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ABSTRACT

Kudryashov in [Phys. Lett. A 342 (2005) 99-106] used simplest nonlinear differential equations like the Riccati equation, the equation for the Jacobi elliptic function to present a new approach for searching exact solutions of nonlinear partial differential equations. As application, he obtained a kind of exact solutions to the Fisher equation. In this letter, more explicit exact solitary wave solutions to the Fisher equations are given by adding to this new approach, the general Riccati equation and the general Jacobi elliptic equation instead of simplified form of the both equations.

1 INTRODUCTION

The original Fisher equation constitutes a (1+1)-dimensional system of evolution equation with both physical and mathematical significance. Physically, Fisher equation describes the propagation of mutant genes [1]. This equation occurs in flame propagation, in branching Browian motion process and in nuclear reactor theory [2]. Mathematically Fisher equation forms a typical and rich system where an efficient factorization procedure directly related to the factorization of its nonlinear polynomial part [3], and other ingenious techniques of the modern theory of nonlinear waves are applicable [4].

Fisher equation possesses many families of exact solutions namely: the kink solutions first obtained by Ablowitz and Zeppetella [2] with a series solutions method, the supersymmetric kink, which was a widths one and half times greater than the original Fisher Kink [2]. Another novel family of solutions is the exact solitary waves found by Kudryashov [4], using its new approach. This approach takes two ideas into account; the first idea applies simplest nonlinear differential equations (the Riccati equation, the equation for the Jacobi elliptic function and so on) that have lesser order than the equation studied. The second idea uses possible singularities of general solution for the equation studied. Owing the use of Riccati equation, the simplify elliptic equation and the (2.8) in [4] reduce or restrict the class of solutions. The aim of this note is to correct this deficiency in the Kudryashov beautiful's work [4]. Therefore, the more general is appropriately proposed ansatz, the more general and more formal solutions of nonlinear partial differential equations (NLPDEs) will be obtained. But it seems also to be important to know how to obtain more new solutions of NLPDEs under known ansatz. To make the method proposed by Kydryashov [4] more powerful, its equations (2.3) and (2.4) should be replaced by

$$E_1(Q) = Q_z - qQ^2 - pQ - r = 0, \qquad (\mathbf{1})$$

and

$$E_2(Q) = Q_z^2 - lQ^4 - aQ^3 - bQ^2 - cQ - d, \qquad (\mathbf{2})$$

respectively. Here the constants l and q in the difference of the Kudryashov's work [4] can take values different from 1 and -1 respectively. In the following note, only the equation (2.9) in its step 2 will be used as it's more complete than Eq.(2.8). By adding the general Riccati equation (1), the general elliptic equation (2) and the form (2.9), i.e.

$$y(z) = A_0+A_1Q+...+A_nQ^n+(B_1+...+B_{n-1}Q^{n-2})Q_z+D_1(\frac{Q_z}{Q})+D_2(\frac{Q_z}{Q})^2+...+D_n(\frac{Q_z}{Q})^n, \qquad (\mathbf{3})$$

where $Q(z)$ is the general solution of Eq.(1) or Eq.(2), A_k $(k = 0, ..., n)$, B_k $(k = 1, ..., n-1)$ and D_k $(k = 1, ..., n)$ are coefficients that should be found. We will show that it's possible to obtain solutions which are more rich and new. Moreover, we will show that the solutions obtained by the author of [4] are just special cases of our solutions. It should be noted that both formulas (1) and (2) can be used in the form (2.9). The following amendments should be considered: The equation (2.8) should be removed from the second step and third step in [4].

Theorem 2.1. will be: Let Q(z) be a solution of Eq.(1). than the equations

$$Q_{zz} = 2q^2Q^3 + 3pqQ^2 + (p^2 + 2qr)Q + pr, \qquad (\mathbf{4})$$

$$Q_{zzz} = 6q^2Q^4 + 12pq^2Q^3 + (7p^2q + 8q^2r)Q^2 + (p^3 + 8pqr)Q + p^2r + 2qr^2, \qquad (\mathbf{5})$$

have special solutions that are expressed via the general solution of Eq.(1).

