LINEARIZATION AND GLOBAL DYNAMICS

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ABSTRACT

In this paper we show how the spectral theory of linear skew-product flows may be used to study the following three questions in the qualitative theory of dynamical systems: (1) When is an ω-limit set or an attractor a manifold? (2) Under which conditions will a dynamical system undergo a Hopf-Landau bifurcation from a k-dimensional torus to a $(k + 1)$-dimensional torus? (3) When is a vector field in the vicinity of a compact invariant manifold smoothly conjugate to the linearized vector field and how smooth is the conjugacy?

I. Introduction

Much of the current research into the qualitative behavior of dynamical systems is concerned with two fundamental problems involving the asymptotic behavior of the motions. The first of these problems is to describe the attractors or, more generally, the ω-limit sets of the motions. If one knows the structure of the ω-limit set, then one has essentially complete information about the given motion.

The second problem arises when one is studying dynamical systems which depend on a parameter. Once again one is interested in the attractors, but now one wants to study their dependence on the underlying parameter. In this study one encounters two correlated theories. First there is a perturbation theory in which the goal is to find sufficient conditions for the attractors to appear to be unchanged. Secondly there is a bifurcation theory where the objective is to describe such phenomena as period-doubling, Hopf-Landau bifurcations and the occurrence of "strange" attractors.

Our main objective in this lecture is to illustrate how the classical techniques of linearization can be used to address these problems of global dynamics. Specifically we are concerned with the question of linearization in the vicinity of a compact invariant manifold or, more generally, near a bounded motion in a dynamical system. The linearization theory we require is primarily a theory of linearization near a time-varying solution.

As an illustration of the power of these linearization techniques, we will address here three specific problems. The first of these, which we study in Section III, is the question of determining when an attractor or a ω-limit set is a manifold.

In Section IV we study the second of these problems by illustrating how the technique of linearization can be used to develop a bifurcation theory for invariant manifolds. Specifically we will describe conditions under which a k-dimensional torus may undergo a Hopf-Landau bifurcation to a $(k + 1)$-dimensional torus. This bifurcation theory is not simply a linear theory, but it also depends on the occurrence of certain irremovable nonlinear terms. Such nonlinearities give rise to "normal forms" for differential equations, which in turn form the basis for developing various bifurcation theories. The study of these normal forms in the vicinity of a smooth invariant manifold is an important chapter in the development of a qualitative description of dynamical systems.

The first step in a study of normal forms is the question of smooth linearization near an invariant manifold. This theory of smooth linearization, which completes our triad of problems, is described in Section V.

The theory we will describe here is valid for all compact invariant manifold M or, more generally, for all bounded solutions ϕ of the underlying differential equation. For the most part the new contributions of our theory occur when M (or ϕ) is not a fixed point or a periodic orbit. One noteworthy exception occurs in the linearization theory in Section V. As a corollary of our methods, we are able to give a definitive answer to the question of determining whether there is a C^N -linearization $(1 \leq N < \infty)$ of a (nonlinear) vector field in the vicinity of a hyperbolic fixed point or periodic orbit.

Before turning to the mathematical details, we wish to express our sincere gratitude to Robert Sacker for the essential role he played in the development of these theories. Many of the ideas described below find their origins in our collaborations with Dr. Sacker.

II. The Spectrum

We want to study the concept of a linear skew-product flow π defined on a vector bundle $\underline{E}$ over a compact base space M , Sacker and Sell (1974) and Selgrade (1975). Let us begin with a specific example which will be of interest later in the lecture.

Consider a smooth vector field or ordinary differential equation

$$X' = f(X) \tag{1}$$

defined in some smooth Riemannian manifold W , which we assume (for simplicity) to be an open set in a fixed Euclidean space R^n . Let M denote a given compact invariant set for (1). For example one may have

$$M = \mathrm{Hull}(\phi) = \mathrm{Closure}\{\phi(t) : t \in R\}$$

where ϕ is a solution of (1) with range in a compact subset of W .

