THE GRADIENT FLOW MOTION OF BOUNDARY VORTICES

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Abstract. We consider the gradient flow of an energy functional describing boundary vortices in thin magnetic films. We obtain motion laws for the singularities in all time scalings by using the method of Γ-convergence of gradient flows.

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1. Introduction and main results

Kohn and Slastikov [10] derived the following thin-film limit of the micromagnetic energy functional: In a certain scaling, the three-dimensional micromagnetic energy Γ-converges to

$$\frac{1}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega}(m\cdot\nu)^2 \tag{1.1}$$

among maps $m \in H^1(\Omega, S^1)$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain representing the thin film, and ν is normal to $\partial\Omega$. The asymptotic behavior as $\varepsilon \to 0$ of (1.1) on a simply connected domain Ω was studied in [13], where it was shown that the energy of minimizers diverges logarithmically, and critical points satisfying a logarithmic energy bound converge to singular harmonic maps. The position of the singularities was shown for some classes of critical points including minimizers to be governed by a renormalized energy. These results are similar to those of Bethuel, Brezis and Hélein [4] for the

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Ginzburg-Landau functional, which is perhaps not surprising if one considers that the singularities in [4] and in [13] arise from the same topological phenomenon, see [13] for some more discussion.

In this article, we focus on the motion law for these boundary vortices, more specifically, on the gradient flow, and will show that the appropriately time-rescaled equations converge in a certain sense to the motion of the boundary vortices by the gradient flow of the renormalized energy. Other rescalings lead to trivial motion laws for the boundary vortices.

Again, there are strong similarities of these results to those in the theory of gradient flow motion of interior vortices as studied by Jerrard and Soner [9] and Lin [14], [15]. Their proofs rely on PDE methods to study the gradient flow. We will use the method of Γ-convergence of gradient flows developed by Sandier and Serfaty [16], which allows us to work mostly with energy estimates. The work [16] relies on a "product estimate" from [17] which helps to separate space- and time-variables. We prove an analogous result by somewhat different methods, and extend a compactness theorem of Alberti, Bouchitté and Seppecher [3] to the noncoercive case. Another proof of the compactness result has been recently given by Garroni and Müller [8]. Our proof reduces the problem to the one-dimensional case that has been treated in [11].

We expect that our main results for the gradient flow carry over to the renormalized energy of Cabré and Cónsul [5] where other penalty terms than those in [13] can be treated thanks to the uniqueness result of Cabré and Solà-Morales [6].

As in [13], we will use the fact that maps $m \in H^1(\Omega, S^1)$ can be lifted via $m = e^{iu}$ to $u \in H^1(\Omega, S^1)$. Using this lifting, we can rewrite the energy (1.1) as

$$\mathscr{F}^\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(u-g). \tag{1.2}$$

Here g corresponds to ν by $\nu = ie^{ig}$, and can be chosen as smooth as ν except for one jump of height -2π. We will examine the more general case where g is a function with a single jump of height $-2\pi D$, with $D \geq 0$, corresponding to the map e^{ig} of degree D. For regularity, we assume that $\partial\Omega \in C^{2,\alpha}$ and $e^{ig} \in C^1$. By $(\partial\Omega)^N_*$, we denote the set of N-tuples $(a_1, \dots, a_N) \in (\partial\Omega)^N$ such that $a_i \neq a_j$ for $i \neq j$.

Definition 1.1. For $\vec{a} = (a_1, \dots, a_N) \in (\partial\Omega)^N_*$ and $\vec{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$ with $\sum_{i=1}^N d_i = 2D$, we let $u_* = u_*(\vec{a}, \vec{d})$ denote the harmonic function on Ω that satisfies $\sin(u_* - g) = 0$ on $\partial\Omega$ and jumps by $-d_i\pi$ at a_i.

The compactness result of [11] suggests the following definition for the "sense of convergence" necessary for the application of the theory of Γ-convergence of gradient flows:

Definition 1.2. Fix $\vec{d} \in \mathbb{Z}^N$. We say that a sequence (u_ε) of functions in $H^1(\Omega)$ *converges in singularities* to $\vec{a} \in (\partial\Omega)^N$ if the boundary traces satisfy $u_\varepsilon \to u_*(\vec{a}, \vec{d})$ in $L^2(\partial\Omega)$. We will write $u_\varepsilon \overset{S}{\rightharpoonup} \vec{a}$.

Definition 1.3. For $\vec{a} \in (\partial\Omega)^N_*$ and $\vec{d} \in \mathbb{Z}^N$ we define the *renormalized energy* as

$$W(\vec{a}, \vec{d}) = \frac{1}{2} \lim_{\rho \to 0} \left(\int_{\Omega_\rho} |\nabla u_*|^2 - \pi \sum_{i=1}^N d_i^2 \log \frac{1}{\rho} \right), \tag{1.3}$$

where $\Omega_\rho = \Omega \setminus \bigcup_{i=1}^N B_\rho(a_i)$.

The renormalized energy can be expressed via the solution of a linear boundary value problem for the Laplacian, see [13, Proposition 7.1].

We can now state our main result:

Theorem 1.4. *Let $0 < T \leq \infty$ and let (u_ε) be a sequence of solutions of*

$$\lambda_\varepsilon \partial_t u_\varepsilon = \Delta u_\varepsilon \quad \text{in } \Omega \times (0, T) \tag{1.4}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(u_\varepsilon - g) \quad \text{on } \partial\Omega \times (0, \infty). \tag{1.5}$$

For the initial conditions we assume that $u_\varepsilon(0) \overset{S}{\rightharpoonup} \vec{a} = (a_1, \dots, a_N) \in (\partial\Omega)^N_*$ with $\vec{d} = (d_1, \dots, d_N) \in \{\pm 1\}^N$. Furthermore, u_ε is supposed to be initially well-prepared*, meaning that*

$$\mathscr{F}^\varepsilon(u_\varepsilon(0)) - \frac{\pi N}{2} \log \frac{1}{\varepsilon} - \frac{\pi N}{2}(1 - \log 2) \leq W(\vec{a}, \vec{d}) + o(1) \tag{1.6}$$

as $\varepsilon \to 0$.

Depending on the asymptotic behavior of λ_ε, we then have:

(i) *If $\lambda_\varepsilon = \frac{1}{\log \frac{1}{\varepsilon}}$, then there exists a time $T^* > 0$ such that for all $t \in [0, T^*)$, there holds $u_\varepsilon(t) \overset{S}{\rightharpoonup} \vec{a}(t)$, with the same $\vec{d}$. Furthermore, the $\vec{a}(t)$ satisfy the motion law*

$$\frac{da_i}{dt} = -\frac{2}{\pi} \frac{\partial}{\partial a_i} W(\vec{a}(t), \vec{d}) \tag{1.7}$$

in the tangent space at a_i to $\partial\Omega$. As $t \to T^*$, there exist $i \neq j$ such that $a_i(t)$ and $a_j(t)$ converge to the same point. The energy of $u_\varepsilon(t)$ satisfies the expansion

$$\mathscr{F}^\varepsilon(u_\varepsilon(t)) = \frac{\pi N}{2} \log \frac{1}{\varepsilon} + \frac{\pi N}{2}(1 - \log 2) + W(\vec{a}(t), \vec{d}) + o(1) \tag{1.8}$$

as $\varepsilon \to 0$.

(ii) *If $\lambda_\varepsilon \log \frac{1}{\varepsilon} \to 0$ as $\varepsilon \to 0$, then for almost every $t \in [0, T)$ we have $u_\varepsilon(t) \overset{S}{\rightharpoonup} \vec{a}(0)$, so there is no motion.*

(iii) *If $\lambda_\varepsilon \log \frac{1}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, then for almost every $t \in [0, \infty)$ we have $u_\varepsilon(t) \overset{S}{\rightharpoonup} \vec{b}$ with $\nabla W(\vec{b}) = 0$, so the system instananeously jumps into a critical point.*

The proof is based on the technique of Γ-convergence of gradient flows [16] that we will apply to the functionals

$$\mathscr{E}^\varepsilon(u) = \mathscr{F}^\varepsilon(u) - \frac{\pi N}{2}(\log\frac{1}{\varepsilon} + 1 - \log 2) \tag{1.9}$$

and the limit functional

$$\mathscr{E}(\vec{a}) = W(\vec{a}, \vec{d}). \tag{1.10}$$

The PDE with the nonlinear boundary condition is the gradient flow of $\mathscr{E}^\varepsilon$ with respect to the norm $\sqrt{\lambda_\varepsilon}\,\|\cdot\|_{L^2}$, which we will use as the spaces X_ε in the terminology of [16]. With $\langle\cdot,\cdot\rangle$ denoting the $L^2(\Omega)$ scalar product, it is the strong form of $\lambda_\varepsilon\,\langle\partial_t u, \varphi\rangle = -d\mathscr{E}^\varepsilon(u)(\varphi)$, which is the condition for being the gradient flow.

