PARTITIONS FOR SPECTRAL (FINITE) VOLUME RECONSTRUCTION IN THE TETRAHEDRON

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Partitions for Spectral (Finite) Volume Reconstruction in the Tetrahedron

Qian-Yong Chen*

Abstract

In this paper, we compute partitions of the tetrahedron for up to the fourth-order spectral volume reconstruction. Certain optimization is made to these partitions and previously obtained partitions of lower dimensional simplex. These optimized partitions have the smallest Lebesgue constants among currently known spectral volume partitions.

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1 Introduction

Spectral volume reconstruction is a key element of the recently proposed spectral volume method [10, 11] for hyperbolic conservation laws. Analogous to the well known fact that the quality of polynomial interpolation depends on the interpolation points set, the quality of spectral volume reconstruction in the simplex is determined by the partition of the simplex [3]. Since the development of the spectral volume method, some research has been done on the partition generation. For example, by following the idea of Chen and Babuška [1, 2], Wang and Liu computed the so called mean L^2 optimal partitions for up to the seventh order spectral volume reconstruction of the one-dimensional simplex [9]. Several systematic techniques based on the Voronoi diagram and its variants have also been developed in [3] for both the one and two-dimensional simplex. More recently, a linear partition and a quadratic partition of the tetrahedron were constructed in [7].

However, partitions for high order spectral volume reconstruction on the tetrahedron are still unavailable. In this paper we compute up to the fourth-order partitions of the three-dimensional simplex, S^3. These partitions are based on the idea we proposed in [3], i.e., building the partition through extensive use of the geometry structure of the interpolation points in the simplex, such as the symmetry and layering structure. The idea can be extended to generate higher order partitions of the tetrahedron. Optimization within the framework of building these partitions is also made for these partitions and the previously obtained partitions of the one and two-dimensional simplex.

Denote $\mathbf{P}^n(S^3)$ as the space of polynomials of degree up to n in three variables. The dimension of this approximation space is

$$N_n = \dim \mathbf{P}^n(S^3) = \binom{3+n}{3} = \frac{(n+1)(n+2)(n+3)}{6}.$$

$N = N_n$ and $\mathbf{P}^n = \mathbf{P}^n(S^3)$ will be used to simplify the notations if there is no confusion. Assume $\{p_1(x,y,z), \cdots, p_N(x,y,z)\}$ forms a complete basis of $\mathbf{P}^n(S^3)$.

Given any continuous function $u(x,y,z)$ on S^3, i.e., $u \in C(S^3)$, the computation of its n-th order spectral volume reconstruction on S^3 consists of two steps:

1. Construct a partition Π_n of S^3 with N non-overlapping sub-cells (only polyhedron sub-cells

are considered):

$$S^3 = C_1 \cup \cdots \cup C_N.$$

2. Find a projection $\mathcal{I}_{\Pi_n} u = \sum_{i=1}^{N} \; a_i \; p_i(x, y, z) \in \mathbf{P}^n$, which shares the same average as u on all the sub-cells, i.e.,

$$\frac{1}{V_i} \int_{C_i} (\mathcal{I}_{\Pi_n} u) \; dV = \frac{1}{V_i} \int_{C_i} u(x, y, z) dV, \qquad i = 1, \cdots, N, \tag{1}$$

where V_i denotes the area of sub-cell C_i.

Denote $\bar{u}_i$ as the average of $u(x, y, z)$ over sub-cell C_i, i.e.,

$$\bar{u}_i = \frac{1}{V_i} \int_{C_i} u(x, y, z) \; dV, \qquad i = 1, \cdots, N.$$

Rewrite the system (1) into a matrix form: $\mathbf{A}\mathbf{a} = \mathbf{u}$ with $\mathbf{a} = (a_1, \cdots, a_N)^T$ and $\mathbf{u} = (\bar{u}_1, \cdots, \bar{u}_N)^T$. The reconstruction matrix $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{pmatrix} \frac{1}{V_1} \int_{C_1} p_1(x, y, z) dV & \cdots & \frac{1}{V_1} \int_{C_1} p_N(x, y, z) dV \\ \cdots & \cdots & \cdots \\ \frac{1}{V_N} \int_{C_N} p_1(x, y, z) dV & \cdots & \frac{1}{V_N} \int_{C_N} p_N(x, y, z) dV \end{pmatrix}. \tag{2}$$

Assuming the reconstruction matrix is nonsingular, the projection $\mathcal{I}_{\Pi_n} u$ can be expressed in the Lagrange form, $\mathcal{I}_{\Pi_n} u = \sum_{i=1}^{N} \bar{u}_i \; L_i(x, y, z)$, where the cardinal basis functions $\mathbf{L} = (L_1, \cdots, L_N) = (p_1, \cdots, p_N) \; \mathbf{A}^{-1}$.

Then we equip the space $\mathbf{P}^n$ and $C(S^3)$ with an L^∞ norm (supremum-norm, denoted as $\|\cdot\|$) and the induced functional norm

$$\|\mathcal{I}_{\Pi_n}\| \; = \; \sup_{\|u\| \neq 0} \frac{\|\mathcal{I}_{\Pi_n} u\|}{\|u\|}.$$

Since $|\bar{u}_i| \leq \|u\|$ for $i = 1, \cdots, N$, one can show that

$$\|\mathcal{I}_{\Pi_n}\| = \max_{(x,y,z) \in S^3} \sum_{i=1}^{N} |L_i(x, y, z)| \, .$$

From the linearity of the projection operator $\mathcal{I}_{\Pi_n}$ and the fact that $\mathcal{I}_{\Pi_n} f = f, \forall f \in \mathbf{P}^n(S^3)$, it is easy to verify that the error of spectral volume reconstruction can be bounded as

$$\|u - \mathcal{I}_{\Pi_n} u\| \leq (1 + \Lambda(\Pi_n)) \; \|u - u^*\|, \tag{3}$$

where u^* is the optimal approximating polynomial whose existence is guaranteed by the continuity of $u(x, y, z)$ [4]. And

$$\Lambda(\Pi_n) = \|\mathcal{I}_{\Pi_n}\| = \max_{(x,y)\in S^3} \sum_{i=1}^{N} |L_i(x, y, z)| \tag{4}$$

is called the Lebesgue constant of the operator $\mathcal{I}_{\Pi_n}$.

According to (3), the partitions with small Lebesgue constants are preferred. It is rather difficult to directly build good high order partitions, specially for three dimensional spectral volume reconstructions, because there are too many parameters such as the position of points, the number of edges for each sub-cell and the topology of the sub-cells. For another reconstruction problem, the polynomial interpolation, several almost optimal sets have been obtained (see [8, 5, 1, 2] and reference therein). At a glimpse, it seems that the methodology of [1, 2] can be used to optimize the spectral volume partition. However, as shown in [9], the mean L^2 optimal partitions are even not very satisfactory for the one-dimensional case. So in this paper we do not compute the mean L^2 optimal partitions. Instead, we only compute partitions based on the polynomial interpolation points, and try to optimize these partitions within the framework of constructing them. In addition, we try to minimize the number of total faces of the partition whenever it is possible.

The rest of the paper includes two sections. In Sec. 2, we describe the algorithm of computing up to the fourth order partition of the tetrahedron. Section 3 is devoted to the optimization of the partitions in Sec. 2 and the partitions of the lower dimensional simplex developed in [3].

2 Partitions of the Tetrahedron

In this section, we propose an algorithm to compute symmetric partitions for the spectral volume reconstruction in the tetrahedron. Similar to the technique developed in [3] for the lower dimensional simplex, this algorithm exploits the geometry structure of interpolation points on the tetrahedron (points from [6] are used in this paper). The algorithm is described in a recursive fashion in the sense of high order partitions being based on lower order partitions. In specific, when building high order partitions, we first group all the interpolation points except those on a single tetrahedron face into a new points set. Then we construct some sub-cells from the new points set with the algorithm for the one-order lower partition. We will explain it in detail in the following.

The symmetry property of the partitions is extensively used in the algorithm. But unlike polynomial interpolation, we cannot first compute the possible number of different symmetric points such as four-fold or six-fold symmetric points, as the authors did in [2]. It is because the total number of vertices for any order partition is not a fixed number, which is a direct consequence of the fact that a face can have any number (≥ 3) of vertices. For a similar reason, the total number of faces is also not a fixed number for a given order partition of the tetrahedron. So in the following, we will try to minimize, besides the Lebesgue constant, the number of faces as that is proportional to the work load of the spectral volume method [11, 10].

2.1 The First Order Partition

Our first order partition is the same as that given in [7]. But we describe our algorithm within a more general setting so that the algorithm can be used to generate higher order partitions.

Follow the idea of [3], we build the partition from the polynomial interpolation points on the tetrahedron in a certain way such that each sub-cell contains an input point. For the first order partition, the input points are simply the vertices of the tetrahedron. So a sub-cell is needed for each vertex of the tetrahedron. The four sub-cells can be computed in the exactly same way because of symmetry. Hence we only describe how to construct the sub-cell for one vertex. Figure 1 shows the sub-cell for vertex A in tetrahedron $ABCD$ (point D is behind the scene). This sub-cell consists of

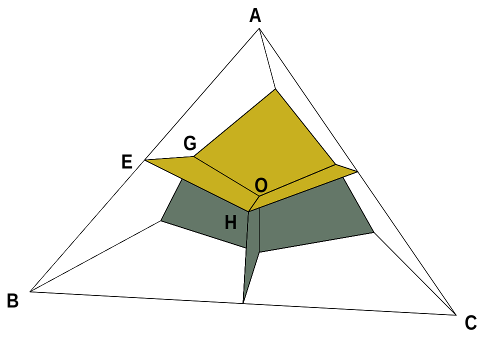

Figure 1: First order partition of the tetrahedron.

three interior 'faces' and three faces which are on the faces of the tetrahedron. Again by symmetry, the three interior faces can be constructed in a similar way. One such interior face is $EHOG$, in which O is inside the tetrahedron, E is on the edge AB, and G and H are on the face ABD and ABC respectively. In this face, we require that point G has the same barycenter coordinates in ΔABD as the point H in ΔABC. Moreover, as shown in Fig. 1, the three remaining faces will be fixed after building the interior faces. So it suffices to specify how to choose points O, E and G, in order to compute the sub-cell for vertex A, thus the whole first order partition. In the following, bold symbols represent the coordinates of points in column vector form.