Proof. Theorem 2.1 is proved by differentiation of (1) with respect to z and substitution Q_z from Eq.(1) into expressions obtained.

Theorem 2.2. Let $Q(z)$ be a solution of Eq.(2). Than the equations

$$Q_{zz} = 2lQ^3 + \frac{3}{2}aQ^2 + bQ + \frac{c}{2}, \qquad (\mathbf{6})$$

$$Q_{zzz} = 6lQ^2Q_z + 3aQQ_z + bQ_z, \qquad (\mathbf{7})$$

have special solution that are expressed via the general solution of Eq. (2).

Proof. Theorem 2.2 is proved along similar lines than Theorem 2.1. Substituting dependence y on Q (3) into Eq.(1), and taking Theorem 2.1 into account, we obtain

$$M[y] = \sum_{k=-N}^{N} P_k(p, q, r, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n)Q^k. \qquad (\mathbf{8})$$

Consider the system of algebraic equations with respect to $p,\ q,\ r,\ A_k$ $(k = 0, ..., n)$, B_k $(k = 1, ..., n-1)$ and D_k $(k = 1, ...n)$

$$P_k(p, q, r, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n) = 0, \quad (k = 0, ...2N). \qquad (\mathbf{9}).$$

As a consequence of the choice for the dominant terms of Eq.(8) we have $A_n \neq 0$ at the first step of the method. Assume there are solutions of the system of equations (9). If Eqs.(9) are satisfied then

$$M(y) = 0, \qquad (\mathbf{10})$$

and y by formula (3) is the solution of the given NLPDEs. Here $Q(z)$ is the general solution of Eq.(1). M is defined in [4]
Substituting (3) into Eq.(2) and taking theorem 2.2. into consideration we have the equality in the form

$$M(y) = \sum_{k=-N}^{N} P_k(l, a, b, c, d, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n)Q^k +$$

$$\sum_{k=2-N}^{N-2} S_k(l, a, b, c, d, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n)Q^k Q_z. \quad (\mathbf{11})$$

Suppose we have solutions of the system of equations.

$$P_k(l, a, b, c, d, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n) = 0, \quad (k = 0, ..., 2N), \quad (\mathbf{12})$$

$$S_k(l, a, b, c, d, A_0, ...A_n, B_1, ...B_{n-1}, D_1, ..., D_n) = 0, \quad (k = 1, ..., 2N-4), \quad (\mathbf{13})$$

then y expressed by the formula (3) allows us to have exact solutions of the NLPDEs in consideration.

2 Exact solutions waves of the Fisher equation.

Let begin as in [4] with the equation (3.1). After using the formula (3.2), equation (3.3) is obtained in [4]. Then from (3.3), using the travelling wave reduction Eq.(3.4) in [4] is reduced to Eq.(3.5) namely

$$y_{zz} + C_0 y_z + y - y^2 = O, \qquad z = x - C_0 t, \quad (\mathbf{14})$$

where C_0 is a constant to be determined later. Let us look for solution of Eq.(14) in the form (3) at $n = 2$.

$$y(z) = A_0 + A_1 Q + A_2 Q^2 + B_1 Q_z + D_1(\frac{Q_z}{Q}) + D_2(\frac{Q_z}{Q})^2. \quad (\mathbf{15})$$

Taking Eq.(1) into consideration formula (15) can be written in the form

$$y(z) = (A_0 + rB_1 + pD_1 + p^2 D_2 + 2qrD_2) + (A_1 + pB_1 + qD_1 + 2qpD_2)Q+$$

$$(A_2 + qB_1 + q^2 D_2)Q^2 + \frac{rD_1 + 2prD_2}{Q} + \frac{r^2 D_2}{Q^2}. \quad (\mathbf{16})$$