The limit functional is defined on $(\partial\Omega)^N_*$, which is an open subset of the (flat) Riemannian manifold $(\partial\Omega)^N$. The approach of [16] for Euclidean limit spaces carries over to this situation without changes. As the limiting norm on the tangent space which is identified with $\mathbb{R}^N$ we use the constant Riemannian metric $\sqrt{\frac{2}{\pi}}\,\|\cdot\|_{\mathbb{R}^N}$.

To carry out the program of [16], we need to prove compactness and a lower bound in space variables only for every time t, which will be done in Section 2. Then we need to prove that the vortices move H^1 in time, and show that the time-derivative of the vortex motion is a lower bound in L^2 for the rescaled time-derivatives of the solutions u_ε. This is achieved in Section 5. Finally, we need to construct for given vortex motion an approximating sequence u_ε corresponding to this motion and satisfying some limiting inequalities, which will be the content of Section 6.

With these preparations, our Theorem 1.4 now follows from the abstract Theorem 1.4 and Proposition 1.5 in [16] just as Theorem 1.6 from there does: The result holds for small time and continues to hold until the vortices collide.

2. The lower bound in space

In this section, we restate some results of [11] and generalize some results of [13] from the case of critical points to more general sequences.

Theorem 2.1. *If (u_ε) is a sequence of functions with $\mathscr{F}^\varepsilon(u_\varepsilon) \le M \log\frac{1}{\varepsilon}$, then there exists a sequence (z_ε) in $\pi\mathbb{Z}$ such that $v_\varepsilon = u_\varepsilon - z_\varepsilon$ has a subsequence that converges in singularities to some $\vec{a} \in (\partial\Omega)^N$ for some $\vec{d} \in \mathbb{Z}^N$.*

Proof. By the results of [11], u_ε is precompact up to translation in all $L^p(\partial\Omega)$. The accumulation points v_* satisfy $v_* - g \in BV(\partial\Omega, \pi\mathbb{Z})$, hence can be written as $v_* = u_*(\vec{a}, \vec{d})$ for some $N \in \mathbb{N}$, $\vec{d} \in \mathbb{Z}^N$ and $\vec{a} \in (\partial\Omega)^N$. □

Theorem 2.2. *If (u_ε) are functions with $\mathscr{F}^\varepsilon(u_\varepsilon) \le M \log \frac{1}{\varepsilon}$ that converge in singularities to $\vec{a}$ for some $\vec{d} \in \{\pm 1\}^N$, then*

$$\liminf_{\varepsilon\to 0} \left(\mathscr{F}^\varepsilon(u_\varepsilon) - \frac{\pi N}{2} \left(\log \frac{1}{\varepsilon} + 1 - \log 2 \right) \right) \ge W(\vec{a}, \vec{d}) \tag{2.1}$$

Proof. Without loss of generality, we may assume all the u_ε to be harmonic It follows that $u_\varepsilon \to u_* = u_*(\vec{a}, \vec{d})$ in $L^2(\Omega)$. For small $\rho > 0$ we set $\Omega_\rho := \Omega \setminus \bigcup_{j=1}^N B_\rho(a_i)$. Now

$$\mathscr{F}^\varepsilon(u_*; \Omega_\rho) \le \liminf_{\varepsilon\to 0} \mathscr{F}^\varepsilon(u_\varepsilon; \Omega_\rho), \tag{2.2}$$

since this is trivial if the right-hand side is $+\infty$, and otherwise, a subsequence converges weakly in $H^1(\Omega_\rho)$, and the limit has to be u_* by the $L^2(\Omega)$ convergence. The weak lower semicontinuity of the Dirichlet integral now implies (2.2).

By a diagonal argument, we can find a sequence $\rho_j \to 0$ such that for a subsequence $u_\varepsilon \to u_*$ in $L^2(\partial\Omega_{\rho_j})$ for all j. For such ρ and ε, we define a new sequence v_ε^ρ as the minimizer of $\mathscr{F}^\varepsilon(w)$ among all functions w such that $w = v_\varepsilon^\rho$ on Ω_ρ. By a construction using the limit function of [13], it can be shown that

$$\liminf_{\rho\to 0} \inf_{w_\varepsilon \to u_* \text{in } L^2(\partial B_\rho \cap \Omega)} \liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(w_\varepsilon; \Omega_\rho) \le \frac{\pi}{2} \left(\log \frac{\rho}{\varepsilon} + 1 - \log 2 \right). \tag{2.3}$$

We need to show equality. On v_ε^ρ, we can use the approach of [13]. By a suitable refinement of Proposition 3.5 there to include mixed boundary conditions, it is possible to show that the vortex set of v_ε^ρ is not near $\partial B_\rho \cap \Omega$ since otherwise, the corner version of the lower bound from Proposition 3. 10 there (see also Proposition 4.17 of [12]) would contradict (2.3).

Continuing as in the case without boundary conditions from [13], it follows that $v_\varepsilon^\rho \rightharpoonup v_*^\rho$ in $H^1_{\mathrm{loc}}(B_\rho \cap \Omega)$, where $\sin^2(v_*^\rho - g) = 0$ on $B_\rho \cap \partial\Omega$ with a single jump point. By the analog of the renormalized energy theorem 8.6 of [13], we obtain that the nonsingular part of the energy is given by the renormalized energy of v_*^ρ. As $\rho \to 0$, this energy is minimal when the vortex is in the center. This shows we have equality in (2.3), which implies (2.1). □

Lemma 2.3. Assume that $u_\varepsilon \overset{S}{\rightharpoonup} \vec{a}$, $\mathscr{E}^\varepsilon(u_\varepsilon) \le W(\vec{a}, \vec{d}) + D_\varepsilon$, D_ε bounded.

Then for $\rho > 0$ such that $B_\rho(a_i)$ are disjoint and setting as usual $\Omega_\rho = \Omega \setminus \bigcup B_\rho(a_i)$, we have

$$\frac{1}{2}\int_{B_\rho(a_i)\cap\Omega}|\nabla u_\varepsilon|^2 = \frac{\pi}{2}\log\frac{1}{\varepsilon}+O(1) \tag{2.4}$$

$$\frac{1}{2\varepsilon}\int_{\partial\Omega\cap\partial\Omega_\rho}\sin^2(u_\varepsilon-g)\le D_\varepsilon \tag{2.5}$$

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\varepsilon-\nabla u_*|^2\le D_\varepsilon+o_\varepsilon(1) \tag{2.6}$$

$$\frac{1}{2\varepsilon}\int_{B_\rho(a_i)\cap\partial\Omega}\sin^2(u_\varepsilon-g)\le\frac{\pi}{2}\log\frac{1}{\rho}+O_\varepsilon(1) \tag{2.7}$$

Proof. We have with $C_0=\frac{\pi}{2}(1-\log 2)$

$$\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2+\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(u_\varepsilon-g)\le\frac{\pi N}{2}\log\frac{1}{\varepsilon}+NC_0+W(\vec{a})+D_\varepsilon. \tag{2.8}$$

From the proof of Theorem 2.2 we have the lower bound

$$\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\varepsilon|^2\ge\frac{\pi N}{2}\log\frac{1}{\rho}+W(\vec{a})+o_\rho(1) \tag{2.9}$$

and the bound

$$\frac{1}{2}\int_{\Omega\cap B_\rho(a_i)}|\nabla u_\varepsilon|^2+\frac{1}{2\varepsilon}\int_{\partial\Omega\cap B_\rho(a_i)}\sin^2(u_\varepsilon-g)\ge\frac{\pi}{2}\log\frac{\rho}{\varepsilon}+C_0+o_\rho(1). \tag{2.10}$$

Combining these, we see that

$$\frac{1}{2\varepsilon}\int_{\partial\Omega_\rho\cap\partial\Omega}\sin^2(u_\varepsilon-g)\le D_\varepsilon+o_\rho(1). \tag{2.11}$$

Since for any fixed ρ_0 we have $\partial\Omega_{\rho_0}\cap\partial\Omega\subset\partial\Omega_\rho\cap\partial\Omega$ for all $\rho<\rho_0$, we can let $\rho\to0$ on the right-hand side of (2.11) and obtain (2.5).

We similarly see that for fixed ρ and with $D_\varepsilon=O(1)$,

$$\frac{1}{2}\int_{\Omega\cap B_\rho(a_i)}|\nabla u_\varepsilon|^2+\frac{1}{2\varepsilon}\int_{\partial\Omega\cap B_\rho(a_i)}\sin^2(u_\varepsilon-g)=\frac{\pi}{2}\log\frac{1}{\varepsilon}+O_\varepsilon(1), \tag{2.12}$$

hence (2.4). Comparing with (2.10), we obtain (2.7).

For (2.6), we need that — similar to the discussion in Chapter I of [4] — another definition of W can be given by using instead of u_* the function $\tilde{u}_\rho$ which is harmonic, equal to u_* on $\partial\Omega\cap\partial\Omega_\rho$ and has $\frac{\partial\tilde{u}_\rho}{\partial\nu}=0$ on $\partial B_\rho(a_i)\cap\Omega$.