Note that the points E, H, O and G are not always coplanar. We choose the points O, E and G with form

$$\begin{cases} \mathbf{O} &= r\ \mathbf{B} + r\ \mathbf{C} + r\ \mathbf{D} + (1-3r)\ \mathbf{A}, \\ \mathbf{E} &= s\ \mathbf{B} + (1-s)\ \mathbf{A}, \\ \mathbf{G} &= t\ \mathbf{B} + t\ \mathbf{D} + (1-2t)\ \mathbf{A}, \end{cases} \tag{5}$$

where $0 < r, s, t < 1$. For the first order partition, we have $r = 1/4, s = 1/2$ and $t = 1/3$ according to the symmetry property. So O, E and G are the mass center of the tetrahedron $ABCD$, edge AB and face ABD respectively. The partition consists of four hexahedrons, each of which has six quadrilateral faces as shown in Fig. 1. (From **Lemma** 1, one can verify that the points O, G, E and H are coplanar.) The data set of the partition is given in Table 1-3. The Lebesgue constant for such partition is $95/26$ (see [7]).

Lemma 1 *The points O, G, E and H are coplanar if and only if* $(st + rt - 2rs) = 0$.

Proof: According to symmetry, $\mathbf{H} = t\ \mathbf{B} + t\ \mathbf{C} + (1-2t)\ \mathbf{A}$. So

$$\begin{pmatrix} \mathbf{O} & \mathbf{G} & \mathbf{E} & \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} 1-3r & 1-2t & 1-s & 1-2t \\ r & t & s & t \\ r & 0 & 0 & t \\ r & t & 0 & 0 \end{pmatrix}.$$

The points O, G, E and H are coplanar if and only if the above matrix is singular. The determinant of the above matrix is equal to $t(st + rt - 2rs)$, which finishes the proof. ■

Table 1: The first order partition of the tetrahedron: the barycenter coordinates of the vertices.

i				
1	0.0000000000	0.0000000000	1.0000000000	0.0000000000
2	0.0000000000	0.0000000000	0.5000000000	0.5000000000
3	0.0000000000	0.5000000000	0.5000000000	0.0000000000
4	0.0000000000	0.3333333333	0.3333333333	0.3333333333
5	0.0000000000	1.0000000000	0.0000000000	0.0000000000
6	0.0000000000	0.5000000000	0.0000000000	0.5000000000
7	0.0000000000	0.0000000000	0.0000000000	1.0000000000
8	0.2500000000	0.2500000000	0.2500000000	0.2500000000
9	0.3333333333	0.3333333333	0.3333333333	0.0000000000
10	0.3333333333	0.0000000000	0.3333333333	0.3333333333
11	0.3333333333	0.3333333333	0.0000000000	0.3333333333
12	0.5000000000	0.5000000000	0.0000000000	0.0000000000
13	0.5000000000	0.0000000000	0.5000000000	0.0000000000
14	0.5000000000	0.0000000000	0.0000000000	0.5000000000
15	1.0000000000	0.0000000000	0.0000000000	0.0000000000

2.2 The Second Order Partition

Second-order polynomial interpolation on the tetrahedron needs ten interpolation points. We choose four tetrahedron vertices and six tetrahedron edge middle points as the interpolation points (see Fig. 2). For the four tetrahedron vertices, we construct a sub-cell for each of them as we did for the first order partition in the last section. For example, when building the sub-cell for vertex A, we treat A as a vertex of a small tetrahedron $AIJH$ (Fig. 2), and employ the algorithm described

Table 2: The first order partition of the tetrahedron: the faces. Each row lists the indices of vertices for one face. M denotes the number of vertices of each individual face.

i	M				
1	4	9	13	15	12
2	4	10	14	15	13
3	4	11	14	15	12
4	4	8	11	14	10
5	4	8	11	12	9
6	4	8	10	13	9
7	4	1	13	10	2
8	4	1	13	9	3
9	4	1	3	4	2
10	4	3	9	8	4
11	4	2	10	8	4
12	4	3	5	6	4
13	4	5	12	11	6
14	4	3	9	12	5
15	4	4	8	11	6
16	4	6	11	14	7
17	4	2	7	6	4
18	4	2	10	14	7

Table 3: The first order partition of the tetrahedron: the sub-cells. Each row lists the indices of faces of one sub-cell. M denotes the number of faces of each individual sub-cell.

i	M						
1	6	1	2	3	4	5	6
2	6	7	8	9	10	11	6
3	6	12	13	14	5	10	15
4	6	16	17	18	11	4	15

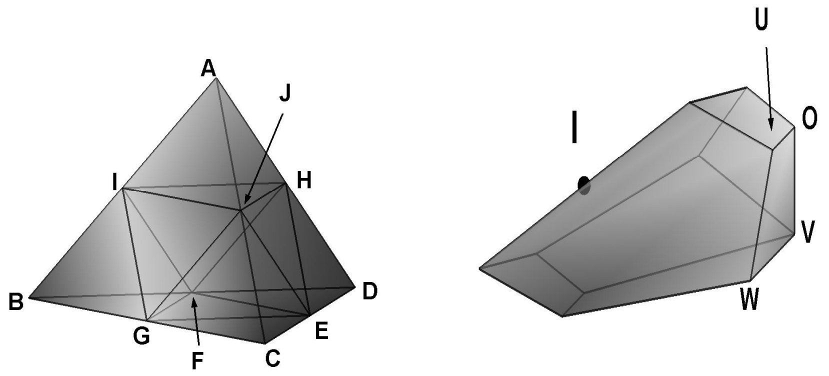

Figure 2: The second order partition. Left: a tetrahedron with input points; Right: the sub-cell that includes point I.

in the last section to construct the sub-cell containing A. Note that the three parameters r, s and t are not constant this time. We consider only the parameters which ensure that each of these four sub-cells has six quadrilateral faces (i.e., condition of **Lemma** 1 is satisfied). In Sec. 3.3, we will vary r, s, t with the above constraint to minimize the Lebesgue constant of the partition.

For the six remaining points, it is enough to explain how to compute the sub-cell for one point (e.g., point I) as we will build the other five sub-cells in a symmetric way. In our second order partition, the sub-cell (Fig. 2) for point I has eight faces: two quadrilateral faces, two pentagon faces, and four more symmetric quadrilateral 'faces'. Among those faces, the two quadrilateral faces also separately belong to the sub-cell including point A and B. The two pentagon faces are on the tetrahedron face ABC and ABD respectively, and they will be fixed after building the four symmetric faces. In Fig. 2, we mark one of these four faces as $UOVW$. In this face, the point U is also on the triangle face AIJ, and O is inside the small tetrahedron $AIJH$. In fact, these two points have already been specified when building the sub-cell for vertex A. So the only remaining work is to choose points W and V. We put point W inside the triangle IJG and V inside the octahedron $IJHEFG$. In particular, we choose

$$\begin{cases} \mathbf{W} & = \frac{\mathbf{I}+\mathbf{J}+\mathbf{G}}{3}, \\ \mathbf{V} & = \frac{\mathbf{I}+\mathbf{J}+\mathbf{G}+\mathbf{E}+\mathbf{H}+\mathbf{F}}{6}, \\ \mathbf{O} & = r\,\mathbf{I} + r\,\mathbf{J} + r\,\mathbf{H} + (1-3r)\,\mathbf{A}, \\ \mathbf{U} & = t\,\mathbf{I} + t\,\mathbf{J} + (1-2t)\,\mathbf{A}, \end{cases} \tag{6}$$

where the specification of O and U are also included for completeness. These four points always form a quadrilateral according to the following lemma.

Lemma 2 *The four points, W, V, O, and U as defined in* (6)*, are always coplanar.*

Proof: According to the distribution of the input points, there exists λ, η such that

$$\begin{cases} \mathbf{I} &= \lambda\ \mathbf{A} + (1-\lambda)\ \mathbf{B}, \qquad\qquad \mathbf{G} = \eta\ (\mathbf{B}+\mathbf{C}) + (1-2\eta)\mathbf{A}, \\ \mathbf{J} &= \lambda\ \mathbf{A} + (1-\lambda)\ \mathbf{C}, \qquad\qquad \mathbf{E} = \eta\ (\mathbf{C}+\mathbf{D}) + (1-2\eta)\mathbf{A}, \\ \mathbf{H} &= \lambda\ \mathbf{A} + (1-\lambda)\ \mathbf{D}, \qquad\qquad \mathbf{F} = \eta\ (\mathbf{B}+\mathbf{D}) + (1-2\eta)\mathbf{A}. \end{cases} \tag{7}$$

Substitute the above formula into (6) to obtain

$$\begin{pmatrix}\mathbf{U} & \mathbf{O} & \mathbf{W} & \mathbf{V}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D}\end{pmatrix} \begin{pmatrix} 1-2t+2t\lambda & 1-3r+3r\lambda & \frac{1-2\eta+2\lambda}{3} & \frac{3+3\lambda-6\eta}{6} \\ t(1-\lambda) & r(1-\lambda) & \frac{1+\eta-\lambda}{3} & \frac{1-\lambda+2\eta}{6} \\ t(1-\lambda) & r(1-\lambda) & \frac{1+\eta-\lambda}{3} & \frac{1-\lambda+2\eta}{6} \\ 0 & r(1-\lambda) & 0 & \frac{1-\lambda+2\eta}{6} \end{pmatrix}.$$

The matrix in the above is singular, which proves the lemma. ■

Remark 1 *For the points which are used to compute the second order partition in this section, one have $\lambda = 1/2, \eta = 1/3$. These special values for λ and η are not used in the proof. So the lemma also applies to certain faces of the third and fourth-order partitions which are generated with the algorithm in this section.*

2.3 The Third Order Partition

To compute the third order partition, we choose the interpolation points from [6]. All these points are on the surface of the tetrahedron. In the upper left part of Fig. 3, we plot the points which are on one tetrahedron face. Using the algorithm to compute the second order partition, a sub-cell can be constructed for all the points shown in the picture except P. For example, when building sub-cells for the three points on the top of the picture, we ignore all the interpolation points on the lowest layer (i.e., the ten points on the bottom tetrahedron face). Then the number of remaining points will be ten, which is exactly the number of second order interpolation points. Treat these ten points as the input points and use the algorithm in Sec. 2.2 to construct the sub-cells for the three top points shown in the picture. Similar procedure can be done for all other points except P.