Substituting (16) into Eq.(14) we obtain

$$D_2^{(1)} = 6, \qquad D_2^{(2)} = 0. \tag{17}$$

Consider the case $D_2 = D_2^{(1)} = 6$. We have

$$D_1 = -\frac{6}{5}(5p + C_0), \quad A_0 = \frac{1}{50}(25 + 25p^2 - 400qr - 50rB_1 + 30C_0p - C_0^2),$$

$$A_2 = -6q^2 - qB_1, \qquad A_2 = -qB_1. \tag{18}$$

Assuming $A_2 = -6q^2 - qB_1$, we have

$$A_1 = \frac{1}{5}(-30pq - 5pB_1 + 6qC_0), \quad C_0 = \pm 5\sqrt{p^2 - 4qr},$$

$$p = \epsilon\frac{\sqrt{1+24qr}}{\sqrt{6}}, \quad p = \epsilon\frac{\sqrt{-1+24qr}}{\sqrt{6}}, \quad \epsilon = \pm 1. \tag{19}$$

where $r,\ q$ and B_1 are arbitrary constants. as a result, We have the exact solution of (14) in the form

$$y(z) = \frac{1}{2} + \frac{p^2}{2} + 4qr - \frac{3pC_0}{5} - \frac{C_0^2}{50} + \frac{6r(p - \frac{C_0}{5})}{Q} + \frac{6r^2}{Q^2}, \tag{20}$$

where $Q(z)$ is determined by appendix (1). Assuming $p = \epsilon\frac{\sqrt{1+24qr}}{\sqrt{6}}$, we find that $C_0 = \pm 5\frac{\sqrt{6}}{6}$ from (19), and we have several special solutions of the Fisher equation determined by the sign of the parameter $p^2 - 4qr$, which is equal to $\frac{1}{6}$ thus positive. Finally, the exact solutions for the Fisher equation are written as

$$y(z) = \frac{1}{2}(1 + 12qr \mp \epsilon\sqrt{1 + 24qr}) + \sqrt{6}r(\mp 1 + \epsilon\sqrt{1 + 24qr})Q^{-1} + 6r^2Q^{-2}. \tag{21}$$

$Q(z)$ is given by appendix (1) in which $p,\ p^2 - 4qr$ are replaced by $\epsilon\frac{\sqrt{1+24qr}}{\sqrt{6}}$ and $\frac{1}{6}$ respectively, $q,\ r$ and C_2 are arbitrary constants.

Now, if we assume $p = \epsilon\frac{\sqrt{-1+24qr}}{\sqrt{6}}$ this leads to $C_0 = \pm 5I\frac{\sqrt{6}}{6}$ from (19), and the parameter $p^2 - 4qr$ is equal to $-\frac{1}{6}$. We have other several special periodic wave solutions of the Fisher equation which can be deduced from the solitary wave solutions given in Appendix. We will omit them in the present note.

In the case $A_2 = -qB_1$ and $p = 0$, we have

$$A_1 = 12qC_0, \quad C_0 = \pm 20\sqrt{-qr}, \quad qr = \epsilon\tfrac{1}{96}, \quad \epsilon = \pm 1. \qquad \textbf{(22)}$$

Assuming $\epsilon = -1$, and the two values of $C_0 = \pm 5\frac{\sqrt{6}}{6}$, the exact solutions of the Fisher equation in this case is determined by the formula

$$y(z) = \frac{3}{8} \mp \frac{1}{16\sqrt{6}r}Q + \frac{1}{1536r^2}Q^2 \mp \frac{\sqrt{6}r}{Q} + \frac{6r^2}{Q^2}, \qquad \textbf{(23)}$$

where $Q(z)$ is determined by the appendix in which $p,\ qr$ may be replaced by 0, $\frac{-1}{96}$ respectively, $C_2,\ r$ are arbitrary constants.
Now assuming $\epsilon = 1$, and the two values of $C_0 = \pm 5I\frac{\sqrt{6}}{6}$, the exact solutions of the Fisher equation are the periodic waves, with the $Q(z)$ function deduced from the Appendix, $C_2,\ r$ are arbitrary constants; we omit them in the present work.