Now we can calculate

$$\int_{\Omega_\rho}|\nabla u_\varepsilon-\nabla\tilde{u}_\rho|^2=\int_{\Omega_\rho}|\nabla u_\varepsilon|^2+|\nabla\tilde{u}_\rho|^2-2\nabla u_\varepsilon\cdot\nabla\tilde{u}_\rho. \tag{2.13}$$

Now $\int_{\Omega_\rho}|\nabla\tilde{u}_\rho|^2=\int_{\partial\Omega_\rho}\tilde{u}_\rho\frac{\partial\tilde{u}_\rho}{\partial\nu}$ and $\int_{\Omega_\rho}\nabla u_\varepsilon\cdot\nabla\tilde{u}_\rho=\int_{\partial\Omega_\rho}u_\varepsilon\frac{\partial\tilde{u}_\rho}{\partial\nu}\longrightarrow\int_{\partial\Omega_\rho}\tilde{u}_\rho\frac{\partial\tilde{u}_\rho}{\partial\nu}$, hence

$$\int_{\Omega_\rho}|\nabla u_\varepsilon-\nabla\tilde{u}_\rho|^2=\int_{\Omega_\rho}|\nabla u_\varepsilon|^2-|\nabla\tilde{u}_\rho|^2+o_\varepsilon(1), \tag{2.14}$$

and this is $\leq 2D_\varepsilon + o_\rho(1) + o_\varepsilon(1)$. Fixing ρ_0 again and using $\Omega_{\rho_0} \subset \Omega_\rho$ for $\rho < \rho_0$, we can again let $\rho \to 0$, whence $\tilde{u}_\rho \to u_*$ and we obtain (2.6). □

3. Compactness in 3D

In this section, we prove that sequences of functions on three-dimensional domains satisfying a logarithmic energy bound have compact boundary traces. These results are adaptations from the work of Alberti, Bouchitté, and Seppecher [3], with changes resulting from our use of the compactness theory for noncoercive periodic potentials from [11] instead of that for coercive potentials from [2]. Other proofs of these results were given in a different context and by somewhat different methods by Garroni and Müller [8]. We wil later apply these theorems to domains that are products of a two-dimensional space domain and a time interval.

Let $\Omega \subset \mathbb{R}^3$ be a bounded set with C^1 boundary. For $B \subset \mathbb{R}^3$, $C \subset \partial B$ we define the following functional:

$$F_\varepsilon(u; B; C) = \frac{1}{2}\int_B |\nabla u|^2 + \frac{1}{2\varepsilon}\int_C V(u) d\mathscr{H}^2, \tag{3.1}$$

where $V : \mathbb{R} \to [0, \infty)$ is a π-periodic, continuous function with $V^{-1}(0) = \pi\mathbb{Z}$. In our applications, we will use $V(t) = \sin^2 t$. In the proofs, we will often make use of the fact that with V, also $V^\mu = \mu V$ for $\mu > 0$ satisfies the same assumptions.

Theorem 3.1. *Let (u_ε) be a sequence in $H^1(\Omega)$ such that $F_\varepsilon(u_\varepsilon; \Omega; \partial\Omega) \leq M \log\frac{1}{\varepsilon}$. Then the boundary traces of u_ε are bounded in $L^2(\partial\Omega)$ and precompact in $L^1(\partial\Omega)$, with every cluster point belonging to $BV(\partial\Omega, \pi\mathbb{Z})$. If $u_\varepsilon \to u$ in $L^1(\partial\Omega)$, then*

$$\int_{\partial\Omega} |Du| \leq \liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}}\int_\Omega |\nabla u_\varepsilon|^2 \tag{3.2}$$

Remark 3.2. It follows from Theorem 3.1 that u_ε is in fact precompact in all $L^p(\partial\Omega)$ with $1 \leq p < 2$.

To prove Theorem 3.1, we will locally flatten the boundary and then reduce the statement to the one-dimensional case by slicing. We start by stating the corresponding one-dimensional results:

For $I \subset \mathbb{R}$ an interval set

$$G_\varepsilon(u; I) := \frac{1}{4\pi\log\frac{1}{\varepsilon}}\int_{I\times I}\left|\frac{u(x)-u(x')}{x-x'}\right|^2 dxdx' + \frac{1}{2\varepsilon\log\frac{1}{\varepsilon}}\int_I V(u) \tag{3.3}$$

Definition 3.3. For a measurable function u on a set A we define the distribution function λ_u by

$$\lambda_u(t) = |\{x \in A : |u(x)| > t\}|. \tag{3.4}$$

and the *median* $m(u)$ (with respect to $\pi\mathbb{Z}$) by

$$m(u) = \max\left\{q \in \pi\mathbb{Z} : |\{u - q > 0\}| \geq \frac{|A|}{2}\right\}. \tag{3.5}$$

It is clear that

$$\frac{|A|}{2}|m(u)|^2 \le \int_{\{u>m(u)\}} u^2 \le \|u\|^2_{L^2(A)}. \tag{3.6}$$

Lemma 3.4. There exist constants $C_1, C_2 > 0$ and $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$, any $u \in L^1(I)$ such that $G_\varepsilon(u; I) < \infty$ satisfies

$$\lambda_u(t - m(u)) \le C_1 e^{-\frac{C_2|t|}{\sqrt{G_\varepsilon(u)}}}(|I| \vee 1) \tag{3.7}$$

Proof. For ε sufficiently small, this follows from closely reexamining the proof of Proposition 2.11 in [11] (The functional given there is for small $|I|$ equivalent to the one considered here). □

Proposition 3.5. *Let (u_ε) be a sequence in $L^1(I)$ such that $G_\varepsilon(u_\varepsilon; I) \le M < \infty$ and such that (u_ε) is bounded in some L^p for $p > 1$. Then (u_ε) is relatively compact in $L^1(I)$, every cluster point belongs to $BV(I, \pi\mathbb{Z})$, and the following inequality holds for every sequence (u_ε) with $u_\varepsilon \to u$ in $L^1(I)$:*

$$\int_I |Du| \le 2 \liminf_{\varepsilon\to 0} G_\varepsilon(u; I). \tag{3.8}$$

Proof. This is basically the content of Proposition 2.13 of [11]. □

Corollary 3.6. By replacing V with $V^\mu = \mu V$ and letting $\mu \to 0$, it follows that (3.8) in fact holds without the penalty term.

We define for $r > 0$ the sets $D_r := B_r(0) \cap \{x_3 > 0\} \subset \mathbb{R}^3$ and $E_r := B_r(0) \cap \{x_3 = 0\}$.

Proposition 3.7. *Let $u_\varepsilon \in H^1(D_r)$ be a sequence of functions satisfying the energy bound*

$$F_\varepsilon(u_\varepsilon; D_r; E_r) \le M \log\frac{1}{\varepsilon}. \tag{3.9}$$

Then the traces of u_ε on E_r are bounded in $L^2(E_{\gamma r})$ and precompact in $L^1(E_{\gamma r})$, and every cluster point belongs to $BV(E_{\gamma r}, \pi\mathbb{Z})$, where $\gamma = \frac{1}{\sqrt{3}}$.

Furthermore, if $u_\varepsilon \to u$ in $L^1(E_{\gamma r})$, then

$$\liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}} \int_{D_{\gamma r}} F_\varepsilon(u_\varepsilon; D_{\gamma r}; E_{\gamma r}) \ge \frac{1}{2}\left|\int_{E_{\gamma r}} Du\right|. \tag{3.10}$$

Proof. Let C be a maximal cube inscribed in B_r and $H = C \cap D_r$. Let $Q = C \cap E_r$. From the geometrical setup we see that the maximal circle in $P = \{x_3 = 0\}$ inscribed in Q has radius $\frac{r}{\sqrt{3}}$. Let $e \in P$ be a unit vector parallel to a side of Q. Let M denote the orthogonal complement of e in P and p the projection of $\mathbb{R}^3$ onto M. We set $Q_e = p(Q)$. For every $y \in Q_e$, we let $Q^y = p^{-1}(y) \cap Q$ and $H^y = p^{-1}(y) \cap H$. Just as in [3], we can use

Fubini's theorem and some facts on slicing of Sobolev functions found in the appendix of that paper to show that the traces u^y of u satisfy

$$\begin{aligned}
&\frac{1}{\log\frac{1}{\varepsilon}} F_\varepsilon(u;H;Q) \\
&\geq \int_{Q_e}\left(\frac{1}{4\pi\log\frac{1}{\varepsilon}}\int_{Q^y\times Q^y}\left|\frac{u^y(x)-u^y(x')}{x-x'}\right|^2 dxdx' + \frac{1}{2\varepsilon\log\frac{1}{\varepsilon}}\int_{Q^y} V(u^y)\right) dy \\
&= \int_{Q_e} G_\varepsilon(u^y;Q^y)dy.
\end{aligned}$$