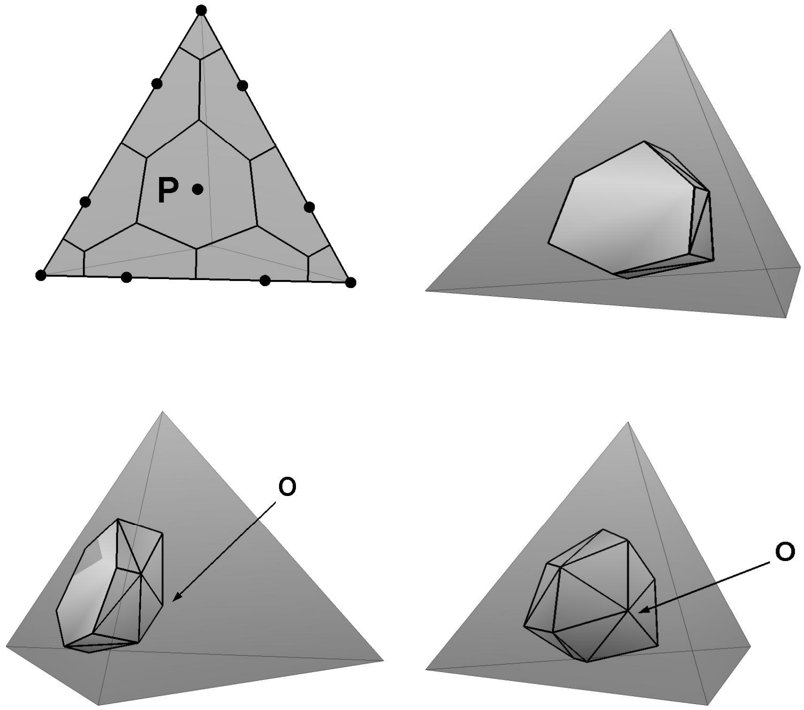

Figure 3: The third order partition of the tetrahedron: a sub-cell.

So the only additional work is to build a sub-cell containing point P for each tetrahedron face. We compute these four sub-cells as follows. Observing that a single connected volume will be left after excluding those sub-cells which have already been built, we choose the mass center of the original tetrahedron as a new vertex, and then connect this new vertex to certain vertices of the single volume to divide it into four identical sub-cells. One of such sub-cell is shown in Fig. 3. In the upper right, it shows a view from the side with point P. In the lower left, it is a view after rotating the tetrahedron from right to left for a small angle, where point O is the mass center of the tetrahedron. Finally in the lower right, we rotate it a little more to show the sub-cell from the opposite side. This sub-cell has one hexagonal face and 18 triangular faces. Till now we have computed a third order partition of the tetrahedron. In Sec. 3.3, we will optimize the partition by moving the input points around.

2.4 The Fourth Order Partition

The fourth order interpolation points set [6] has 35 points, only one of which is inside the tetrahedron. Figure 4 shows the points which are on one face of the tetrahedron. For each surface point, a sub-cell can be constructed with the algorithm for the third-order partition. For example, for the six points shown in the top left of Fig. 4, we can simply apply the algorithm for the third-order partition on the points set which includes all the original interpolation points except those on the bottom tetrahedron face. So it seems that we only need a new technique to construct a sub-cell for the interior point. However, even this new technique is not necessary because after building all the other sub-cells, there will be left a single connected volume which is just the sub-cell for the interior point. Figure 4 shows this sub-cell from different perspective. On the top right, it is a view from the reader's side. The bottom left shows a view after rotating the tetrahedron from right to left for a certain angle. The bottom right is another view after further rotation. This sub-cell has 24 triangular faces. The data for the optimized version of the partition will be given in Sec. 3.3.

Remark 2 *The recursive algorithm can be used to compute higher order partitions of the tetrahedron.*

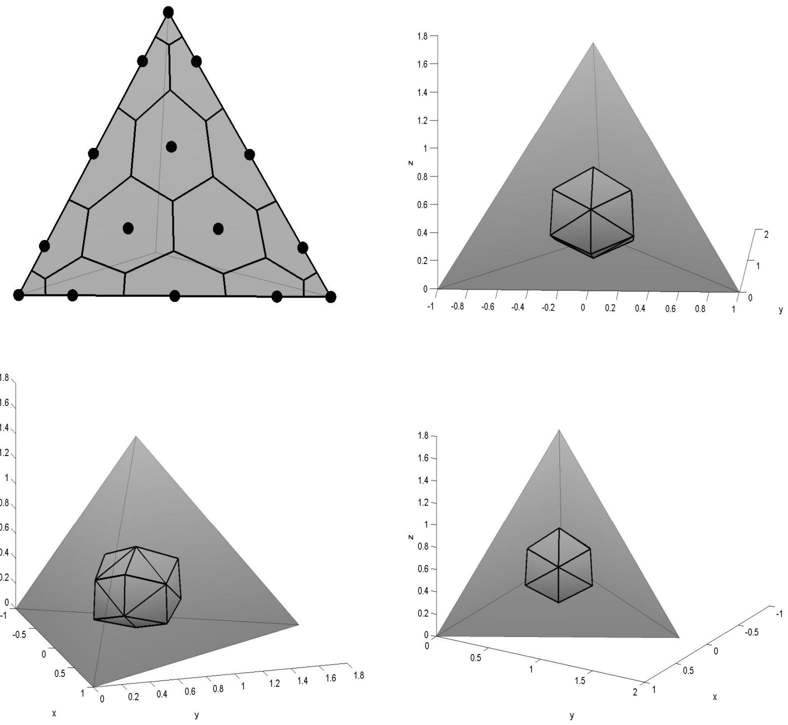

Figure 4: The interior sub-cell of the fourth order partition of the tetrahedron.

3 Partition Optimization

In this section, we optimize the partitions given in [3] and the three-dimensional partitions computed in Sec.2.

3.1 One Dimensional Partitions

In [3], we developed several one-dimensional partitions from the Chebyshev and Legendre Gauss-Lobatto points by using middle points of two neighboring input points as partition vertices. However, those partitions have larger Lebesgue constants than the partitions given in [9].

So in this paper, instead of simply using middle points of two neighboring input points as endpoints of sub-cells, we construct a partition in which most input points sit at the center of the sub-cells and the sub-cells have increasing size from the boundary to the interior. In specific, consider an even order partition with input points $-1 = x_0 < x_1 < \cdots < x_{2n} = 1$. Take $d_i = x_i - x_{i-1}, i = 1, \cdots, 2n$. Assume r be the portion of interval $[x_{n-1}, x_n]$ that is allocated to the sub-cell including x_n. So the $(1-r)d_n$ of $[x_{n-1}, x_n]$ is allocated to the sub-cell x_{n-1}, i.e., $y_n = x_{n-1} + (1-r)d_n$ is a vertex of the partition. In order to make x_{n-1} be the center of a sub-cell, $y_{n-1} = x_{n-1} - (1-r)d_n$ should also be a vertex of the partition. Hence $(1-r)d_n$ of the interval $[x_{n-2}, x_{n-1}]$ is allocated to the sub-cell including x_{n-1}. Since we require the sub-cells closer to the middle have larger size, one can have

$$\begin{cases} rd_n \geq (1-r)d_n \\ (1-r)d_n \geq \frac{d_{n-1}}{2} \end{cases} \qquad \Longrightarrow \qquad 1 - \frac{d_{n-1}}{2d_n} \geq r \geq \frac{1}{2}.$$

Continue the above process until we get y_0 such that x_0 is the middle point of $[y_0, y_1]$. Repeat it for the other half interpolation points, $\{x_{n+1}, \cdots, x_{2n}\}$, to get the whole partition. Note that the partition will cover a region larger than the original domain $[-1, 1]$. A partition of $[-1, 1]$ can be easily obtained by a linear mapping. Similar procedure can be done for odd order partitions.

In the algorithm, r is the only parameter to be optimized. With the above algorithm, we compute the optimal partitions from the Legendre-Gauss-Lobatto, Chebyshev-Gauss-Lobatto points, and the optimal polynomial interpolation set [1]. These optimized partitions are denoted as Π_{LGL},

Π_{CGL}, and Π_{T_1} respectively, among which Π_{LGL} is the smallest for partitions of most order. The results are shown in Tab. 4-5.