For $\theta \in M$ we let $\theta \cdot t$ denote the solution of (1) that satisfies $\theta \cdot 0 = \theta$. Let $A = Df$ denote the linear part of f (i.e. the Jacobian matrix) and let $\Phi(\theta,t)$ denote the fundamental solution matrix of

$$x' = A(\theta \cdot t)x \ , \tag{2}$$

with $\Phi(\theta,0) = I$. Then

$$\pi(x,\theta,t) = (\Phi(\theta,t)x \ , \ \theta \cdot t) \tag{3}$$

is a linear skew-product flow on $R^n \times M$, or equivalently, Φ is a co-cycle on M , Ellis and Johnson (1982). The shifted flow associated with Eq. (3) is

$$\pi_\lambda(x,\theta,t) = (\Phi_\lambda(\theta,t)x \ , \ \theta \cdot t)$$

where $\Phi_\lambda(\theta,t) = e^{-\lambda t}\Phi(\theta,t)$ and λ is a real parameter. In other words, $\Phi_\lambda(\theta,t)$ is the fundamental matrix solution of

$$x' = [A(\theta \cdot t) - \lambda I]x \ .$$

The concept of a linear skew-product flow extends directly to a vector bundle $\underline{E}$ over M where $\theta \cdot t$ is a flow on M , see Sacker and Sell (1974). In this case $\Phi(\theta,t)$ is a linear mapping from the fibre $\underline{E}(\theta)$ over $\theta \in M$ to the fibre $\underline{E}(\theta \cdot t)$. For example if M is a compact invariant manifold and one restricts the vectors x in Eq. (2) to be tangent vectors to M , then π becomes the induced linearized flow on the tangent bundle TM $(= \underline{E})$ generated by (1).

We say that a linear skew-product flow π admits an exponential dichotomy over M if there is a projector $\hat{P}:\underline{E} \to \underline{E}$ and constants $K \geq 1$, $\alpha > 0$ such that

$$|\Phi(\theta,t)P(\theta)\Phi^{-1}(\theta,s)| \leq Ke^{-\alpha(t-s)} \ , \quad s \leq t$$

$$|\Phi(\theta,t) \ [I - P(\theta)] \ \Phi^{-1}(\theta,s)| \leq Ke^{-\alpha(s-t)} \ , \quad t \leq s \ .$$

Recall that a projector is a continuous mapping $\hat{P}:\underline{E} \to \underline{E}$ that satisfies $\hat{P}(x,\theta) = (P(\theta)x,\theta)$ where $P(\theta)$ is a linear projection on the fibre $\underline{E}(\theta)$.

The (continuous) <u>spectrum</u> $\Sigma = \Sigma(M)$ of π over M is defined as the collection of all $\lambda \varepsilon R$ for which the shifted flow π_λ fails to admit an exponential dichotomy over M. The complement $\rho(M) = R - \Sigma(M)$ is the <u>resolvent set</u>. If $\lambda \varepsilon \rho(M)$ we let $\underline{S}_\lambda$ and $\underline{U}_\lambda$ denote the range and null space of the projector associated with the exponential dichotomy for π_λ. These are invariant subbundles for π and one has $\underline{E} = \underline{S}_\lambda + \underline{U}_\lambda$ as a Whitney sum. Also we define

$$N_\lambda = \text{fibre-dim } \underline{S}_\lambda ,$$

where the fibre-dimension of any subbundle $\underline{V}$ of $\underline{E}$ is defined by

$$\text{fibre-dim } \underline{V} = \dim \underline{V}(\theta) .$$

Note that N_λ is monotone nondecreasing in λ for $\lambda \varepsilon \rho(M)$.

If M is connected then the Spectral Theorem assures us that $\Sigma(M)$ is the union of k nonoverlapping compact intervals $I_1, \ldots, I_k$, where $1 \leq k \leq n$ and $n = \text{fibre-dim } \underline{E}$. Also associated with each spectral interval I_i there is an invariant subbundle $\underline{V}_i$ of $\underline{E}$, where $n_i = \text{fibre-dim } \underline{V}_i$ satisfies $n_i \geq 1$, $1 \leq i \leq k$, with $n = n_1 + \ldots + n_k$. Furthermore if $\mu, \lambda \varepsilon \rho(M)$ with $\mu < \lambda$ and the open interval (μ, λ) contains precisely one spectral interval I_i, then one has

$$N_\lambda - N_\mu = \text{fibre-dim } \underline{V}_i . \tag{4}$$

See Sacker and Sell (1978) for more details.