From this we obtain the L^2 bound as follows: By Fubini's theorem and since $2\int_0^\infty t\lambda(t) = \int_{Q^y} u^2$, we can calculate $\|u_\varepsilon\|^2_{L^2(Q)}$ as

$$\|u_\varepsilon\|^2_{L^2(Q)} = \int_{Q_e}\int_{Q^y}|u_\varepsilon^y|^2 = \int_{Q_e} 2\int_0^\infty t\lambda_{u_\varepsilon^y}(t)dtdy \tag{3.11}$$

We estimate the integrand in the y-integral. To avoid clutter, we write λ for $\lambda_{u_\varepsilon^y}$, m for $m(u_\varepsilon^y)$ and G for $G_\varepsilon(u_\varepsilon^y;Q^y)$. We have, using (3.7)

$$\begin{aligned}
2\int_0^\infty t\lambda(t)dt &= 2\int_m^\infty (t-m)\lambda(t-m)dt \\
&\leq 2\int_{-\infty}^\infty (|t|+|m|)\lambda(|t-m|)dt \\
&\leq 4C_1\left(\int_0^\infty te^{-\frac{C_2t}{\sqrt{G}}}dt + |m|\int_0^\infty e^{-\frac{C_2t}{\sqrt{G}}}dt\right) \\
&= 4C_1\left(\frac{G}{C_2^2} + \frac{|m|\sqrt{G}}{C_2}\right).
\end{aligned}$$

Using Young's inequality, we can bound this for any $\alpha > 0$ by

$$\frac{4C_1\alpha}{C_2}|m|^2 + \frac{C_1(1+4\alpha)}{C_2^2\alpha}G. \tag{3.12}$$

Since $|Q^y| = \frac{2r}{\sqrt{3}}$ for all $y \in Q_e$, we can estimate by (3.6)

$$\int_{Q_e}|m(u_\varepsilon^y)|^2 \leq \frac{\sqrt{3}}{r}\int_{Q_e}\int_{Q^y}|u_\varepsilon^y|^2 = \frac{\sqrt{3}}{r}\|u_\varepsilon^y\|^2_{L^2(Q)}. \tag{3.13}$$

From this and (3.12) we obtain for any $\alpha > 0$

$$\|u_\varepsilon\|^2_{L^2(Q)} \leq \frac{4\sqrt{3}C_1\alpha}{C_2 r}\|u_\varepsilon\|^2_{L^2(Q)} + \frac{C_1(1+4\alpha)}{C_2^2\alpha}\int_{Q_e} G_\varepsilon(u_\varepsilon^y;Q^y)dy. \tag{3.14}$$

Choosing an appropriate $\alpha > 0$, we obtain

$$\|u_\varepsilon\|^2_{L^2(Q)} \leq \frac{C}{r}\int_{Q_e} G_\varepsilon(u_\varepsilon^y;Q^y)dy \leq \frac{CM}{r}. \tag{3.15}$$

To show the precompactness of (u_ε) in $L^1(Q)$, we use Theorem 3.9. The approximating family of functions will be given slice-wise by

$$w^y_{\varepsilon,\delta} = \begin{cases} u^y_\varepsilon & \text{for } y \in Q_e \text{ with } G_\varepsilon(u^y_\varepsilon, Q^y) \le C_\delta \text{ and } m(u^y_\varepsilon) < C_\delta \\ 0 & \text{else,} \end{cases} \tag{3.16}$$

for some C_δ to be chosen below. Using Hölder's inequality we see that

$$\begin{aligned} \int_Q |u_\varepsilon - w_{\varepsilon,\delta}| &\\ &\le \int_{\{y\in Q_e, G_\varepsilon(u^y_\varepsilon, e^y) > C_\delta\}} |u^y_\varepsilon| dy + \int_{\{y\in Q_e, m(u^y_\varepsilon)>C_\delta\}} |u^y_\varepsilon| dy \\ &\le \|u_\varepsilon\|_{L^2} \left(|\{y \in Q_e, G_\varepsilon(u^y_\varepsilon, e^y) > C_\delta\}|^{\frac{1}{2}} + |\{y \in Q_e, m(u^y_\varepsilon) > C_\delta\}|^{\frac{1}{2}} \right). \end{aligned} \tag{3.17}$$

By the "weak-L^1" bound

$$|\{y \in Q_e, G_\varepsilon(u^y_\varepsilon, e^y) > C_\delta\}| \le \frac{M}{C_\delta}, \tag{3.18}$$

and a similar bound resulting from (3.13) for $m(u^y_\varepsilon)$, we see from (3.17) and the L^2 bound (3.15) that we can choose C_δ such that $\|u_\varepsilon - w_{\varepsilon,\delta}\| \le \delta$.

The functions $w^y_{\varepsilon,\delta}$ now satisfy $G_\varepsilon(w^y_{\varepsilon,\delta}, Q^y) \le C_\delta$ for every $y \in Q_e$, and the one-dimensional theory applies: From Lemma 3.4 we see that $(w^y_{\varepsilon,\delta} - m(w^y_{\varepsilon,\delta})$ is bounded in all L^p, so we can use the boundedness of $m(w^y_{\varepsilon,\delta}$ and Proposition 3.5 to obtain that for every δ, the family $\varepsilon \mapsto w^y_{\varepsilon,\delta}$ is compact in $L^1(Q^y)$. By Theorem 3.9, this shows that in fact (u_ε) is compact in $L^1(Q)$.

It remains to prove that if $u_\varepsilon \to u$ in $L^1(E)$ for $E = E_{\gamma r}$, then $u \in BV(E, \pi\mathbb{Z})$, and inequality (3.10) holds. Slicing again (using E_e and E^y as we did Q_e and Q^y above) and using Fatou's lemma, we obtain that

$$M \ge \liminf_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} F_\varepsilon(u_\varepsilon; D; E) \ge \int_{E_e} \liminf_{\varepsilon\to 0} G_\varepsilon(u^y_\varepsilon; E^y) dy. \tag{3.19}$$

We now finish as in the proof of Proposition 4.7 of [3]: Since $u_\varepsilon \to u$ in $L^1(E)$, we have (possibly for a subsequence) that $u^y_\varepsilon \to u^y$ in $L^1(E^y)$ for a.e. $y \in E^e$. From Proposition 3.5 we obtain $u^y \in BV(E^y, \pi\mathbb{Z})$ and

$$\liminf_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} F_\varepsilon(u_\varepsilon; D; E) \ge \frac{1}{2} \int_{E_e} |Du^y| dy. \tag{3.20}$$

Using Proposition 6.9 of [3], generalized from characteristic functions to $\pi\mathbb{Z}$-valued functions or using Section 5.10 of [7], we obtain that $u \in BV(E, \pi\mathbb{Z})$, and

$$\liminf_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} F_\varepsilon(u_\varepsilon; D; E) \ge \frac{1}{2} \int_E \langle Du, e \rangle, \tag{3.21}$$

from which (3.10) follows for e parallel to $\int_E Du$. ∎

Lemma 3.8. For $u \in BV(\partial\Omega, \pi\mathbb{Z})$, the jump set S_u is countably $\mathscr{H}^1$-rectifiable, and both the jump $[u]$ and the normal ν_{S_u} are approximately continuous $\mathscr{H}^1$-a.e. on S_u.

Proof. It suffices to show this for characteristic functions that are in BV. For these, the conclusion follows from the countable rectifiability of S_u (see e.g. [7, Section 5.9, Theorem 1]). □

Proof of Theorem 3.1. As in [3, p. 26], we cover the compact set $\partial\Omega$ by finitely many balls B_i centered on $\partial\Omega$ such that $\Omega \cap B_i$ is the image of a half-ball under a map Ψ^i with isometry defect less than 1. From here we obtain the $L^2(\partial\Omega)$ boundedness and $L^1(\partial\Omega)$ compactness results by Proposition 3.7. To prove (3.2), we can again proceed as in [3] (see (4.29)–(4.30) there), just using Lemma 3.8 and replacing the use of ν_{S_u} by that of Du.

Since the estimate with V is valid for all V, we can use $V^\mu = \mu V$ and let $\mu \to 0$ to obtain that the bound holds for the Dirichlet integral alone, as remarked in [1]. □

In the proof above we have used the following version of a compactness theorem found [3], showing that the result stated there remains true without an a priori L^∞ bound if we have a better control on the approximation.

We consider functions in $L^1(A)$, where A is a bounded subset of $\mathbb{R}^N$. Take a unit vector $e \in \mathbb{R}^N$. Let $M = e^\perp$ be its orthogonal complement. Let A_e be the projection of A onto M. For every $y \in M$, set $A_e^y := \{t \in \mathbb{R} : y + te \in A\}$. For a function u, we denote its trace on A_e^y by u_e^y, so $u_e^y(t) = u(y + te)$.

We will consider families (v^n) of functions, parametrized by $n \in \mathscr{N}$, where $\mathscr{N}$ is some index set.