Table 4: Lebesgue constants for several one-dimensional partitions.

order	Π_{LGL}	r	Π_{CGL}	r	Π_{T_1}	r
2	1.685	0.877	1.685	0.877	1.685	0.877
3	1.823	0.884	1.823	0.872	1.823	0.890
4	1.950	0.533	2.132	0.656	2.674	0.500
5	2.108	0.562	2.416	0.542	2.337	0.500
6	2.291	0.542	2.560	0.500	2.241	0.593
7	2.449	0.554	2.666	0.500	2.387	0.602
8	2.545	0.514	2.834	0.571	3.898	0.500
9	2.644	0.521	3.015	0.518	3.514	0.500
10	2.748	0.512	3.105	0.500	3.190	0.541
11	2.840	0.519	3.180	0.500	2.920	0.552
12	2.918	0.509	3.268	0.534	4.711	0.500
13	2.991	0.511	3.384	0.509	4.343	0.500
14	3.063	0.504	3.450	0.500	4.020	0.522
15	3.124	0.507	3.509	0.500	3.738	0.529
16	3.190	0.507	3.571	0.519	5.312	0.500
17	3.248	0.507	3.653	0.505	4.969	0.500
18	3.303	0.500	3.705	0.500	4.661	0.513
19	3.354	0.505	3.753	0.500	4.384	0.519
20	3.404	0.507	3.801	0.512		

Table 5: Node sets of the partitions Π_{LGL}. Only positive interior points are listed.

order	x_i
2	0.7812757765
3	0.8797040574
4	0.3358143267
	0.9232452092
5	0.5414344225
	0.9518023783
6	0.2485522653
	0.6680769262
	0.9550635604
7	0.4131856053
	0.7482430030
	0.9628667435
8	0.1837943613
	0.5315587191
	0.8025200275

Table 5 (Contd.)

order	x_i
	0.9700321565
9	0.3247408153
	0.6207881103
	0.8408055127
	0.9784046134
10	0.1498237400
	0.4354893164
	0.6831261231
	0.8693871112
	0.9790260552

3.2 Two Dimensional Partitions

We only optimize the partitions of up to the fourth order in the triangle which are given in [3]. Different constraint will be applied when optimizing partitions of different order.

3.2.1 The Second Order Partition of the Triangle

Figure 5 shows the second order partition from [3]. We seek an optimal partition whose vertices on each triangle edge are just the vertices of the one-dimensional second-order partition computed in Sec. 3.1. So only three vertices of the partition, points E, F, G, are not fixed. Because of symmetry, points E, F, G are determined by a single parameter. Specially, their barycenter coordinates can be expressed as $(r, r, 1-2r), (r, 1-2r, r), (1-2r, r, r)$. Under the above constraint, the optimal partition (Fig. 6) has Lebesgue constant 3.0630, and the partition data is listed in Tab. 6-7. We also tried to move around the partition vertices which are on the triangle edges. There is little difference in the optimal Lebesgue constant.

Table 6: The second order partition of the triangle: the barycenter coordinates of the vertices.

i			
1	1.0000000000	0.0000000000	0.0000000000
2	0.8906378883	0.1093621117	0.0000000000
3	0.8269977508	0.0865011246	0.0865011246
4	0.8906378883	0.0000000000	0.1093621117
5	0.1093621117	0.8906378883	0.0000000000
6	0.0865011246	0.8269977508	0.0865011246
7	0.3333333334	0.3333333334	0.3333333331

Table 6 (Contd.)

i			
8	0.0865011246	0.0865011246	0.8269977508
9	0.1093621117	0.0000000000	0.8906378883
10	0.0000000000	1.0000000000	0.0000000000
11	0.0000000000	0.8906378883	0.1093621117
12	0.0000000000	0.1093621117	0.8906378883
13	0.0000000000	0.0000000000	1.0000000000

Table 7: The second order partition of the triangle: the sub-cells. Each row lists the indices of vertices in counterclockwise direction for one sub-cell. M denotes the number of vertices of each individual face.

i	M					
1	4	9	8	12	13	
2	5	4	3	7	8	9
3	5	8	7	6	11	12
4	4	1	2	3	4	
5	5	3	2	5	6	7
6	4	6	5	10	11	

3.2.2 The Third Order Partition of the Triangle

In [3], we propose a technique to compute the partitions from the layering structure of the input points. In that algorithm, we first generate a triangularization from the input points. The centroids of the triangles in the triangularization are then used as the vertices of the partition. (See [3] for further details.) Here we apply the same algorithm to compute the third order partition except that a different point instead of the centroid in each triangle is chosen to be a vertex of the partition. In particular, we choose a weighted average of the triangle vertices, with the weight being the largest barycenter coordinate of each triangle vertex. Within the above algorithm, we move around the input points while keeping its symmetry property to obtain a partition with the smallest Lebesgue constant. The optimal partition obtained this way is shown in Fig. 6, and its Lebesgue constant is 3.2129. The partition data is given in Tab.8-9. We also tried to compute the optimal partition under the constraint similar to Sec. 3.2.1. The result is not as good as the one we give here.

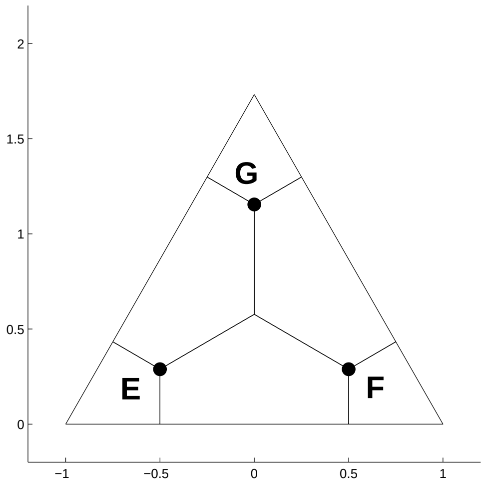

Figure 5: The second order partition of the triangle.

Table 8: The third order partition of the triangle: the barycenter coordinates of the vertices.

i			
1	0.9574919119	0.0212540441	0.0212540441
2	0.4747800545	0.4747800545	0.0504398910
3	0.8437475098	0.0781262451	0.0781262451
4	0.4747800545	0.0504398911	0.4747800544
5	0.0212540441	0.9574919119	0.0212540441
6	0.0781262451	0.8437475098	0.0781262451
7	0.0504398911	0.4747800545	0.4747800544
8	0.0781262451	0.0781262451	0.8437475097
9	0.0212540441	0.0212540441	0.9574919119
10	1.0000000000	0.0000000000	0.0000000000
11	0.0000000000	0.0000000000	1.0000000000
12	0.0000000000	1.0000000000	0.0000000000
13	0.9673771699	0.0326228301	0.0000000000
14	0.5000000000	0.5000000000	0.0000000000
15	0.0326228301	0.9673771699	0.0000000000
16	0.9673771699	0.0000000000	0.0326228301
17	0.5000000000	0.0000000000	0.5000000000
18	0.0326228301	0.0000000000	0.9673771699
19	0.0000000000	0.9673771699	0.0326228301
20	0.0000000000	0.5000000000	0.5000000000
21	0.0000000000	0.0326228301	0.9673771699

Table 9: The third order partition of the triangle: the sub-cells.

i	M						
1	4	10	13	1	16		
2	5	1	13	14	2	3	
3	5	16	1	3	4	17	
4	5	2	14	15	5	6	
5	6	4	3	2	6	7	8
6	5	17	4	8	9	18	

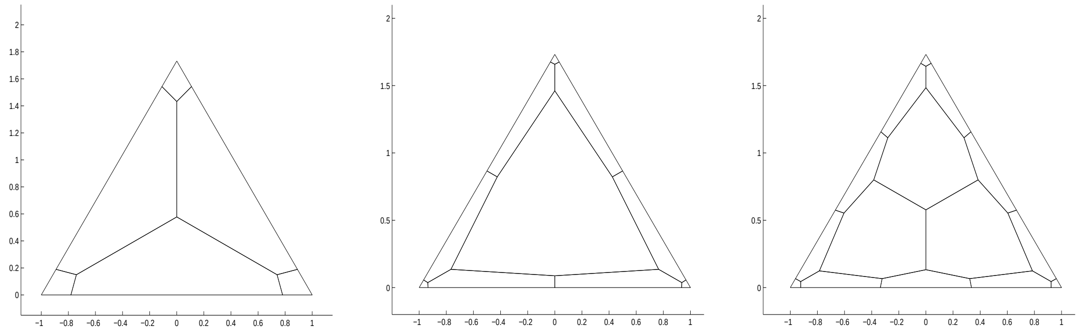

Figure 6: Several two-dimensional partitions. Left: second order; Middle: third order; Right: fourth order.

partition vertices to get a small Lebesgue constant. The partition shown in Fig. 6 has Lebesgue constant 4.0563. Note that this partition is not optimal. But when perturbing the partition vertices, only a few percent change is observed in the Lebesgue constant. The partition data is given in Tab. 10-11.