If M is a smooth compact invariant submanifold for the flow generated by (1) on W, then there are three spectra (Σ, Σ_T and Σ_N) which we wish to study. First there is the <u>full</u> spectrum Σ, which is the spectrum of π on the full bundle $R^n \times M$. The tangent bundle TM is an invariant subbundle for the linearized flow π . By restricting π to the tangent bundle TM one obtain the <u>tangential</u> spectrum Σ_T . Next let $\underline{N}$ denote any subbundle in $R^n \times M$ which is complementary to TM . Th linearized flow π the induces an associated flow π_N on $\underline{N}$ and the spectrum of π_N , is the <u>normal</u> spectrum Σ_N . As shown in Sacker and Sell (1980), the normal spectrum Σ_N is independent of the choice of the normal subbundle $\underline{N}$. One can compute Σ_T and Σ_N by using the projections of the Jacobian matrix A in the tangential and normal diretions, respectively. Some of the properties of the three spectra are the following (cf. Sacker and Sell (1980)):

1) $\Sigma_T \cup \Sigma_N \subseteq \Sigma$.
2) Usually (but not always) one has $\Sigma = \Sigma_T \cup \Sigma_N$.
3) If $M \neq$ point, then $0 \in \Sigma_T$ and $0 \in \Sigma$.

<u>Remarks</u> 1. The theory we describe here extends readily to the study of homeomorphisms, diffeomorphisms and, in general, linear skew-product flows with discrete time t . We will not develop the discrete version of this theory in this report. Instead we invite the reader to consult the references cited above.

2. This notion of the continuous spectrum is very closely related to the ergodic concept of the Oseledec spectrum, which is based on the theory of Lyapunov exponents, see Oseledec (1968) and Ruelle (1979). The connections between these two concepts is described in Johnson and Sell (1983). By exploiting these interconnections Perry (1983) has developed numerical algorithms for approximating the spectral intervals and the associated spectral subbundles. These numerical methods can then be used to study various bifurcation phenomena.

III. Hyperbolic Almost Periodic Motions

We can now address the first of the three questions posed above. A somewhat more general version of this question is to ask when does the ω-limit set of a given trajectory lie on an invariant submanifold. A classical answer to this question is given in terms of the first integrals of the differential system. However even in the presence of first integrals one can rephrase the question by restricting to submanifolds of low dimension. In this rather general form, it seems overly optimistic to expect that there is any situation where this rather subtle problem can be resolved by studying the linearized equations above. Nevertheless this does occur in study of almost periodic solutions.

Let $\phi(t)$ be an almost periodic solution of (1) and let $M = \mathrm{Hull}(\phi)$. The Pontryagin Duality Theorem assures us that the topological dimension of M agrees with the algebraic dimension of M. (The latter is the dimension of the Fourier-Bohr frequency module.) Let k denote this dimension. Let Σ denote the spectrum of the linearized flow on $R^n \times M$.

First we note that if $k \geq 1$, then $\lambda = 0 \in \Sigma$. In this case, let I_0 denote the spectral subinterval that contains $\lambda = 0$ and let $\underline{V}_0$ denote the associated spectral subbundle. It is shown in Sell (1978) that fibre-dim $\underline{V}_0 \geq k$. Futhermore if fibre-dim $\underline{V}_0 = k$, then M is Lipschitz homeomorphic to the k-dimensional torus T^k and $\phi(t)$ is a quasi-periodic solution. These conditions on k can be checked by using (4).

The proof of these assertions, which can be found in Sell (1978), is based on ideas developed by Pliss (1964).

