B \Psi_1(x,m) \, dx = \Omega_1,
\]

because \( u(x) \leq m \) for all \( x \in p_1 \). Thus the inequality (2.33) is established. The inequality (2.34) can be proved in the similar way. The proof of Theorem 1 is completed.

2.2. Solutions without limitations on their gradient (the case when \( r(G) = + \infty \)).

Theorem 2. Let \( u(x) \in W^{(n)}_2(B) \cap C(\overline{B}) \) be a solution of Euler-Lagrange equation (1.5). We assume that Assumption A (see Subsection 1.2) is fulfilled, \( G = p^n \) and

\[
\omega_1 < (\Phi_1, p^n) \quad (2.37)
\]

\[
\omega_2 < (\Phi_2, p^n) \quad (2.38)
\]

where

\[
m = \inf_{\partial B} u(x) \quad , \quad M = \sup_{\partial B} u(x)
\]
\[ \omega_1 = n^{-n} \int_B \psi_1(x,m) \, dx, \quad \omega_2 = n^{-n} \int_B \psi_2(x,M) \, dx \] (2.39)

and the numbers

\[ \omega(\Phi_k^*, P^n) = \text{mes} \, Q^n(\phi_k^*(P^n))^* \quad (k = 1, 2) \] (2.40)

are the total curvatures of the convex hypersurfaces \( w = \Phi_k(p) \) in the space \( P^n \times \mathbb{R} \) (see Subsection 1.1).

Then the inequalities

\[ m - b_1(\omega_1) \text{diam } B \leq u(x) \leq M + b_2(\omega_2) \text{diam } B \] (2.41)

hold for all \( x \in B \).

**Proof.** As we mentioned above it is sufficient to establish only the inequality

\[ m - b_1(\omega_1) \text{diam } B \leq u(x) \] (2.41 a)

for all \( x \in \overline{B} \). The proof can be derived directly from Theorem 1. Let \( K_1 \) be the convex cone of revolution constructed in the proofs of Lemmas 1 and 2. Then \( h = m - u_\circ > 0 \)***) is the height of \( K_1 \) and the normal image of \( K_1 \) is the \( n \)-ball: \( |p| \leq h(\text{diam } B)^{-1} \) in the space \( P^n \). In the proofs of Lemma 2 and Theorem 1 it was established that this ball is contained in the ball: \( |p| \leq b_1(\omega_1) \), if the inequality (2.37) is hold. Hence

\[ \frac{h}{\text{diam } B} \leq b_1(\omega_1) \] (2.42)

Since \( h = m - u_\circ \), then (2.42) leads to the desired inequality (2.41 a) for all \( x \in \overline{B} \). The second inequality in (2.41) can be obtained in the similar way. The proof of Theorem 2 is completed.

*) The case \( \omega(\Phi_k^*, P) = +\infty \) is not excluded.

**) The definitions of the functions \( b_k(t) \), \( c_k(p) \), \( (k = 1, 2) \) were given by (2.19 - 20) (see Subsection 2.1).

***) \( u_\circ = \inf_{\overline{B}} u(x) \). As we noted above only the case \( u_\circ < m \) is interesting.
2.3. Formula for the total area of the tangential mapping of a complete infinite convex hypersurface \( w = \Phi(p) \). In this Subsection we assume that the closed domain \( G \) is either the \( n \)-ball \( |p| \leq a, (0 < a < +\infty) \) or the entire space \( \mathbb{R}^n \). Then

\[
U(r(G)) = G
\]  

(2.43)

Hence

\[
\text{mes}^n(\mathcal{J}_\Phi(U(r(G)))) = \lim_{\delta \to 0^+} \text{mes}^n(\mathcal{J}_\Phi(U(\delta))) = \omega(\Phi,G)
\]  

(2.44)

where \( \omega(\Phi,G) \) is the total curvature of a convex hypersurface \( w = \Phi(p) \) (see Subsection 1.1). Thus inequalities (2.28-29) and (2.37-38) coincide.

Now we assume that \( w = \Phi(p) \) is a complete infinite convex hypersurface. We denote by \( K \) the asymptotic cone of \( \Phi \) (see the definition and properties of asymptotic cone in [ ]). Since

\[
\mathcal{J}_\Phi(G) = \mathcal{J}_K(G),
\]  

(2.45)

then

\[
\omega(\Phi,G) = \text{mes}^n(\mathcal{J}_K(G)).
\]  

(2.46)

It is well known that \( K = \mathcal{Q}_\Phi \), where \( Q_\Phi \) is a solid convex cone in \( \mathbb{R}^n \times \mathbb{R} \) with the vertex at any point \( M \) of the convex hypersurface \( w = \Phi(p) \). More precisely \( Q_\Phi \) is the union of all rays starting from \( M \) and situated over the convex hypersurface \( \Phi \). The solid cone \( Q_\Phi \) can have dimension \( 1, 2, 3, \ldots, n+1 \).

If \( G \) is a bounded set, then \( Q_\Phi \) is necessary a ray orthogonal to \( \mathbb{R}^n \) (we remind that \( \Phi \) is a complete infinite hypersurface in \( \text{int} G \)). Clearly

\[
\mathcal{J}_K(G) = Q^n
\]

for such degenerate one-dimensional convex cone \( Q_\Phi \).

If \( K \) is projected one-to-one on \( \mathbb{R}^n \), then \( K \) is a non-degenerate
n-dimensional convex cone and \( \mathcal{K}_\Phi(G) \) is a bounded closed convex set in \( Q^n \). Hence \( 0'''' \in \text{int} \mathcal{K}_\Phi(G) \), where \( 0'''' \) is the origin of \( Q^n \) and
\[
\text{mes}_{Q^n} \mathcal{K}_\Phi(G) < + \infty. \quad (2.47)
\]
The formula (2.46) is very useful for calculations of the total area of tangential mapping.