In the case $D_2 = D_2^{(1)} = 0$, we have

$$D_1 = 0, \qquad B_1 = 0, \qquad A_1 = \tfrac{6}{5}(5pq + qC_0), \qquad A_2 = 6q^2$$
$$A_0 = \tfrac{1}{50}(25 + 25p^2 + 200qr + 30pC_0 - C_0^2), \qquad r = \tfrac{25p^2 - C_0^2}{100q}, \qquad \textbf{(24)}$$

$$C_0^{(1)} = \tfrac{5\sqrt{6}}{6}, \ \ C_0^{(2)} = -\tfrac{5\sqrt{6}}{6}, \ \ C_0^{(3)} = \tfrac{5I\sqrt{6}}{6}, \ \ C_0^{(4)} = -\tfrac{5I\sqrt{6}}{6}. \qquad \textbf{(25)}$$

Therefore, special solution of the Fisher equation using $C_0^{(1)}$ and $C_0^{(2)}$ can be written in the form

$$y(z) = \frac{1}{10}(\frac{15}{6} \pm 5\sqrt{6}p + 15p^2) + 6q(\pm\frac{\sqrt{6}}{6} + p)Q + 6q^2Q^2. \qquad \textbf{(26)}$$

where $Q(z)$ is determined by the Appendix in which $r,\ p^2 - 4qr$ are replaced $\frac{25p^2 - C_0^2}{100q}$ and $\frac{1}{6}$ respectively, $C_2,\ p$ and q are arbitrary constants.

Using the formula (3) at $n = 2$, we obtain another solution of the Fisher equation. These solutions are found at C_0 that are determined by the formulas (25). For example, the solution of the Fisher equation at $C_0^{(1)}$ and $C_0^{(2)}$ takes the form

$$y(z) = \frac{1}{2}(1 - \frac{D_2}{3}) + (\pm\sqrt{\frac{3l}{2}} + \frac{1}{4}a(6 - 4D_2))Q + l(3 - D_2)Q^2 + 3\sqrt{l}Q_z \pm \sqrt{\frac{3}{2}}\frac{Q_z}{Q} + D_2(\frac{Q_z}{Q})^2, \qquad \textbf{(27)}$$

or

$$y(z)=\frac{1}{2}(1-\frac{D_2}{3})+(\mp\sqrt{\frac{3l}{2}}+\frac{1}{4}a(6-4D_2))Q+l(3-D_2)Q^2-3\sqrt{l}Q_z\pm\sqrt{\frac{3}{2}}\frac{Q_z}{Q}+D_2(\frac{Q_z}{Q})^2, \quad (\mathbf{28})$$

where

$$Q_z^2=\frac{1}{6}Q^2+aQ^3+lQ^4. \quad (\mathbf{29})$$

The solution of Eqs.(27) - (29) take the form

$$Q(z)=\frac{-3a\tanh^2(\frac{\sqrt{6}}{6}(z-C_1))+3a\pm\sqrt{3}\sqrt{(3a^2-2)F_1(z)}}{3(2l\tanh^2(\frac{\sqrt{6}}{6}(z-C_1))-3a^2)},$$

$$F_1(z)=\frac{3a^2-2l}{3a^2-2}\tanh^2(\frac{\sqrt{6}}{6}(z-C_1))(\tanh^2(\frac{\sqrt{6}}{6}(z-C_1))-1). \quad (\mathbf{30})$$

The exact solitary waves of the Fisher equation with four arbitrary constants $a,\ C_1,\ D_2$ and $l>0$ are obtained. It should be noted that if one uses special values of the arbitrary constants, for example $l=1,\ D_2=0$, in Eqs.(27),(29) and (30) he can obtain exact solutions that were obtained by Kudryashov [4].