Table 10: The fourth order partition of the triangle: the barycenter coordinates of the vertices.

i			
1	1.0000000000	0.0000000000	0.0000000000
2	0.9616226046	0.0383773954	0.0000000000
3	0.9488301395	0.0255849303	0.0255849303
4	0.9616226046	0.0000000000	0.0383773954
5	0.6679071634	0.3320928367	0.0000000000
6	0.6422746261	0.3193479785	0.0383773954
7	0.8568233902	0.0715883049	0.0715883049
8	0.6422746261	0.0383773954	0.3193479785
9	0.6679071634	0.0000000000	0.3320928367
10	0.3320928367	0.6679071634	0.0000000000
11	0.3193479785	0.6422746261	0.0383773954
12	0.4616226046	0.4616226046	0.0767547908
13	0.3333333334	0.3333333334	0.3333333331
14	0.4616226046	0.0767547908	0.4616226046
15	0.3193479785	0.0383773954	0.6422746261
16	0.3320928367	0.0000000000	0.6679071634
17	0.0383773954	0.9616226046	0.0000000000
18	0.0255849303	0.9488301395	0.0255849303
19	0.0715883049	0.8568233902	0.0715883049
20	0.0383773954	0.6422746261	0.3193479785
21	0.0767547908	0.4616226046	0.4616226046
22	0.0383773954	0.3193479785	0.6422746261
23	0.0715883049	0.0715883049	0.8568233902

Table 10 (Contd.)

i			
24	0.0255849303	0.0255849303	0.9488301395
25	0.0383773954	0.0000000000	0.9616226046
26	0.0000000000	1.0000000000	0.0000000000
27	0.0000000000	0.9616226046	0.0383773954
28	0.0000000000	0.6679071634	0.3320928367
29	0.0000000000	0.3320928367	0.6679071634
30	0.0000000000	0.0383773954	0.9616226046
31	0.0000000000	0.0000000000	1.0000000000

Table 11: The fourth order partition of the triangle: the sub-cells.

i	M						
1	4	25	24	30	31		
2	5	16	15	23	24	25	
3	5	24	23	22	29	30	
4	5	9	8	14	15	16	
5	6	14	13	21	22	23	15
6	5	22	21	20	28	29	
7	5	4	3	7	8	9	
8	6	7	6	12	13	14	8
9	6	13	12	11	19	20	21
10	5	20	19	18	27	28	
11	4	1	2	3	4		
12	5	2	5	6	7	3	
13	5	6	5	10	11	12	
14	5	10	17	18	19	11	
15	4	18	17	26	27		

3.3 Three Dimensional Partitions

For the second order partition, we use the optimized second order partition of the triangle computed in Sec. 3.2.1 on the faces of the tetrahedron. In order to minimize the number of total faces, the partition vertices inside the tetrahedron are chosen such that the condition of **Lemma** 1 is satisfied. Then the partition is actually fixed, which is shown in Fig. 7. The partition has 37 vertices and 48 faces, and its Lebesgue constant is 5.0814. The partition data is listed in Tab. 12-14.

For the third and fourth order partitions, we simple employ the algorithm given in Sec. 2.3 and 2.4 to compute the partition, and try to find the one with minimal Lebesgue constant by moving

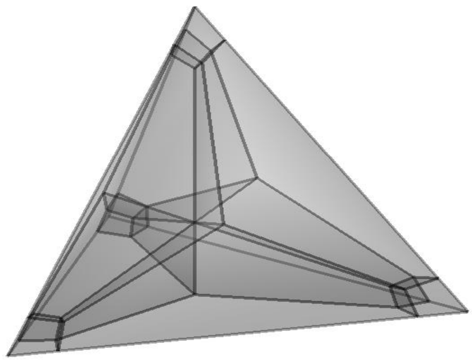

Figure 7: Second order partition of tetrahedron.

the input points around. The smallest Lebesgue constants we found for the third and fourth order partitions are 6.8725 and 7.9940 respectively. The obtained third order partition has 73 vertices and 130 faces, while the fourth order partition has 126 vertices and 252 faces. The partition data is listed in Tab. 15-20. Note that this fourth order partition has some non-convex sub-cells.

Table 12: The second order partition of the tetrahedron: the barycenter coordinates of the vertices.

i				
1	0.0000000000	1.0000000000	0.0000000000	0.0000000000
2	0.0000000000	0.8906378883	0.1093621117	0.0000000000
3	0.0000000000	0.8906378883	0.0000000000	0.1093621117
4	0.0000000000	0.8269977508	0.0865011246	0.0865011246
5	0.0000000000	0.0000000000	1.0000000000	0.0000000000
6	0.0000000000	0.1093621117	0.8906378883	0.0000000000
7	0.0000000000	0.0000000000	0.8906378883	0.1093621117
8	0.0000000000	0.0865011246	0.8269977508	0.0865011246
9	0.0000000000	0.0000000000	0.0000000000	1.0000000000
10	0.0000000000	0.1093621117	0.0000000000	0.8906378883
11	0.0000000000	0.0000000000	0.1093621117	0.8906378883
12	0.0000000000	0.0865011246	0.0865011246	0.8269977508
13	0.0000000000	0.3333333334	0.3333333334	0.3333333331
14	0.0715453331	0.7853640006	0.0715453331	0.0715453331
15	0.0715453331	0.0715453331	0.7853640006	0.0715453331
16	0.0715453331	0.0715453331	0.0715453331	0.7853640006
17	0.0865011246	0.8269977508	0.0865011246	0.0000000000
18	0.0865011246	0.8269977508	0.0000000000	0.0865011246
19	0.0865011246	0.0865011246	0.8269977508	0.0000000000

Table 12 (Contd.)

i				
20	0.0865011246	0.0000000000	0.8269977508	0.0865011246
21	0.0865011246	0.0000000000	0.0865011246	0.8269977508
22	0.0865011246	0.0865011246	0.0000000000	0.8269977508
23	0.1093621117	0.8906378883	0.0000000000	0.0000000000
24	0.1093621117	0.0000000000	0.8906378883	0.0000000000
25	0.1093621117	0.0000000000	0.0000000000	0.8906378883
26	0.2500000000	0.2500000000	0.2500000000	0.2500000000
27	0.3333333334	0.3333333334	0.3333333331	0.0000000000
28	0.3333333334	0.3333333334	0.0000000000	0.3333333331
29	0.3333333334	0.0000000000	0.3333333334	0.3333333331
30	0.7853640006	0.0715453331	0.0715453331	0.0715453331
31	0.8269977508	0.0865011246	0.0865011246	0.0000000000
32	0.8269977508	0.0000000000	0.0865011246	0.0865011246
33	0.8269977508	0.0865011246	0.0000000000	0.0865011246
34	0.8906378883	0.1093621117	0.0000000000	0.0000000000
35	0.8906378883	0.0000000000	0.1093621117	0.0000000000
36	0.8906378883	0.0000000000	0.0000000000	0.1093621117
37	1.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 13: The second order partition of the tetrahedron: the faces. Each row lists the indices of vertices in counterclockwise order for one face. M denotes the number of vertices of each individual face.

i	M				
1	4	31	35	37	34
2	4	32	36	37	35
3	4	33	36	37	34
4	4	30	33	36	32
5	4	30	33	34	31
6	4	30	32	35	31
7	4	1	23	17	2
8	4	1	3	4	2
9	4	1	23	18	3
10	4	3	18	14	4
11	4	14	18	23	17
12	4	2	17	14	4
13	4	5	24	19	6
14	4	5	24	20	7
15	4	5	7	8	6
16	4	7	20	15	8
17	4	6	19	15	8
18	4	15	20	24	19
19	4	9	11	12	10

Table 13 (Contd.)

i	M					
20	4	9	25	21	11	
21	4	9	25	22	10	
22	4	16	22	25	21	
23	4	10	22	16	12	
24	4	11	21	16	12	
25	5	17	27	31	34	23
26	5	18	28	33	34	23
27	4	26	30	31	27	
28	4	26	30	33	28	
29	4	14	26	27	17	
30	4	14	26	28	18	
31	5	19	27	31	35	24
32	5	20	29	32	35	24
33	4	26	30	32	29	
34	4	15	26	27	19	
35	4	15	26	29	20	
36	5	22	28	33	36	25
37	5	21	29	32	36	25
38	4	16	26	28	22	
39	4	16	26	29	21	
40	5	2	6	8	13	4
41	5	2	17	27	19	6
42	4	4	14	26	13	
43	4	8	15	26	13	
44	5	3	10	12	13	4
45	5	3	18	28	22	10
46	4	12	16	26	13	
47	5	7	11	12	13	8
48	5	7	20	29	21	11

Table 14: The second order partition of the tetrahedron: the sub-cells. Each row lists the indices of the faces of one sub-cell. M denotes the number of faces of each individual sub-cell.

i	M								
1	6	1	2	3	4	5	6		
2	6	7	8	9	10	11	12		
3	6	13	14	15	16	17	18		
4	6	19	20	21	22	23	24		
5	8	25	26	5	11	27	28	29	30
6	8	31	32	6	18	27	33	34	35
7	8	36	37	4	22	28	33	38	39
8	8	40	41	12	17	42	29	43	34
9	8	44	45	10	23	42	30	46	38

Table 14 (Contd.)

i	M								
10	8	47	48	16	24	43	35	46	39

Table 15: The third order partition of the tetrahedron: the barycenter coordinates of the vertices.

i				
1	0.0000000000	1.0000000000	0.0000000000	0.0000000000
2	0.0000000000	0.9750000000	0.0250000000	0.0000000000
3	0.0000000000	0.9750000000	0.0000000000	0.0250000000
4	0.0000000000	0.9666666667	0.0166666667	0.0166666667
5	0.0000000000	0.0000000000	1.0000000000	0.0000000000
6	0.0000000000	0.0000000000	0.9750000000	0.0250000000
7	0.0000000000	0.0250000000	0.9750000000	0.0000000000
8	0.0000000000	0.0166666667	0.9666666667	0.0166666667
9	0.0000000000	0.0000000000	0.0000000000	1.0000000000
10	0.0000000000	0.0250000000	0.0000000000	0.9750000000
11	0.0000000000	0.0000000000	0.0250000000	0.9750000000
12	0.0000000000	0.0166666667	0.0166666667	0.9666666667
13	0.0000000000	0.5000000000	0.5000000000	0.0000000000
14	0.0000000000	0.4444444444	0.4444444444	0.1111111111
15	0.0000000000	0.7444444444	0.1277777778	0.1277777778
16	0.0000000000	0.0000000000	0.5000000000	0.5000000000
17	0.0000000000	0.1111111111	0.4444444444	0.4444444444
18	0.0000000000	0.1277777778	0.7444444444	0.1277777778
19	0.0000000000	0.5000000000	0.0000000000	0.5000000000
20	0.0000000000	0.4444444444	0.1111111111	0.4444444444
21	0.0000000000	0.1277777778	0.1277777778	0.7444444444
22	0.0125000000	0.9625000000	0.0125000000	0.0125000000
23	0.0125000000	0.0125000000	0.9625000000	0.0125000000
24	0.0125000000	0.0125000000	0.0125000000	0.9625000000
25	0.0166666667	0.9666666667	0.0000000000	0.0166666667
26	0.0166666667	0.9666666667	0.0166666667	0.0000000000
27	0.0166666667	0.0000000000	0.9666666667	0.0166666667
28	0.0166666667	0.0166666667	0.9666666667	0.0000000000
29	0.0166666667	0.0166666667	0.0000000000	0.9666666667
30	0.0166666667	0.0000000000	0.0166666667	0.9666666667
31	0.0250000000	0.9750000000	0.0000000000	0.0000000000
32	0.0250000000	0.0000000000	0.9750000000	0.0000000000
33	0.0250000000	0.0000000000	0.0000000000	0.9750000000
34	0.0833333333	0.4166666667	0.4166666667	0.0833333333
35	0.0833333333	0.0833333333	0.4166666667	0.4166666667
36	0.0833333333	0.4166666667	0.0833333333	0.4166666667
37	0.1111111111	0.4444444444	0.4444444444	0.0000000000