IV. Perturbation and Bifurcation of Manifolds

Consider next the dynamical system

$$X' = f(X,\alpha) \tag{5.α}$$

on W, where f depends smoothly on X and a parameter α. Assume that for a fixed value, say $\alpha = \alpha_0$, there exists a compact invariant submanifold M_0 for $(5.\alpha_0)$. We want to study the behavior of M_0 as α varies in a neighborhood of α_0. The perturbation theories of Sacker (1969), Fenichel (1972) and Hirsch, Pugh and Shub (1977) describe sufficient conditions under which M_0 can be imbedded in a smooth family of invariant manifolds M_α, for α near α_0, with $M_{\alpha_0} = M_0$. These theories can be summarized in terms of the spectra.

The manifold M_0 is said to be <u>normally hyperbolic of order</u> r, where r is a positive integer, if there exist real numbers a,b with $0 \leq a \leq ra < b$ and such that

$$\begin{aligned} &1) \quad \lambda \in \Sigma_T \implies |\lambda| \leq a\,, \\ &2) \quad \lambda \in \Sigma_N \implies |\lambda| \geq b\,. \end{aligned} \tag{6}$$

If M_0 is normally hyperbolic of order r, then there is a smooth family of invariant manifolds M_α of class C^r defined for α near α_0 with $M_{\alpha_0} = M_0$.

If the assumption of normal hyperbolicity breaks down at α_0, then the behavior of the flow generated by $(5.\alpha)$ near M_0, for α near α_0, can be very complicated, see Chenciner (1983), Meyer (1983), Sell (1983a) and Smale (1967). A full understanding of this behavior, even from a generic point of view, still eludes us. However there are a number of situations where one can obtain some insight. One very interesting case arises in the study of the Hopf-Landau bifurcation of a k-dimensional torus into a (k + 1)-dimensional torus.

Assume that the parameter α is real and that for $\alpha \in I$, where I is an open interval with $0 \in I$, Eq. $(5.\alpha)$ has a family of k-dimensional invariant tori $\tau(\alpha)$ which varies smoothly in α. (Smooth variation means of class C^N, for N sufficiently large.) Next we shall assume that the tori satisfy Hypotheses I and II of Sell (1979), which means that one can find smooth local coordinates $z \in R^{n-k-2}$, $x \in R^2$, $\phi \in T^k$ near $\tau(\alpha)$ so that the Eq. (5.α) can be written in the form

$$\begin{aligned} x' &= A_{11}(\phi,\alpha)x + \alpha A_{12}(\phi,\alpha)z + F(x,z,\phi,\alpha) \\ z' &= \alpha A_{21}(\phi,\alpha)x + [B(\phi) + \alpha A_{22}(\phi,\alpha)]z + H(x,z,\phi,\alpha) \\ \phi' &= G(x,z,\phi,\alpha) . \end{aligned} \tag{7}$$

The terms B and A_{ij} denote matrices of the appropriate dimensions and F and H contain higher order terms in x and z. Furthermore one has $(F,H) = (0,0)$ when $(x,z) = (0,0)$. Also the differential equation $\phi' = G(0,0,\phi,\alpha)$ denotes the restriction of the flow generated by (5.α) to the torus $\tau(\alpha)$. The system (7) is a Hopf-Landau dynamical system.

Let (ρ,θ_0) denote polar coordinates in the x-plane and let $\theta = (\theta_0,\phi) = (\theta_0,\theta_1,\dots,\theta_k)$ denote a typical point in T^{k+1}, where $\phi \in T^k$. For any continuous function $u = u(\theta)$ on T^{k+1} we let $M_\theta[u]$ denote the mean value of u.

The main hypotheses concern the (2×2) matrix A_{11} and the function G. First let us expand A_{11} in terms of α, that is let $A_{11} = \Omega + \alpha W$ where $\Omega = A_{11}(\phi,0)$. Let $w_{ij}(\phi,\alpha)$ denote the entries of $W(\phi,\alpha)$. Consider the following hypotheses:

H1. <u>The (2×2) matrix Ω satisfies</u>

$$\Omega = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$$

<u>where</u> ω_0 <u>is a nonzero constant</u>.

H2. <u>The mean value</u> $W = M_\theta[w_{11} \cos^2 2\pi\,\theta_0 + w_{22} \sin^2 2\pi\,\theta_0]$ (<u>at</u> $\alpha = 0$) <u>is nonzero</u>.