Theorem 3.9. *Let (v^n) be a family of functions in $L^1(A)$. Assume that there exists for every $\delta > 0$ a function w_δ^n that satisfies $\|w_\delta^n\|_{L^1(A_e^y)} \leq \|v^n\|_{L^1(A_e^y)}$ and $\|w_\delta^n - v^n\|_{L^1(A)} \leq \delta$ such that $(w_\delta^n)_e^y$ is precompact in $L^1(A_e^y)$ for $\mathscr{H}^{N-1}$-a.e. $y \in A_e$, and such that $\|w_\delta^n\|_{L^1(A_e^y)} \leq C(e, \delta)$ for all y.*

Then (v^n) is precompact in $L^1(A)$.

Proof. By the assumptions, (w_δ^n) is precompact in $L^1(A)$ so $\sup_n \|w_\delta^n\|_{L^1(A)} \leq C(\delta)$, hence $\sup_n \|v_n\|_{L^1(A)} < \infty$.

Without loss of generality, we assume $|A_e^y| \leq 1$ for all y, e. We extend all functions defined on A to functions on $\mathbb{R}^N$ by 0, and similarly all functions defined on A_e^y to functions on $\mathbb{R}$.

Fix a unit vector e so that the condition of the theorem holds. For $y \in A_e$ and $s > 0$ define

(3.22)
$$\omega_\delta^y(s) = \sup \left\{ \int_{\mathbb{R}} |(w_\delta^n)_e^y(t+h) - (w_\delta^n)_e^y(t)| \, dt : n \in \mathscr{N}, h \in [-s, s] \right\}.$$

By assumption, this is bounded by $2C(e, \delta)$. The precompactness of $((w_\delta^n)_e^y)$ shows by the Fréchet-Kolmogorov theorem that $\omega_\delta^y(s) \to 0$ as $s \to 0$.

We now calculate

$$\begin{aligned}\int_{\mathbb{R}^N} |v^n(x+he) - v^n(x)|\,dx &\le 2\delta + \int_{\mathbb{R}^N} |w^n_\delta(x+he) - w^n_\delta(x)|\,dx \\ &= 2\delta + \int_{A_e} \left(\int_{\mathbb{R}} |(w^n_\delta)^y_e(t+h) - (w^n_\delta)^y_e(t)|\,dt \right) dy \\ &\le 2\delta + \int_{A_e} \omega^y_\delta(|h|)dy. \qquad (3.23)\end{aligned}$$

Now we set $\omega_\delta(s) = \int_{A_e} \omega^y_\delta(s)dy$. Then $\omega_\delta \le 2C(e,\delta)|A_e| \le 2C(e,\delta)$. Also, $\omega_\delta(s) \to 0$ as $s \to 0$ by the corresponding convergence for every ω^y_δ and the dominated convergence theorem. We now define $\omega(s) := \inf_{\delta>0}(2\delta + \omega_\delta(s))$, which is a bounded function with $\omega(s) \to 0$ as $s \to 0$. By (3.23), we have

$$\int_{\mathbb{R}^N} |v^n(x+he) - v^n(x)|\,dx \le \omega(|h|) \qquad (3.24)$$

for all $n \in \mathscr{N}$ and $h \in \mathbb{R}$.

Repeating this construction for N linearly independent vectors $e_1, \dots, e_N$ shows the analog of (3.24) for all of these vectors, and now the Fréchet-Kolmogorov theorem shows that (v^n) is precompact in $L^1(A)$. □

4. A product estimate

In this section, we prove a product estimate similar to that of Sandier and Serfaty [17] that allows us to use the lower bounds of the last section just for specific directions. This will later be useful to separate time- and space-derivatives.

Theorem 4.1. *Let $X, Y \in C^0(\overline{\Omega}, \mathbb{R}^3)$ be vector fields. Let (u_ε) be a sequence of functions such that $F_\varepsilon(u_\varepsilon; \Omega; \partial\Omega) \le M \log\frac{1}{\varepsilon}$ and $u_\varepsilon \to u$ in $L^1(\partial\Omega)$. Then there holds*

$$\liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}} \left(\int_\Omega |X \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \left(\int_\Omega |Y \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \ge \frac{1}{2} \int_{\partial\Omega} \left| D^\perp u \cdot (X \times Y) \right|, \qquad (4.1)$$

where $D^\perp u$ denotes the vector-valued measure obtained by rotating Du in the tangential space to $\partial\Omega$ by $\frac{\pi}{2}$.

Corollary 4.2. Let $G \subset \mathbb{R}^2$ open, with $\partial G \in C^1$, and $I \subset \mathbb{R}$. Then for every $X, Y \in C^0(\overline{G \times I})$ and (u_ε) with $F_\varepsilon(u_\varepsilon; G \times I; \partial G \times I) \le M \log\frac{1}{\varepsilon}$ and $\int_{G \times \partial I} |\nabla u_\varepsilon|^2 \le M \log\frac{1}{\varepsilon}$ there holds

$$\liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}} \left(\int_{G\times I} |X \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \left(\int_{G\times I} |Y \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \ge \frac{1}{2} \int_{\partial G\times I} \left| D^\perp u \cdot (X \times Y) \right|. \qquad (4.2)$$

Proof. This follows from Theorem 4.1 by extending u_ε as $u_\varepsilon\big|_{\partial I}$ on top and bottom of the cylinder to $G \times I_{2\delta}$ for $I_\delta \supset I$ the interval extended by δ on both ends, and extending X and Y to X_δ and Y_δ that are 0 on $G \times \partial I_\delta$. We can choose a C^1 domain Ω_δ with $G \times I_\delta \subset \Omega_\delta \subset G \times I_{2\delta}$. The theorem then applies on Ω_δ, and letting $\delta \to 0$ we obtain the claim. □

Remark 4.3. The results of Theorems 3.1 and 4.1 and Corollary 4.2 also hold *mutatis mutandis* for the functional $\mathscr{F}^\varepsilon$ defined in (1.2): The sequence (u_ε) still satisfies the same compactness properties, just the limit u will be such that $v := u - g \in BV(\partial\Omega, \pi\mathbb{Z})$. Furthermore, the lower bounds (4.1) and (4.2) hold with u on the right-hand sides replaced by v. This can be proved similarly to the argument in Section 3 of [11].

Proposition 4.4. *If X and Y are constant vectors that span $\mathbb{R}^2$, then there exists a bilinear form g on $\mathbb{R}^2$ such that $g(X,X) = \lambda$, $g(Y,Y) = \frac{1}{\lambda}$, $g(X,Y) = 0$, and $\det g = |X \times Y|^{-2}$. Furthermore, we have for any $B \subset \mathbb{R}^2$ and $u \in H^1(B)$*

$$\int_B \frac{1}{\lambda}|X \cdot \nabla u|^2 + \lambda |Y \cdot \nabla u|^2 = |X \times Y| \int_B g(\nabla_g u, \nabla_g u)\sqrt{\det g}. \tag{4.3}$$

Proof. For $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ set

$$g_{11} = \frac{x_2^2 + \lambda^2 y_2^2}{\lambda |X \times Y|^2}$$
$$g_{12} = g_{21} = -\frac{x_1 x_2 + \lambda^2 y_1 y_2}{\lambda |X \times Y|^2}$$
$$g_{22} = \frac{x_1^2 + \lambda^2 y_1^2}{\lambda |X \times Y|^2},$$

where $|X \times Y|^2 = (x_1 y_2 - x_1 y_2)^2 \neq 0$ since X and Y span $\mathbb{R}^2$. This metric satisfies the claims.

For the second part, we observe that ∇_g is defined such that $g(V, \nabla_g u) = du(V) = V \cdot \nabla u$ for all vectors V. Since X and Y are g-orthogonal, we have

$$\nabla_g u = \frac{g(\nabla_g u, X)X}{g(X,X)} + \frac{g(\nabla_g u, Y)Y}{g(Y,Y)},$$

and after a short calculation we obtain (4.3). □

Proposition 4.5. *For any smooth metric g on the upper half-plane $H = \{(x,y) : y > 0\} \subset \mathbb{R}^2$ such that $\det g \geq c > 0$ there is a conformal diffeomorphism $\Phi : H \to H$ extending smoothly to $\{y = 0\}$ such that for any $u \in H^1(A)$, $A \subset H$, there holds*

$$\int_{\Phi(A)} |\nabla u|^2 = \int_A g(\nabla_g(u \circ \Phi), \nabla_g(u \circ \Phi))\sqrt{\det g}. \tag{4.4}$$

Proof. The existence of the conformal map is basically the content of Riemann's mapping theorem. Equation (4.4) just states the conformal invariance of the Dirichlet integral in two dimensions. □

Proposition 4.6. *Let $X, Y \in \mathbb{R}^3$. Let $D = D_r$ be a half-ball and $E = E_r$ the flat part of its boundary as in Proposition 3.7. Then for any $\lambda > 0$*

$$\liminf \frac{1}{\log \frac{1}{\varepsilon}} \int_D \frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq \int_E |D^\perp u \cdot (X \times Y)|. \tag{4.5}$$