Table 15 (Contd.)

i				
38	0.1111111111	0.0000000000	0.4444444444	0.4444444444
39	0.1111111111	0.4444444444	0.0000000000	0.4444444444
40	0.1194444444	0.6416666667	0.1194444444	0.1194444444
41	0.1194444444	0.1194444444	0.6416666667	0.1194444444
42	0.1194444444	0.1194444444	0.1194444444	0.6416666667
43	0.1277777778	0.7444444444	0.1277777778	0.0000000000
44	0.1277777778	0.0000000000	0.7444444444	0.1277777778
45	0.1277777778	0.1277777778	0.0000000000	0.7444444444
46	0.1277777778	0.0000000000	0.1277777778	0.7444444444
47	0.1277777778	0.7444444444	0.0000000000	0.1277777778
48	0.1277777778	0.1277777778	0.7444444444	0.0000000000
49	0.2500000000	0.2500000000	0.2500000000	0.2500000000
50	0.4166666667	0.4166666667	0.0833333333	0.0833333333
51	0.4166666667	0.0833333333	0.0833333333	0.4166666667
52	0.4166666667	0.0833333333	0.4166666667	0.0833333333
53	0.4444444444	0.4444444444	0.1111111111	0.0000000000
54	0.4444444444	0.4444444444	0.0000000000	0.1111111111
55	0.4444444444	0.1111111111	0.0000000000	0.4444444444
56	0.4444444444	0.0000000000	0.1111111111	0.4444444444
57	0.4444444444	0.0000000000	0.4444444444	0.1111111111
58	0.4444444444	0.1111111111	0.4444444444	0.0000000000
59	0.5000000000	0.5000000000	0.0000000000	0.0000000000
60	0.5000000000	0.0000000000	0.0000000000	0.5000000000
61	0.5000000000	0.0000000000	0.5000000000	0.0000000000
62	0.6416666667	0.1194444444	0.1194444444	0.1194444444
63	0.7444444444	0.1277777778	0.1277777778	0.0000000000
64	0.7444444444	0.1277777778	0.0000000000	0.1277777778
65	0.7444444444	0.0000000000	0.1277777778	0.1277777778
66	0.9625000000	0.0125000000	0.0125000000	0.0125000000
67	0.9666666667	0.0166666667	0.0166666667	0.0000000000
68	0.9666666667	0.0000000000	0.0166666667	0.0166666667
69	0.9666666667	0.0166666667	0.0000000000	0.0166666667
70	0.9750000000	0.0250000000	0.0000000000	0.0000000000
71	0.9750000000	0.0000000000	0.0250000000	0.0000000000
72	0.9750000000	0.0000000000	0.0000000000	0.0250000000
73	1.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 16: The third order partition of the tetrahedron: the faces.

i	M				
1	4	67	71	73	70
2	4	68	72	73	71

Table 16 (Contd.)

i	M					
3	4	69	72	73	70	
4	4	66	69	72	68	
5	4	66	69	70	67	
6	4	66	68	71	67	
7	4	1	3	4	2	
8	4	1	31	25	3	
9	4	1	31	26	2	
10	4	22	26	31	25	
11	4	2	26	22	4	
12	4	3	25	22	4	
13	4	5	32	27	6	
14	4	5	32	28	7	
15	4	5	7	8	6	
16	4	7	28	23	8	
17	4	6	27	23	8	
18	4	23	28	32	27	
19	4	9	33	29	10	
20	4	9	11	12	10	
21	4	9	33	30	11	
22	4	11	30	24	12	
23	4	24	30	33	29	
24	4	10	29	24	12	
25	5	53	63	67	70	59
26	5	54	64	69	70	59
27	4	50	54	59	53	
28	4	62	66	67	63	
29	4	62	66	69	64	
30	3	50	62	63		
31	3	50	62	64		
32	5	2	13	14	15	4
33	5	2	26	43	37	13
34	4	13	37	34	14	
35	4	4	22	40	15	
36	4	22	40	43	26	
37	3	14	34	40		
38	3	34	40	43		
39	5	6	27	44	38	16
40	5	6	16	17	18	8
41	4	16	38	35	17	
42	4	23	41	44	27	
43	4	8	23	41	18	
44	3	35	41	44		
45	3	17	35	41		

Table 16 (Contd.)

i	M					
46	5	29	45	55	60	33
47	5	30	46	56	60	33
48	4	51	56	60	55	
49	4	24	42	45	29	
50	4	24	42	46	30	
51	3	42	51	55		
52	3	42	51	56		
53	5	57	65	68	71	61
54	5	58	63	67	71	61
55	4	52	58	61	57	
56	4	62	66	68	65	
57	3	52	62	65		
58	3	52	62	63		
59	5	3	25	47	39	19
60	5	3	19	20	15	4
61	4	19	39	36	20	
62	4	22	40	47	25	
63	3	36	40	47		
64	3	15	40	36		
65	5	28	48	58	61	32
66	5	27	44	57	61	32
67	4	23	41	48	28	
68	3	41	52	58		
69	3	41	52	57		
70	5	10	19	20	21	12
71	5	10	29	45	39	19
72	4	12	24	42	21	
73	3	20	36	42		
74	3	36	42	45		
75	5	55	64	69	72	60
76	5	56	65	68	72	60
77	3	51	62	64		
78	3	51	62	65		
79	5	26	43	53	59	31
80	5	25	47	54	59	31
81	3	40	50	53		
82	3	40	50	54		
83	5	7	13	14	18	8
84	5	7	28	48	37	13
85	3	14	34	41		
86	3	34	41	48		
87	5	11	30	46	38	16
88	5	11	16	17	21	12

Table 16 (Contd.)

i	M						
89	3	35	42	46			
90	3	17	35	42			
91	6	37	48	58	63	53	43
92	3	49	62	50			
93	3	49	62	52			
94	3	40	50	49			
95	3	34	49	40			
96	3	34	49	41			
97	3	41	52	49			
98	6	14	18	17	21	20	15
99	3	36	49	40			
100	3	35	49	41			
101	3	35	49	42			
102	3	36	49	42			
103	6	38	46	56	65	57	44
104	3	42	51	49			
105	3	49	62	51			
106	6	39	47	54	64	55	45
107	3	63	53	50			
108	3	64	54	50			
109	3	40	15	14			
110	3	43	37	34			
111	3	44	38	35			
112	3	41	18	17			
113	3	55	45	42			
114	3	56	46	42			
115	3	65	57	52			
116	3	63	58	52			
117	3	47	39	36			
118	3	36	20	15			
119	3	58	48	41			
120	3	57	44	41			
121	3	42	21	20			
122	3	45	39	36			
123	3	64	55	51			
124	3	65	56	51			
125	3	53	43	40			
126	3	54	47	40			
127	3	41	18	14			
128	3	48	37	34			
129	3	46	38	35			
130	3	42	21	17			

Table 17: The third order partition of the tetrahedron: the sub-cells.

i	M												
1	6	1	2	3	4	5	6						
2	6	7	8	9	10	11	12						
3	6	13	14	15	16	17	18						
4	6	19	20	21	22	23	24						
5	10	25	26	5	27	28	29	30	31	107	108		
6	10	32	33	11	34	35	36	37	38	109	110		
7	10	39	40	17	41	42	43	44	45	111	112		
8	10	46	47	23	48	49	50	51	52	113	114		
9	10	53	54	6	55	56	28	57	58	115	116		
10	10	59	60	12	61	62	35	63	64	117	118		
11	10	65	66	18	55	67	42	68	69	119	120		
12	10	70	71	24	61	72	49	73	74	121	122		
13	10	75	76	4	48	29	56	77	78	123	124		
14	10	79	80	10	27	36	62	81	82	125	126		
15	10	83	84	16	34	43	67	85	86	127	128		
16	10	87	88	22	41	50	72	89	90	129	130		
17	19	91	30	58	38	81	86	68	92	93	94	95	96
		97	107	110	116	119	125	128					
18	19	98	37	64	45	85	90	73	95	99	96	100	101
		102	109	112	118	121	127	130					
19	19	103	44	69	52	89	78	57	100	97	101	104	105
		93	111	114	115	120	124	129					
20	19	106	51	74	31	77	82	63	104	102	105	92	94
		99	108	113	117	122	123	126					

Table 18: The fourth order partition of the tetrahedron: the barycenter coordinates of the vertices.