Other new solutions of the Fisher equation can be found by chosen in formulas (27) - (29),

$$Q(z)=\frac{\alpha_1\tanh^2(\frac{\sqrt{6}}{12}(z-C_1))-3a\alpha_2}{-3a\alpha_1\tanh^2(\frac{\sqrt{6}}{12}(z-C_1))+\alpha_2}, \quad (\mathbf{31})$$

where $l=\frac{3a^2}{2},\ a,\ C_1,\ D_2,\ \alpha_1$ and α_2 are arbitrary constants.

$$Q(z)=\frac{-1/\sqrt{6}\pm\sqrt{l}}{\eta(\pm\sqrt{l}+\cosh(1/\sqrt{6}(z-C_1))-\sinh(1/\sqrt{6}(z-C_1))}, \quad (\mathbf{32})$$

$$Q(z)=-\frac{1/\sqrt{6}(\cosh(1/\sqrt{6}(z-C_1))+\sinh(1/\sqrt{6}(z-C_1)))}{\eta(\pm\sqrt{l}+\cosh(1/\sqrt{6}(z-C_1))-\sinh(1/\sqrt{6}(z-C_1))}. \quad (\mathbf{33})$$

where $l=\frac{3a^2}{2},\ C_1,\ D_2,\ \eta$ and a are arbitrary constants.

In conclusion, by using the general Riccati equation and the general elliptic equation instead of the simplified Riccati equation and the equation for the Jacobi elliptic function as Kudryashov did in [4], we have succeeded to extend the class of solutions and improve the beautiful method built by the above mentioned author. By doing this The solutions he proposed in [4] revealed to be a particular case of our. We have also found new solutions of the Fisher equation.

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Appendix Solitary wave solutions of the general Riccati equation (1).

As in [5], when $p^2 - 4qr > 0$ and $pq \neq 0$ $(qr \neq 0)$,

$$\begin{cases} Q_1 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\tanh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)\right], \\ Q_2 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\coth\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)\right]; \end{cases}$$

$$\begin{cases} Q_3 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}(z-C_2)\right) \pm i\,sech\left(\sqrt{p^2-4qr}(z-C_2)\right)\right)\right], \\ Q_4 = -\frac{1}{2q}\left[p + \sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}(z-C_2)\right) \pm csch\left(\sqrt{p^2-4qr}(z-C_2)\right)\right)\right], \\ Q_5 = -\frac{1}{4q}\left[2p + \sqrt{p^2-4qr}\left(\tanh\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right) + \coth\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)\right)\right]; \end{cases}$$

$$\begin{cases} Q_6 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2+B^2)(p^2-4qr)} - A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(z-C_2)\right)}{A\sinh\left(\sqrt{p^2-4qr}(z-C_2)\right)+B}\right], \\ Q_7 = \frac{1}{2q}\left[-p - \frac{\sqrt{(B^2-A^2)(p^2-4qr)} + A\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}(z-C_2)\right)}{A\cosh\left(\sqrt{p^2-4qr}(z-C_2)\right)+B}\right]; \end{cases}$$

where A and B are two non-zero real constants and satisfies $B^2 - A^2 > 0$.

$$\begin{cases} Q_8 = \frac{2r\cosh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right) - p\cosh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)}, \\ Q_9 = \frac{-2r\sinh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)}{p\sinh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right) - \sqrt{p^2-4qr}\cosh\left(\frac{\sqrt{p^2-4qr}}{2}(z-C_2)\right)}; \end{cases}$$

$$\begin{cases} Q_{10} = \dfrac{2r\cosh\left(\sqrt{p^2-4qr}(z-C_2)\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}(z-C_2)\right)-p\cosh\left(\sqrt{p^2-4qr}(z-C_2)\right)\pm i\sqrt{p^2-4qr}}, \\ Q_{11} = \dfrac{2r\sinh\left(\sqrt{p^2-4qr}(z-C_2)\right)}{-p\sinh\left(\sqrt{p^2-4qr}(z-C_2)\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}(z-C_2)\right)\pm\sqrt{p^2-4qr}}; \end{cases}$$

$$Q_{12} = \frac{4r\cosh\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)\sinh\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)}{-2p\cosh\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)\sinh\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)+2\sqrt{p^2-4qr}\cosh^2\left(\frac{\sqrt{p^2-4qr}}{4}(z-C_2)\right)-\sqrt{p^2-4qr}}.$$

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