Proof. If X and Y are linearly dependent or both lie in the plane $P := \{x_3 = 0\}$, this is trivial. Otherwise, let P_{XY} be the plane spanned by X and Y. Let $e \in P$ be a unit vector orthogonal to $P_{XY} \cap P$. Let $p : \mathbb{R}^3 \to P_{XY}$ denote the projection parallel to P. We set $E_e := p(E) = P \cap P_{XY} \cap E$ and $E^y := p^{-1}(y) \cap E$ as well as $D^y := p^{-1}(y) \cap D$ for every $y \in E_e$ as before. Using Fubini's theorem and writing $u^y = u^y_\varepsilon$ for the slices of u_ε on D^y, we obtain

$$\begin{aligned} &\int_D \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_E V(u^y) \\ &\qquad = \int_{E_e} \left(\int_{D^y} \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_{E^y} V(u^y) \right) dy. \end{aligned} \tag{4.6}$$

We use (4.3) above and rewrite the integral over D^y in the metric $g = g^\lambda_{XY}$ given by Proposition 4.4. Hence

$$\int_{D^y} \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 = |X \times Y| \int_{D^y} g(\nabla_g u^y, \nabla_g u^y) \sqrt{\det g}. \tag{4.7}$$

We now use Proposition 4.5 and change variables to obtain with $v^y = u^y \circ \Phi$ and $\alpha^\lambda_{XY} = \left|\frac{\partial}{\partial x} \Phi\right|$

$$\begin{aligned} &\int_{D^y} \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_{E^y} V(u^y) \\ &\qquad = |X \times Y| \int_{\Phi^{-1}(D^y)} |\nabla v^y|^2 + \frac{1}{\varepsilon} \int_{\Phi^{-1}(E^y)} \alpha^\lambda_{XY}(x) V(v^y(x)) dx, \end{aligned} \tag{4.8}$$

where $\alpha^\lambda_{XY} \geq \alpha_0 > 0$ since Φ extends to a diffeomorphism on the boundary, and E^y is bounded. For almost every y, we have as in the proof of Proposition 3.7 that $u^y \to u$ in $L^1(E^y)$, and $u \in BV(E^y)$, which translates for v to $v^y_\varepsilon \to v = u \circ \Phi$ and $v \in BV(\Phi^{-1}(E^y))$ Since E^y is open, we can find disjoint two-dimensional half-balls B_i inside $\Phi^{-1}(D^y)$ that cover S_v. On these balls, we can reduce the functional to G_ε on the boundary as before, just changing V to $\alpha_0 V$, and use Corollary 3.6 to obtain

$$\liminf_{\varepsilon \to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_{B_i} |\nabla v^y_\varepsilon|^2 \geq |X \times Y| \int_{C_i} |Dv| \,. \tag{4.9}$$

Changing back to the original variables and integrating over E_e, we obtain the estimate

$$\liminf \frac{1}{\log \frac{1}{\varepsilon}} \int_D \frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq |X \times Y| \int_E |e \cdot Du| \,. \tag{4.10}$$

Since $e \cdot Du = e^\perp \cdot D^\perp u$, $e^\perp$ is parallel to the projection of $X \times Y$ onto P, and $e_3 \cdot Du = 0$, this implies the claim. □

Proof of Theorem 4.1. We first show that for every $\lambda > 0$, there holds

$$\liminf_{\varepsilon\to 0} \int_\Omega \frac{1}{\lambda}|X\cdot\nabla u_\varepsilon|^2 + \lambda|Y\cdot\nabla u_\varepsilon|^2 \geq \int_{\partial\Omega}\Big|D^\perp u\cdot(X\times Y)\Big|. \tag{4.11}$$

From this, (4.1) follows by optimizing over λ. To prove (4.11), we define the measure μ_ε for every Borel set $B\subset\mathbb{R}^3$ by

$$\mu_\varepsilon(B) := \frac{1}{\log\frac{1}{\varepsilon}}\left(\int_{\Omega\cap B}\frac{1}{\lambda}|X\cdot\nabla u_\varepsilon|^2 + \lambda|Y\cdot\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega\cap B}V(u_\varepsilon)\right). \tag{4.12}$$

By the assumption on the energy, μ_ε are equibounded, and for a subsequence converge to a bounded measure μ on $\mathbb{R}^3$ in the sense of measures. Since $\|\mu\|\leq \liminf_{\varepsilon\to 0}\|\mu_\varepsilon\|$, it suffices to show the one-dimensional density estimate

$$\lim_{r\to 0}\frac{1}{r}\mu(B_r(x)) \geq 2\left|D^\perp u\cdot(X\times Y)\right| \tag{4.13}$$

at every point $x\in S_u$ such that $[u]$ and ν_{S_u} are approximately continuous at x. To prove (4.13), we will first suppose X and Y are constant and equal to their values $\overline{X}$, $\overline{Y}$ at x in $B_r(x)\cap\overline{\Omega}$. In this case the claim follows from Proposition 4.6 by mapping $B_r\cap\overline{\Omega}$ to a half-ball, with isometry defect vanishing for $r\to 0$, as in the proof of Theorem 3.1 and its model in [3].

Otherwise, we have $|X-\overline{X}|\leq\delta(r)\to 0$ as $r\to 0$ (by continuity) and hence

$$\begin{aligned}\frac{1}{\log\frac{1}{\varepsilon}}\int_{B_r}&\big||X\cdot\nabla u_\varepsilon|^2 - |\overline{X}\cdot\nabla u_\varepsilon|^2\big| \\ &\leq \frac{1}{\log\frac{1}{\varepsilon}}\int_{B_r}\big|\big((X+\overline{X})\cdot\nabla u_\varepsilon\big)\big((X-\overline{X})\cdot\nabla u_\varepsilon\big)\big| \\ &\qquad\leq (2|\overline{X}|+\delta(r))\delta(r)M \longrightarrow 0 \quad \text{as } r\to 0.\end{aligned} \tag{4.14}$$

Hence (4.13) holds for X and Y if it holds for the constant values $\overline{X}$ and $\overline{Y}$. □

Just as in [17], we can derive the following corollaries:

Corollary 4.7. If $F_\varepsilon(u_\varepsilon)\leq M\log\frac{1}{\varepsilon}$, then $u_\varepsilon \overset{S}{\rightharpoonup} \vec{a}\in(\partial\Omega)^N$ with $d\in\mathbb{Z}^N$, and there holds

$$\liminf_{\varepsilon\to 0}\frac{1}{2\log\frac{1}{\varepsilon}}\int_\Omega|\nabla u_\varepsilon|^2 \geq \liminf_{\varepsilon\to 0}\frac{1}{\log\frac{1}{\varepsilon}}\left(\int_\Omega|\partial_1 u_\varepsilon|^2\int_\Omega|\partial_2 u_\varepsilon|^2\right) \geq \frac{\pi}{2}\sum_{i=1}^N|d_i|. \tag{4.15}$$

Corollary 4.8. If in addition to the assumptions of the previous theorem we have $u_\varepsilon \overset{S}{\rightharpoonup} \vec{a}\in(\partial\Omega)^N_*$ with $\vec{d}\in\{\pm1\}^N$, and $\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2\leq\frac{\pi N}{2}\log\frac{1}{\varepsilon}(1+o(1))$

as $\varepsilon \to 0$, there holds for any $X, Y \in C^0(\overline{\Omega})$

$$\liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}} (X\cdot\nabla u_\varepsilon)(Y\cdot\nabla u_\varepsilon) = \frac{\pi}{2}\sum_{i=1}^N X(a_i)\cdot Y(a_i). \tag{4.16}$$

5. The lower bound in time

In this section we use the approach of Sandier and Serfaty [17], [16] to show how the product estimate leads to H^1 in time motion of the vortices, and the lower bound part required for the application of the theory of Γ-convergence of gradient flows.