i				
1	0.0000000000	1.0000000000	0.0000000000	0.0000000000
2	0.0000000000	0.9400000000	0.0600000000	0.0000000000
3	0.0000000000	0.9400000000	0.0000000000	0.0600000000
4	0.0000000000	0.9200000000	0.0400000000	0.0400000000
5	0.0000000000	0.0000000000	1.0000000000	0.0000000000
6	0.0000000000	0.0000000000	0.9400000000	0.0600000000
7	0.0000000000	0.0600000000	0.9400000000	0.0000000000
8	0.0000000000	0.0400000000	0.9200000000	0.0400000000
9	0.0000000000	0.0000000000	0.0000000000	1.0000000000
10	0.0000000000	0.0600000000	0.0000000000	0.9400000000
11	0.0000000000	0.0000000000	0.0600000000	0.9400000000
12	0.0000000000	0.0400000000	0.0400000000	0.9200000000
13	0.0000000000	0.6900000000	0.3100000000	0.0000000000

Table 18 (Contd.)

i				
14	0.0000000000	0.7133333333	0.2466666667	0.0400000000
15	0.0000000000	0.8400000000	0.0800000000	0.0800000000
16	0.0000000000	0.0000000000	0.6900000000	0.3100000000
17	0.0000000000	0.0400000000	0.7133333333	0.2466666667
18	0.0000000000	0.0800000000	0.8400000000	0.0800000000
19	0.0000000000	0.6900000000	0.0000000000	0.3100000000
20	0.0000000000	0.7133333333	0.0400000000	0.2466666667
21	0.0000000000	0.3100000000	0.0000000000	0.6900000000
22	0.0000000000	0.2466666667	0.0400000000	0.7133333333
23	0.0000000000	0.0800000000	0.0800000000	0.8400000000
24	0.0000000000	0.3100000000	0.6900000000	0.0000000000
25	0.0000000000	0.2466666667	0.7133333333	0.0400000000
26	0.0000000000	0.0000000000	0.3100000000	0.6900000000
27	0.0000000000	0.0400000000	0.2466666667	0.7133333333
28	0.0000000000	0.4600000000	0.4600000000	0.0800000000
29	0.0000000000	0.0800000000	0.4600000000	0.4600000000
30	0.0000000000	0.4600000000	0.0800000000	0.4600000000
31	0.0000000000	0.3333333333	0.3333333333	0.3333333333
32	0.0300000000	0.9100000000	0.0300000000	0.0300000000
33	0.0300000000	0.0300000000	0.9100000000	0.0300000000
34	0.0300000000	0.0300000000	0.0300000000	0.9100000000
35	0.0300000000	0.7250000000	0.2150000000	0.0300000000
36	0.0300000000	0.0300000000	0.7250000000	0.2150000000
37	0.0300000000	0.7250000000	0.0300000000	0.2150000000
38	0.0300000000	0.2150000000	0.0300000000	0.7250000000
39	0.0300000000	0.2150000000	0.7250000000	0.0300000000
40	0.0300000000	0.0300000000	0.2150000000	0.7250000000
41	0.0400000000	0.9200000000	0.0000000000	0.0400000000
42	0.0400000000	0.9200000000	0.0400000000	0.0000000000
43	0.0400000000	0.0000000000	0.9200000000	0.0400000000
44	0.0400000000	0.0400000000	0.9200000000	0.0000000000
45	0.0400000000	0.0400000000	0.0000000000	0.9200000000
46	0.0400000000	0.0000000000	0.0400000000	0.9200000000
47	0.0400000000	0.7133333333	0.2466666667	0.0000000000
48	0.0400000000	0.0000000000	0.7133333333	0.2466666667
49	0.0400000000	0.7133333333	0.0000000000	0.2466666667
50	0.0400000000	0.2466666667	0.0000000000	0.7133333333
51	0.0400000000	0.2466666667	0.7133333333	0.0000000000
52	0.0400000000	0.0000000000	0.2466666667	0.7133333333
53	0.0600000000	0.9400000000	0.0000000000	0.0000000000
54	0.0600000000	0.0000000000	0.9400000000	0.0000000000
55	0.0600000000	0.0000000000	0.0000000000	0.9400000000
56	0.0600000000	0.8200000000	0.0600000000	0.0600000000

Table 18 (Contd.)

i				
57	0.0600000000	0.0600000000	0.8200000000	0.0600000000
58	0.0600000000	0.0600000000	0.0600000000	0.8200000000
59	0.0625000000	0.3125000000	0.3125000000	0.3125000000
60	0.0800000000	0.8400000000	0.0800000000	0.0000000000
61	0.0800000000	0.0000000000	0.8400000000	0.0800000000
62	0.0800000000	0.0800000000	0.0000000000	0.8400000000
63	0.0800000000	0.0000000000	0.0800000000	0.8400000000
64	0.0800000000	0.8400000000	0.0000000000	0.0800000000
65	0.0800000000	0.0800000000	0.8400000000	0.0000000000
66	0.0800000000	0.4600000000	0.4600000000	0.0000000000
67	0.0800000000	0.0000000000	0.4600000000	0.4600000000
68	0.0800000000	0.4600000000	0.0000000000	0.4600000000
69	0.0816666667	0.4183333333	0.4183333333	0.0816666667
70	0.0816666667	0.0816666667	0.4183333333	0.4183333333
71	0.0816666667	0.4183333333	0.0816666667	0.4183333333
72	0.1225000000	0.6325000000	0.1225000000	0.1225000000
73	0.1225000000	0.1225000000	0.6325000000	0.1225000000
74	0.1225000000	0.1225000000	0.1225000000	0.6325000000
75	0.2150000000	0.0300000000	0.0300000000	0.7250000000
76	0.2150000000	0.0300000000	0.7250000000	0.0300000000
77	0.2150000000	0.7250000000	0.0300000000	0.0300000000
78	0.2466666667	0.0400000000	0.0000000000	0.7133333333
79	0.2466666667	0.0000000000	0.0400000000	0.7133333333
80	0.2466666667	0.0400000000	0.7133333333	0.0000000000
81	0.2466666667	0.0000000000	0.7133333333	0.0400000000
82	0.2466666667	0.7133333333	0.0400000000	0.0000000000
83	0.2466666667	0.7133333333	0.0000000000	0.0400000000
84	0.3100000000	0.0000000000	0.0000000000	0.6900000000
85	0.3100000000	0.0000000000	0.6900000000	0.0000000000
86	0.3100000000	0.6900000000	0.0000000000	0.0000000000
87	0.3125000000	0.3125000000	0.3125000000	0.0625000000
88	0.3125000000	0.0625000000	0.3125000000	0.3125000000
89	0.3125000000	0.3125000000	0.0625000000	0.3125000000
90	0.3333333333	0.3333333333	0.3333333333	0.0000000000
91	0.3333333333	0.0000000000	0.3333333333	0.3333333333
92	0.3333333333	0.3333333333	0.0000000000	0.3333333333
93	0.4183333333	0.4183333333	0.0816666667	0.0816666667
94	0.4183333333	0.0816666667	0.0816666667	0.4183333333
95	0.4183333333	0.0816666667	0.4183333333	0.0816666667
96	0.4600000000	0.4600000000	0.0800000000	0.0000000000
97	0.4600000000	0.4600000000	0.0000000000	0.0800000000
98	0.4600000000	0.0800000000	0.0000000000	0.4600000000
99	0.4600000000	0.0000000000	0.0800000000	0.4600000000

Table 18 (Contd.)

i				
100	0.4600000000	0.0000000000	0.4600000000	0.0800000000
101	0.4600000000	0.0800000000	0.4600000000	0.0000000000
102	0.6325000000	0.1225000000	0.1225000000	0.1225000000
103	0.6900000000	0.3100000000	0.0000000000	0.0000000000
104	0.6900000000	0.0000000000	0.3100000000	0.0000000000
105	0.6900000000	0.0000000000	0.0000000000	0.3100000000
106	0.7133333333	0.2466666667	0.0400000000	0.0000000000
107	0.7133333333	0.2466666667	0.0000000000	0.0400000000
108	0.7133333333	0.0000000000	0.2466666667	0.0400000000
109	0.7133333333	0.0400000000	0.2466666667	0.0000000000
110	0.7133333333	0.0400000000	0.0000000000	0.2466666667
111	0.7133333333	0.0000000000	0.0400000000	0.2466666667
112	0.7250000000	0.2150000000	0.0300000000	0.0300000000
113	0.7250000000	0.0300000000	0.2150000000	0.0300000000
114	0.7250000000	0.0300000000	0.0300000000	0.2150000000
115	0.8200000000	0.0600000000	0.0600000000	0.0600000000
116	0.8400000000	0.0800000000	0.0800000000	0.0000000000
117	0.8400000000	0.0800000000	0.0000000000	0.0800000000
118	0.8400000000	0.0000000000	0.0800000000	0.0800000000
119	0.9100000000	0.0300000000	0.0300000000	0.0300000000
120	0.9200000000	0.0400000000	0.0400000000	0.0000000000
121	0.9200000000	0.0000000000	0.0400000000	0.0400000000
122	0.9200000000	0.0400000000	0.0000000000	0.0400000000
123	0.9400000000	0.0600000000	0.0000000000	0.0000000000
124	0.9400000000	0.0000000000	0.0600000000	0.0000000000
125	0.9400000000	0.0000000000	0.0000000000	0.0600000000
126	1.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 19: The fourth order partition of the tetrahedron: the faces.

i	M				
1	4	120	124	126	123
2	4	121	125	126	124
3	4	122	125	126	123
4	4	119	122	125	121
5	4	119	122	123	120
6	4	119	121	124	120
7	4	1	3	4	2
8	4	1	53	41	3
9	4	1	53	42	2
10	4	32	42	53	41
11	4	2	42	32	4

Table 19 (Contd.)