S 3. Applications.

3.1. Hypersurfaces with prescribed mean curvature in the Euclidean space
\( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \). Let the hypersurface \( S \) with prescribed mean curvature \( H \) be a graph of the function \( u(x) \in C^2(\overline{B}) \cap C(\overline{B}) \). We assume that \( H = H(x) \in C(\overline{B}) \), then \( u(x) \) satisfies the Euler-Lagrange equation for the functional
\[
I(u) = \int \left( \sqrt{1 + (Du(x))^2} + nH(x)u(x) \right) \, dx \quad (3.1)
\]
According to Example, considered in Subsection 1.2
\[
\Phi_1(p) = \Phi_2(p) = \left[ 1 + |p|^2 \right]^{1/2} ;
\]
\[
g_1(x,u) = H_+(x) , \quad g_2(x,u) = H_-(x) ;
\]
\[
l_1(x,u) = l_2(x,u) = 1 ;
\]
\[
R_1(p) = R_2(p) = 1
\]
for all \( x \in \overline{B} \), \( u \in \mathbb{R} \) and \( p \in \mathbb{P}^n \), where \( H_+(x) \) and \( H_-(x) \) are correspondingly positive and negative parts of the function \( H(x) \).
The equation of the convex cone $K_{\Phi}$ is $\omega = |p|$ for all $p \in \mathbb{R}^n$. Therefore $\mathcal{K}_{\Phi}(\mathbb{R}^n)$ is the unit $n$-ball $|q| \leq 1$ in $\mathbb{R}^n$. If $\mu_n$ is the $n$-volume of this ball, then inequalities (2.37-38) become

$$\omega_\pm = \int_{B} H_\pm(x) dx < \mu_n$$

(3.2)

because $\omega(\Phi, \mathbb{R}^n) = \mu_n$.

The calculations show that

$$b_1(t) = b_2(t) = \left[ \frac{2}{\mu_n} \right]^{1/2} \left[ \frac{\mu_n}{\mu_n - t} \right]$$

(3.3)

Thus the estimates (2.41) become

$$\inf_{\partial B} u(x) - b_1(\omega_+) \text{diam } B \leq u(x) \leq \sup_{\partial B} u(x) + b_2(\omega_-) \text{diam } B,$$

where $\omega_+$ and $b_1(\omega_+)$, $b_2(\omega_-)$ are defined by (3.2) and (3.3).

Let $B$ be the $n$-dimensional ball of the radius $r_o$ and let $H = H_o = \text{const } 0$. Then the Dirichlet problem

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{u_i}{(1 + (\mu u)^2)^{1/2}} \right) = nH_o$$

(3.4)

$$u|_{\partial B} = 0$$

(3.5)

has only unique solution $u(x) \in C^2(B) \cap C^1(\overline{B})$, moreover this solution is always the spherical segment and the inequalities

$$r_o H_o \leq 1 \quad \text{and} \quad 0 < \mu_n$$

(3.6)

are necessary and sufficient conditions of the solvability of the Dirichlet problem.
(3.4-5). Clearly the inequalities (3.2) become (3.6) for the Dirichlet problem (3.4-5).

Thus the inequalities (3.2) are sharp.

3.2. Spacelike hypersurfaces with prescribed mean curvature in the Lorentz-Minkowski space $\mathbb{M}^{n+1}$. The space $\mathbb{R}^{n+1} = (x,t) = (x_1, x_2, \ldots, x_n, z)$ with the metric

$$ds^2 = \sum_{i=1}^{n} (dx_i)^2 - dz^2$$

(3.7)

is called the Lorentz-Minkowski space. This space will be denoted by $\mathbb{M}^{n+1}$. Let $S$ be a hypersurface such that $ds^2$ restricts to a positive definite on $S$. Such $S$ are called spacelike. If $S$ is the graph of the function $z = u(x)$, then $S$ is spacelike if and only if

$$\left| Du(x) \right| < 1$$

(3.8)

for any $x = (x_1, x_2, \ldots, x_n)$ belonging to the domain of the function $u(x)$. The spacelike hypersurfaces in $\mathbb{M}^{n+1}$ were studied by Calabi [3] and by Cheng and Yau [4] in the investigations the Bernstein conjecture in $\mathbb{M}^{n+1}$.

The spacelike solutions $u(x) \in C^2(\mathbb{B})$ of the Euler-Lagrange equation for the functional

$$I(u) = \int_{\mathbb{B}} \left[ (1 - (Du)^2)^{1/2} - nH(x) u \right] dx$$

(3.9)

have prescribed mean curvature $H(x)$. Clearly $H(x) \in C(\mathbb{B})$ for such hypersurfaces. According to Assumption A we have

$$\Phi_1(p) = \Phi_2(p) = (1 - |p|^2)^{1/2}$$

for all $p \in \text{int} G$, where $\text{int} G$ is the $n$-unit ball $|p| < 1$ in $\mathbb{F}^n$. It is also clear that
\[ g_1(x,u) = H_+(x) , \quad g_2(x,u) = H_-(x) ; \]

\[ l_1(x,u) = l_2(x,u) = 1 ; \]

\[ R_1(p) = R_2(p) = 1 \]

for all \( x \in \overline{B} \), \( u \in \mathbb{R} \) and \( p \in \text{int} \ G \). Clearly

\[
\mathcal{J}_\Phi (\text{int} \ G) = Q^n
\]

Therefore inequalities (2.28-29) become

\[
\int_{\overline{B}} H_+^n(x) dx < + \infty
\]

Thus there is a big difference between the solutions of the Dirichlet problem for the mean curvature equation in Euclidean and Lorentz-Minkowski spaces.

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