We will need the following norm on measures on $\partial\Omega$:

$$\|\mu\|_1 := \sup\left\{\left|\int_{\partial\Omega}\zeta\mu\right| : \int_{\partial\Omega}\zeta = 0, \left|\frac{\partial}{\partial\tau}\zeta\right| \le 1\right\}. \tag{5.1}$$

Theorem 5.1. *Let (u_ε) with $u_\varepsilon = u_\varepsilon(x,t) : \Omega\times[0,T]\to\mathbb{R}$ be a sequence of functions such that*

$$\int_{\Omega\times(0,T)} |\partial_t u_\varepsilon|^2 \le M\log\frac{1}{\varepsilon} \tag{5.2}$$

and for every $t\in[0,T]$ there holds

$$\mathscr{F}^\varepsilon(u_\varepsilon(\cdot,t)) \le M\log\frac{1}{\varepsilon} \tag{5.3}$$

for some $M>0$. Then (u_ε) converges in $L^1(\partial\Omega\times(0,T))$ to a function u with $v = u-g \in BV(\partial\Omega\times(0,T))$. The measures $\mu = \partial_\tau v$ and $\sigma = \partial_t v$ satisfy $\mu\in L^\infty((0,T),\mathscr{M}(\partial\Omega))$ and $\sigma\in L^2((0,T),\mathscr{M}(\partial\Omega))$. Furthermore, $\mu\in C^{0,\frac12}([0,T],(\mathscr{M}(\partial\Omega),\|\cdot\|_1))$ and for every space-vector field X and every continuous function f there holds

$$\liminf_{\varepsilon\to 0}\frac{1}{\log\frac{1}{\varepsilon}}\left(\int_{\Omega\times(0,T)}|X\cdot\nabla u_\varepsilon|^2\int_{\Omega\times(0,T)} f^2|\partial_t u_\varepsilon|^2\right)^{\frac12} \ge \frac12\left|\int_{\partial\Omega\times[0,T]} x_\nu f\sigma\right|. \tag{5.4}$$

Proof. We follow the proof of Theorem 3 in [17]. We use a coordinate system on $\partial\Omega\times[0,T]$ given by $\hat{e}_\nu, \hat{e}_\tau, \hat{e}_t$, where $\hat{e}_t$ is the unit vector pointing in time-direction, $\hat{e}_\nu = \nu$ is the outer normal to $\partial\Omega$, and $\hat{e}_\tau$ is a tangent vector to $\partial\Omega$ with $\hat{e}_t\cdot(\hat{e}_\nu\times\hat{e}_\tau) = 1$. We split the measure Du on $\partial\Omega$ as $Du = \partial_t u\hat{e}_t + \partial_\tau u\hat{e}_\tau$. So $D^\perp u = -\partial_\tau u\hat{e}_t + \partial_t u\hat{e}_\tau$. The equation (5.4) is now a direct consequence of (4.2) and Remark 4.3. Setting $f=1$ and using the estimates (5.2)–(5.3), we see that

$$\left|\int_{\partial\Omega\times[0,T]}\sigma x_\nu\right|^2 \le C\int_0^T \|X\|^2_{L^\infty(\Omega)}. \tag{5.5}$$

Choosing X as $X = x_\nu \hat{e}_\nu$ on $\partial\Omega\times[0,T]$, and extending to Ω with $\|X\|_{L^\infty(\Omega)} = \|x_\nu\|_{L^\infty(\partial\Omega)}$, we see that

$$\left|\int_{\partial\Omega\times[0,T]} \sigma x_\nu\right|^2 \leq C\int_0^T \|x_\nu\|^2_{L^\infty(\partial\Omega)}\,, \tag{5.6}$$

which shows by duality that $\sigma \in L^2([0,T],\mathscr{M}(\partial\Omega))$, so for every X with $|X|\leq 1$ there holds

$$\left|\int_{\partial\Omega\times[t_1,t_2]} (X\cdot\nu)\sigma\right| \leq C\sqrt{t_2-t_1}, \tag{5.7}$$

where $C = \|\sigma\|_{L^2([0,T],\mathscr{M}(\partial\Omega))}$.

Now we choose a vector field $\zeta \in C^1(\overline{\Omega};\mathbb{R}^3)$ with $\left|\frac{\partial}{\partial\tau}\zeta\right| \leq 1$ on $\partial\Omega$ and $\int_{\partial\Omega}\zeta = 0$. We have, using the fact that distributional derivatives commute and $\partial_t\zeta = 0$

$$\begin{aligned}
\int_{\partial\Omega\times[t_1,t_2]} \sigma\partial_\tau\zeta &= \int_{\partial\Omega\times[t_1,t_2]} \partial_t v\partial_\tau\zeta\\
&= -\int_{\partial\Omega\times[t_1,t_2]} \partial_\tau\partial_t v\zeta\\
&= -\int_{\partial\Omega\times[t_1,t_2]} \partial_t\partial_\tau v\zeta\\
&= -\int_{\partial\Omega\times[t_1,t_2]} \partial_t(\partial_\tau v\zeta)\\
&= -\int_{\partial\Omega\times[t_1,t_2]} \partial_t(\mu\zeta)\\
&= \int_{\partial\Omega}\zeta\mu(t_2) - \int_{\partial\Omega}\zeta\mu(t_2).
\end{aligned}$$

Hence

$$\left|\int_{\partial\Omega} (\mu(t_2)-\mu(t_1))\,\zeta\right| \leq \|\sigma\|_{L^2([0,T],\mathscr{M}(\partial\Omega))}\sqrt{t_2-t_1}, \tag{5.8}$$

which shows that $t\mapsto\mu(t)$ is Hölder continuous with respect to the $\|\cdot\|_1$-norm, and

$$[\mu]_{C^{0,\frac{1}{2}}([0,T],(\mathscr{M}(\partial\Omega),\|\cdot\|_1)} \leq \|\sigma\|_{L^2([0,T],\mathscr{M}(\partial\Omega))}\,. \tag{5.9}$$

□

Proposition 5.2. *If (u_ε) satisfy $\mathscr{F}^\varepsilon(u_\varepsilon(t)) \leq M\log\frac{1}{\varepsilon}$ for all $t\in[0,T]$, $\int_{\Omega\times[0,T]}|\partial_t u_\varepsilon|^2 \leq C\log\frac{1}{\varepsilon}$, and $u_\varepsilon \to u$ in $L^1(\partial\Omega)$ with $u-g\in BV(\partial\Omega,\pi\mathbb{Z})$, then $\sigma = \partial_\tau(u-g)$ is of the form $\sigma(t) = \pi\sum_{i=1}^{n(t)} d_i(t)\delta_{a_i(t)}$ for some $a_i\in\partial\Omega$, $d_i\in\mathbb{Z}$.*

In addition, for any $\zeta \in C^1(\overline{\Omega})$, the map $t \mapsto \int_{\partial\Omega} \zeta\mu(t)$ is of class $H^1((0,T))$.

If in addition there holds $\sum_i |d_i(t)| \le \sum_i |d_i(0)|$ for all $t \in [0,T)$, $d_i(0) \in \{\pm 1\}$, and $a_i(0)$ are distinct, then there exists a time $T^* \in (0,T]$ and $n = n(0)$ maps $a_i(t) \in H^1((0,T^*), \partial\Omega)$ such that the $a_i(t)$ are distinct for $0 \le t < T^*$ and $\mu(t) = \pi \sum_i d_i(0)\delta_{a_i(t)}$.

If $T^* < T$, then there exists $i \ne j$ such that $\lim_{t\to T^*} a_i(t) = \lim_{t\to T^*} a_j(t)$.

Proof. This follows by using Theorem 5.1 just as Propositions 3.2 and 3.3 of [16]. □

Proposition 5.3. *If in addition to the conditions of the last proposition there holds*

$$\int_\Omega |\nabla u_\varepsilon|^2 \le \frac{\pi}{2} \sum_i \log \frac{1}{\varepsilon}(1 + o(1)),$$

then for all intervals $[t_1, t_2] \subset [0,T]$ on which $a_i(t)$ remain distinct,

$$\liminf_{\varepsilon\to 0} \frac{1}{\log\frac{1}{\varepsilon}} \int_{t_1}^{t_2} \int_\Omega |\partial_t u_\varepsilon|^2 \ge \frac{\pi}{2} \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 \tag{5.10}$$

Proof. This follows as Corollary 7 in [17] from the proof of Theorem 5.1 and Corollary 4.8. □

6. Construction of a "recovery sequence"

Here we perform the construction necessary for the application of the gradient flow Γ-convergence theorem. Our construction follows the same general idea as that of [16] (pushing the vortices in the direction of the limit flow). However, we need to refine the construction since isometric pushing as in [16] only works for constant curvature.

Theorem 6.1. *Let (u_ε) be a sequence with $u_\varepsilon \overset{S}{\rightharpoonup} \vec{a} \in (\partial\Omega)^N_*$ with respect to $\vec{d} \in \{\pm 1\}^N$. Assume that $\mathscr{F}^\varepsilon(u_\varepsilon) - \frac{\pi N}{2} \log\frac{1}{\varepsilon} \le C$. Let $\vec{V} = (V_1, \dots, V_N)$ be a collection of tangent vectors to $\partial\Omega$ at a_i, and let $\vec{b}(t) \in (\partial\Omega)^N_*$ be such that $\vec{b}(0) = \vec{a}$ and $\frac{d\vec{b}}{dt}(0) = \vec{V}$.*

Then there exist $v_\varepsilon = v_\varepsilon(x,t)$ such that $v_\varepsilon(0) = u_\varepsilon(0)$ and a locally bounded function G on $(\partial\Omega)^N_*$ such that

$$\frac{1}{\log\frac{1}{\varepsilon}} \int_\Omega |\partial_t v_\varepsilon(0)|^2 = \frac{\pi}{2} \left|\vec{V}\right|^2 + o(1) \tag{6.1}$$

and

$$\frac{d}{dt}\Big|_{t=0} \mathscr{E}^\varepsilon(v_\varepsilon(t)) = \frac{d}{dt}\Big|_{t=0} \mathscr{E}(\vec{b}(t)) + G(\vec{a})D_\varepsilon + o(1), \tag{6.2}$$

where $D_\varepsilon = \mathscr{E}^\varepsilon(u_\varepsilon) - \mathscr{E}(\vec{a})$ is the energy excess of u_ε.