H3. <u>There is a vector</u> $\hat{\omega} = (\omega_1,\ldots,\omega_k)$ <u>and a smooth function</u> $L(x,z,\phi,\alpha)$ <u>such that</u> $G = \hat{\omega} + \alpha L$, <u>and the</u> $(k+1)$-<u>dimensional vector</u> $\omega = (\omega_0,\omega_1,\ldots,\omega_k)$, <u>where</u> ω_0 <u>is given by</u> H1 <u>above, satisfies the nonresonance condition</u> $|n \cdot \omega| \geq c|n|^{-\delta}$ <u>for integral vectors</u> $n = (n_0,n_1,\ldots,n_k) \neq 0$. (Here c and δ are positive constants that do not depend on n .)

In the theorem we state next, reference is made to a constant K . This constant, which is expressed as a mean value, depends upon the low-order terms (i.e. order ≤ 3) in the Taylor series expansion of (7). The formula for K appears in Sell (1979, Eq. (4.10)). The proof of this theorem relies on an invariant manifold theorem due to Hale (1961).

<u>Theorem 1.</u> <u>If Hypotheses</u> H1-H3 <u>are satisfied and the constant</u> K <u>is nonzero, then there is a unique family of</u> $(k+1)$-<u>dimensional invariant tori</u> $\hat{\tau}(\alpha)$ <u>defined for</u> $\alpha \cdot \mathrm{sgn}(WK) < 0$ <u>and one has</u> $\hat{\tau}(\alpha) \to \tau(0)$ <u>as</u> $\alpha \to 0$. <u>Furthermore if</u> $W > 0$, $K < 0$ <u>and the tori</u> $\tau(\alpha)$ <u>are asymptotically stable for</u> $\alpha < 0$, <u>then the bifurcating tori</u> $\hat{\tau}(\beta)$ <u>are asymptotically stable for</u> $\beta > 0$.

<u>Remarks</u> 3. For $k = 1$ this result is essentially due to Sacker (1964) who uses a weaker form of the nonresonance condition H3. Also see Marsden and McCracken (1976) and Ruelle and Takens (1971).

4. Basically the same theorem, formulated for mappings instead of differential equations, appears in Chenciner and Iooss (1979). Earlier versions of this result are cited in Haken (1981).

5. A recent paper of Flockerzi (1983) shows that the conclusion of Theorem 1 remains valid in some cases when both W and K vanish. In these cases Eq. (7) admits different normal forms.

V. Linearization Near a Compact Invariant Manifold

We shall begin this section by studying a nonlinear vector field

$$x' = Ax + F(x) \tag{8}$$

in the vicinity of a fixed point $x = 0$. We seek sufficient conditions for the existence of a smooth curvilinear coordinate system with the property that the vector field is linear when written in terms of the new coordinate system. Given such a linearization theory, a natural question then is to determine the smoothness of the new curvilinear coordinate system. Also, if the new coordinate system is lacking in smoothness, we want to determine the obstacles to smooth linearization. As we will now show, we can give a satisfactory and definitive resolution of this problem, when $x = 0$ is hyperbolic.

The differential equation (8) is said to admit a C^N-linearization near $x = 0$ if there is a C^N-diffeomorphism $H:V_1 \to V_2$, where V_1 and V_2 are neighborhoods of $x = 0$, that satisfies the following two properties:

(i) $H(0) = 0$.

(ii) Whenever $x(t)$ is a solution of (8) with $x(t) \in V_1$ for t in some interval I, then $y(t) = H(x(t))$ is a solution of

$$y' = Ay \tag{9}$$

for $t \in I$. Similarly, whenever $y(t)$ is a solution of (9) with $y(t) \in V_2$ for $t \in I$, then $x(t) = H^{-1}(y(t))$ is a solution of (8) for $t \in I$. (The mapping $y = H(x)$ above is referred to as a C^N-conjugation between (8) and (9).)