i	M					
12	4	3	41	32	4	
13	4	5	54	43	6	
14	4	5	54	44	7	
15	4	5	7	8	6	
16	4	7	44	33	8	
17	4	6	43	33	8	
18	4	33	44	54	43	
19	4	9	55	45	10	
20	4	9	11	12	10	
21	4	9	55	46	11	
22	4	11	46	34	12	
23	4	34	46	55	45	
24	4	10	45	34	12	
25	5	103	123	120	116	106
26	5	103	123	122	117	107
27	4	103	107	112	106	
28	4	115	119	120	116	
29	4	115	119	122	117	
30	3	106	116	115		
31	3	107	117	115		
32	5	2	13	14	15	4
33	5	2	42	60	47	13
34	4	13	47	35	14	
35	4	4	32	56	15	
36	4	32	56	60	42	
37	3	14	35	56		
38	3	35	56	60		
39	5	6	43	61	48	16
40	5	6	16	17	18	8
41	4	16	48	36	17	
42	4	33	57	61	43	
43	4	8	33	57	18	
44	3	36	57	61		
45	3	17	36	57		
46	5	45	62	78	84	55
47	5	46	63	79	84	55
48	4	75	79	84	78	
49	4	34	58	62	45	
50	4	34	58	63	46	
51	3	58	75	78		
52	3	58	75	79		
53	5	104	124	121	118	108
54	5	104	124	120	116	109

Table 19 (Contd.)

i	M					
55	4	104	109	113	108	
56	4	115	119	121	118	
57	3	108	118	115		
58	3	109	116	115		
59	5	3	41	64	49	19
60	5	3	19	20	15	4
61	4	19	49	37	20	
62	4	32	56	64	41	
63	3	37	56	64		
64	3	15	56	37		
65	5	44	65	80	85	54
66	5	43	61	81	85	54
67	4	76	81	85	80	
68	4	33	57	65	44	
69	3	57	76	80		
70	3	57	76	81		
71	5	10	21	22	23	12
72	5	10	45	62	50	21
73	4	21	50	38	22	
74	4	12	34	58	23	
75	3	22	38	58		
76	3	38	58	62		
77	5	105	125	122	117	110
78	5	105	125	121	118	111
79	4	105	111	114	110	
80	3	110	117	115		
81	3	111	118	115		
82	5	42	60	82	86	53
83	5	41	64	83	86	53
84	4	77	83	86	82	
85	3	56	77	82		
86	3	56	77	83		
87	5	7	24	25	18	8
88	5	7	44	65	51	24
89	4	24	51	39	25	
90	3	18	57	39		
91	3	39	57	65		
92	5	11	46	63	52	26
93	5	11	26	27	23	12
94	4	26	52	40	27	
95	3	40	58	63		
96	3	23	58	40		
97	5	82	96	106	103	86

Table 19 (Contd.)

i	M						
98	5	83	97	107	103	86	
99	3	77	93	96			
100	3	77	93	97			
101	3	93	112	106			
102	3	93	112	107			
103	5	13	24	25	28	14	
104	5	13	47	66	51	24	
105	3	25	39	69			
106	3	39	69	66			
107	3	14	35	69			
108	3	35	69	66			
109	5	16	48	67	52	26	
110	5	16	26	27	29	17	
111	3	40	70	67			
112	3	27	40	70			
113	3	36	70	67			
114	3	17	36	70			
115	5	78	98	110	105	84	
116	5	79	99	111	105	84	
117	3	94	114	110			
118	3	94	114	111			
119	3	75	94	98			
120	3	75	94	99			
121	5	81	100	108	104	85	
122	5	80	101	109	104	85	
123	3	76	95	100			
124	3	76	95	101			
125	3	95	113	108			
126	3	95	113	109			
127	5	19	49	68	50	21	
128	5	19	21	22	30	20	
129	3	38	71	68			
130	3	22	38	71			
131	3	37	71	68			
132	3	20	37	71			
133	6	90	101	109	116	106	96
134	4	87	93	96	90		
135	4	87	95	101	90		
136	3	102	115	112			
137	3	102	115	113			
138	3	93	112	102			
139	3	87	102	93			
140	3	87	102	95			

Table 19 (Contd.)

i	M						
141	3	95	113	102			
142	6	14	28	31	30	20	15
143	4	28	69	59	31		
144	4	30	71	59	31		
145	3	35	72	56			
146	3	37	72	56			
147	3	35	72	69			
148	3	59	72	69			
149	3	59	72	71			
150	3	37	72	71			
151	6	48	67	91	100	81	61
152	4	67	91	88	70		
153	4	88	95	100	91		
154	3	36	73	57			
155	3	57	76	73			
156	3	36	73	70			
157	3	70	88	73			
158	3	73	95	88			
159	3	73	95	76			
160	6	50	68	92	98	78	62
161	4	89	94	98	92		
162	4	68	92	89	71		
163	3	58	75	74			
164	3	38	74	58			
165	3	74	94	75			
166	3	74	94	89			
167	3	71	89	74			
168	3	38	74	71			
169	6	91	100	108	118	111	99
170	4	88	94	99	91		
171	3	102	115	114			
172	3	88	102	95			
173	3	88	102	94			
174	3	94	114	102			
175	6	49	68	92	97	83	64
176	4	89	93	97	92		
177	3	56	77	72			
178	3	71	89	72			
179	3	72	93	89			
180	3	72	93	77			
181	6	51	66	90	101	80	65
182	4	66	90	87	69		
183	3	39	73	57			

Table 19 (Contd.)

i	M						
184	3	73	95	87			
185	3	69	87	73			
186	3	39	73	69			
187	6	22	30	31	29	27	23
188	4	29	70	59	31		
189	3	40	74	58			
190	3	59	74	71			
191	3	59	74	70			
192	3	40	74	70			
193	6	92	98	110	117	107	97
194	3	89	102	94			
195	3	89	102	93			
196	6	47	66	90	96	82	60
197	3	72	93	87			
198	3	69	87	72			
199	6	17	29	31	28	25	18
200	3	59	73	69			
201	3	59	73	70			
202	6	52	67	91	99	79	63
203	3	70	88	74			
204	3	74	94	88			
205	3	115	112	106			
206	3	115	112	107			
207	3	56	15	14			
208	3	60	47	35			
209	3	61	48	36			
210	3	57	18	17			
211	3	78	62	58			
212	3	79	63	58			
213	3	115	113	108			
214	3	115	113	109			
215	3	64	49	37			
216	3	37	20	15			
217	3	80	65	57			
218	3	81	61	57			
219	3	58	23	22			
220	3	62	50	38			
221	3	115	114	110			
222	3	115	114	111			
223	3	82	60	56			
224	3	83	64	56			
225	3	39	25	18			
226	3	65	51	39			

Table 19 (Contd.)

i	M			
227	3	63	52	40
228	3	40	27	23
229	3	96	82	77
230	3	97	83	77
231	3	106	96	93
232	3	107	97	93
233	3	69	28	25
234	3	66	51	39
235	3	69	28	14
236	3	66	47	35
237	3	67	52	40
238	3	70	29	27
239	3	67	48	36
240	3	70	29	17
241	3	110	98	94
242	3	111	99	94
243	3	98	78	75
244	3	99	79	75
245	3	100	81	76
246	3	101	80	76
247	3	108	100	95
248	3	109	101	95
249	3	68	50	38
250	3	71	30	22
251	3	68	49	37
252	3	71	30	20

Table 20: The fourth order partition of the tetrahedron: the sub-cells.

i	M										
1	6	1	2	3	4	5	6				
2	6	7	8	9	10	11	12				
3	6	13	14	15	16	17	18				
4	6	19	20	21	22	23	24				
5	10	25	26	5	27	28	29	30	31	205	206
6	10	32	33	11	34	35	36	37	38	207	208
7	10	39	40	17	41	42	43	44	45	209	210
8	10	46	47	23	48	49	50	51	52	211	212
9	10	53	54	6	55	56	28	57	58	213	214
10	10	59	60	12	61	62	35	63	64	215	216
11	10	65	66	18	67	68	42	69	70	217	218
12	10	71	72	24	73	74	49	75	76	219	220

Table 20 (Contd.)

i	M												
13	10	77	78	4	79	29	56	80	81	221	222		
14	10	82	83	10	84	36	62	85	86	223	224		
15	10	87	88	16	89	43	68	90	91	225	226		
16	10	92	93	22	94	50	74	95	96	227	228		
17	12	97	98	84	27	99	100	101	102	229	230	231	232
18	12	103	104	89	34	105	106	107	108	233	234	235	236
19	12	109	110	94	41	111	112	113	114	237	238	239	240
20	12	115	116	79	48	117	118	119	120	241	242	243	244
21	12	121	122	67	55	123	124	125	126	245	246	247	248
22	12	127	128	73	61	129	130	131	132	249	250	251	252
23	17	133	30	58	134	101	135	126	136	137	138	139	140
		141	205	214	231	248							
24	17	142	37	64	143	107	144	132	145	146	147	148	149
		150	207	216	235	252							
25	17	151	44	70	152	113	153	123	154	155	156	157	158
		159	209	218	239	245							
26	17	160	51	76	161	119	162	129	163	164	165	166	167
		168	211	220	243	249							
27	17	169	57	81	153	125	170	118	137	171	141	172	173
		174	213	222	242	247							
28	17	175	63	86	162	131	176	100	146	177	150	178	179
		180	215	224	230	251							
29	17	181	69	91	135	124	182	106	155	183	159	184	185
		186	217	226	234	246							
30	17	187	75	96	144	130	188	112	164	189	168	190	191
		192	219	228	238	250							
31	17	193	80	31	161	117	176	102	171	136	174	194	195
		138	206	221	232	241							
32	17	196	85	38	134	99	182	108	177	145	180	197	198
		147	208	223	229	236							
33	17	199	90	45	143	105	188	114	183	154	186	200	201
		156	210	225	233	240							
34	17	202	95	52	152	111	170	120	189	163	192	203	204
		165	212	227	237	244							
35	24	194	173	195	139	140	172	197	179	198	148	149	178
		200	185	201	157	158	184	203	191	204	166	167	190

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