Proof. We want to "push" the vortices along $\partial\Omega$. The "pushing" will need to be nearly isometric on the boundary and infinitesimally conformal in the interior near the vortices in order not to change the energy of the vortex cores.

We define the family of diffeomorphisms $\chi_t : \overline{\Omega} \to \overline{\Omega}$ as the solutions of the flow given by the vector field λ of Proposition 6.2, i.e. $\frac{d}{dt}\chi_t(x) = \lambda(\chi_t(x))$ and $\chi_0(x) = x$.

Let $u_*^t = u_*(\chi_t(a_1), \ldots, \chi_t(a_N))$ denote the singular harmonic function jumping by $-\pi d_i$ at $\chi_t(a_i)$ and $u_* = u_*^0 = u_*(\vec{a})$, and set $\psi_t = u_*^t \circ \chi_t - u_*$.

Now we define $v_\varepsilon(x,t)$ via $v_\varepsilon(\chi_t(x),t) = u_\varepsilon(x) + \psi_t(x)$. Then we calculate $\mathscr{F}^\varepsilon(v_\varepsilon)$ by changing variables as

(6.3)
$$\mathscr{F}^\varepsilon(v_\varepsilon) = \frac{1}{2}\int_\Omega |(\nabla v_\varepsilon)\circ\chi_t|^2 \det D\chi_t + \frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(v_\varepsilon\circ\chi_t - g\circ\chi_t)\frac{\partial}{\partial\tau}(\tau\cdot\chi_t)$$
$$= \frac{1}{2}\int_\Omega \left|D\chi_t^{-1}\nabla(u_\varepsilon+\psi_t)\right|^2 \det D\chi_t + \frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(u_\varepsilon+\psi_t - g\circ\chi_t)\frac{\partial}{\partial\tau}(\tau\cdot\chi_t).$$

Differentiating and using the definition of χ_t and $\psi_0 = 0$, we obtain

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_\Omega \left|D\chi_t^{-1}\nabla(u_\varepsilon+\psi_t)\right|^2 \det D\chi_t = \int_\Omega (-D\lambda\nabla u_\varepsilon)\cdot\nabla u_\varepsilon + \frac{1}{2}|\nabla u_\varepsilon|^2 \operatorname{div}\lambda + \frac{d}{dt}\Big|_{t=0}\nabla\psi_t\cdot\nabla u_\varepsilon. \tag{6.4}$$

In the balls $B_\rho(a_i)$ where λ is holomorphic, we have $-D\lambda\nabla u_\varepsilon\cdot\nabla u_\varepsilon + \frac{1}{2}\operatorname{div}\lambda|\nabla u_\varepsilon|^2 = 0$ so

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 = \int_{\Omega_\rho} -D\lambda\nabla u_\varepsilon\cdot\nabla u_\varepsilon + \frac{1}{2}\operatorname{div}\lambda|\nabla u_\varepsilon|^2 + \int_\Omega \frac{d}{dt}\Big|_{t=0}\nabla\psi_t\cdot\nabla u_\varepsilon \tag{6.5}$$

On the other hand,

$$\int_{\chi_t(\Omega_\rho)} |\nabla u_*^t|^2 = \int_{\Omega_\rho} \left|(\nabla u_*^t)\circ\chi_t\right| \det D\chi_t, \tag{6.6}$$

and since $\nabla\psi_t = D\chi_t\nabla u_*^t\circ\chi_t - \nabla u_*$, this can be rewritten as

$$\int_{\Omega_\rho} \left|D\chi_t^{-1}(\nabla\psi_t + \nabla u_*)\right|^2 \det D\chi_t.$$

Differentiating, we obtain

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_{\chi_t(\Omega_\rho)} |\nabla u_*^t|^2 = \int_{\Omega_\rho} (-D\lambda\nabla u_*)\cdot\nabla u_* + \frac{1}{2}|\nabla u_*|^2 \operatorname{div}\lambda + \frac{d}{dt}\Big|_{t=0}\nabla\psi_t\cdot\nabla u_*. \tag{6.7}$$

Comparing this with (6.5) and using (2.6) and (2.14) of Lemma 2.3, we see that the difference is bounded by $G(\vec{a})D_\varepsilon$ since

$$\lim_{\rho\to 0}\int_{\Omega_\rho}\frac{d}{dt}\Big|_{t=0}\nabla\psi_t\cdot(\nabla u_\varepsilon-\nabla u_*)=o(1),$$

since we can integrate by parts and use that ψ_t is harmonic and $\frac{\partial}{\partial\nu}\frac{d}{dt}\Big|_{t=0}\psi_t\in L^\infty(\partial\Omega)$. For the boundary term, we have

$$\frac{d}{dt}\Big|_{t=0}\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(u_\varepsilon+\psi_t-g\circ\chi_t)\frac{\partial}{\partial\tau}(\tau\cdot\chi_t) \tag{6.8}$$
$$=\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin^2(u_\varepsilon-g)\frac{\partial}{\partial\tau}(\tau\cdot\lambda)+\frac{1}{2\varepsilon}\int_{\partial\Omega}\sin 2(u_\varepsilon-g)\frac{d}{dt}\Big|_{t=0}(u_*^t\circ\chi_t-g\circ\chi_t).$$

Since the last term is 0 and $|\tau\cdot\lambda|\le C\rho$ in $B_\rho\cap\Omega$, we can use (2.5) and (2.7) and obtain that the boundary contribution is bounded by $G(\vec{a})D_\varepsilon+O(\sigma\log\frac{1}{\sigma})$ for every $\sigma<\rho$, and letting $\sigma\to 0$ we obtain (6.2).

The equation (6.1) follows from Corollary 4.8. □

Proposition 6.2. *Let $\Omega\in C^{2,\alpha}$ for some $\alpha>0$, and let $\rho>0$ be such that $B_\rho(a_i)$ are disjoint. Then there exists a vector field $\lambda\in C^1(\overline{\Omega};\mathbb{R}^2)$ with $\lambda(a_i)=(V_i\cdot\tau)\tau$, λ tangential to $\partial\Omega$ everywhere, λ holomorphic in $B_\rho(a_i)\cap\Omega$ and $\frac{\partial}{\partial\tau}(\lambda\cdot\tau)=0$ at the points a_i. The $C^1(\overline{\Omega})$-norm of λ can here be bounded by a function $G(\vec{a})$ that is locally bounded on $(\partial\Omega)^N_*$.*

Proof. Let $h:\mathbb{R}^2_+\to\Omega$ be a conformal map. By Kellogg-Warschawski theorem, it is $C^{2,\alpha}$ up to the boundary. For $a\in\mathbb{R}$, $z\mapsto h(z+a)$ is also such a conformal map, hence the derivative $h'(z)$ is tangent to $\partial\Omega$. If $g=h^{-1}$, this means that $\lambda=\frac{1}{g'}$ is a tangent holomorphic function. With a suitable Möbius transformation, we can achieve the tangential derivative condition at any given point, and patching together gives the desired vector field. □

Lemma 6.3. With $\Omega_\rho(t)=\Omega\setminus\bigcup B_\rho(\chi_t(a_i))$ there holds

$$\lim_{\rho\to 0}\left(\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_{\Omega_\rho(t)}|\nabla u_*^t|^2-\frac{\pi N}{2}\log\frac{1}{\rho}\right) \tag{6.9}$$
$$=\lim_{\rho\to 0}\left(\frac{d}{dt}\Big|_{t=0}\frac{1}{2}\int_{\chi_t(\Omega_\rho)}|\nabla u_*^t|^2-\frac{\pi N}{2}\log\frac{1}{\rho}\right).$$

Proof. Since λ is bounded in C^1, it suffices to show that for the symmetric difference $\Delta_\rho(t)=(\Omega_\rho(t)\setminus\chi_t(\Omega_\rho))\cup(\chi_t(\Omega_\rho)\setminus\Omega_\rho(t))$ there holds $\lim_\rho\to 0\int_{\Delta_\rho(t)}|\nabla u_*^t|^2=0$. However, from the construction of χ_t there exists $a>0$ such that $\Delta_\rho(t)\subset\bigcup_i B_{\rho(1+a\rho)}(a_i(t))\setminus B_{\rho(1-a\rho)}(a_i(t))$. Since $|u_*^t(z)|\le\frac{C}{|z-a_i(t)|}$ near a_i, we can estimate

$$\int_{\Delta_\rho(t)}|\nabla u_*^t|^2\le C\int_{\rho(1-a\rho)}^{\rho(1+a\rho)}\frac{1}{r}dr=C\log\frac{1+a\rho}{1-a\rho}, \tag{6.10}$$

which tends to 0 as $\rho \to 0$. □

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