Let A be an $(n \times n)$ matrix with eigenvalues $\lambda_1,\dots,\lambda_n$ repeated with multiplicities and let $\sum(A) = \{\lambda_1,\dots,\lambda_n\}$. Let $m = (m_1,\dots,m_n)$ be a vector with nonnegative integer entries $m_1,\dots,m_n$, and define $\gamma(\lambda,m)$ by

$$\gamma(\lambda,m) = \lambda - (m_1\lambda_1 + \dots + m_n\lambda_n),$$

where λ is a complex number. Let $|m| = m_1 + \dots + m_n$.

We shall say that A is *hyperbolic* if $\mathrm{Re}\lambda \neq 0$ for all $\lambda \in \Sigma(A)$. A is said to be *stable* if $\mathrm{Re}\lambda < 0$ for all $\lambda \in \Sigma(A)$. A is said to satisfy the *Sternberg condition of order* N , $N \geq 2$, if $\gamma(\lambda,m) \neq 0$ for all $\lambda \in \Sigma(A)$ and all m satisfying $2 \leq |m| \leq N$. We shall say that A satisfies the *strong Sternberg condition of order* N , if A satisfies the Sternberg condition of order N and

$$\mathrm{Re}\ \gamma(\lambda,m) \neq 0 \tag{10}$$

for all $\lambda \in \Sigma(A)$ and all m with $|m| = N$. It is easy to see that if A satisfies the strong Sternberg condition of order $N \geq 2$, then A is hyperbolic. The following two theorems are proved in Sell (1983b).

Theorem 2. Let Eq. (8) *represent a* C^{2N+1}*-vector field on* $U \subset X$ *with* $N \geq 1$, $0 \in U$ *where* $D^P F(0) = 0$ *for* $P = 0,1$. *Assume that* A *is stable. Then the following are valid:*

(A) *Assume that* A *satisfies the strong Sternberg condition of order* $(N + 1)$. *Then* Eq. (8) *admits a* C^N*-linearization.*

(B) *Assume that* $D^P F(0) = 0$ *for* $0 \leq P \leq N$ *and that*

$$\mathrm{Re}\ \gamma(\lambda,m) \neq 0$$

for all $\lambda \in \Sigma(A)$ *and all* m *with* $|m| = N + 1$. *Then* Eq.(8) *admits a* C^N*-linearization.*

If A is hyperbolic and not stable, then one needs to assume that the eigenvalues of A satisfy additional algebraic conditions in order to obtain a C^N-linearization. In particular one has:

Theorem 3. Let Q satisfy $2N < Q < 2N + 1$ and let F be of class C^{2N+Q} on $U \subset X$ with $N > 1$, $0 \varepsilon U$ and $D^P F(0) = 0$ for $P = 0,1$. Then the following are valid:

(A) Assume that A satisfies the strong Sternberg condition of order Q. Then Eq. (8) admits a C^N-linearization.

(B) Assume that $D^P F(0) = 0$ for $0 < P < Q - 1$, and that

$$\operatorname{Re} \gamma(\lambda,m) \neq 0$$

for all $\lambda \varepsilon \sum(A)$ and all m with $|m| = Q$. Then Eq. (8) admits a C^N-linearization.

Remarks 6. Sternberg (1957, 1958) studies the question of finding sufficient conditions that Eq. (8) admits a C^s-linearization. He showed that there is a function $V(s,\lambda_1,\ldots,\lambda_n) > 0$ with the property that if A is hyperhyperbolic and satisfies the Sternberg condition of order N where $N > s + V$, then Eq. (8) admits a C^s-linearization. While there are several alternate proofs of Sternberg's Theorem [cf. Chen (1963), Hartman (1964), Nelson (1969), and Takens (1971)], the implicit formulae for N and V are very complicated. See Hartmen (1964, p. 257), for example. Our theorems assert that $V \equiv s$ (in the hyperbolic case) and $V \equiv 1$ (in the stable case), provided Ineq. (10) is satisfied. The homeomorphic version of these theorems (i.e. $N = 0$) appears in Grobman (1959, 1962) and Hartman (1960, 1963).

7. The result in Theorem 3 is sharp. One can show that for $N > 2$ the differential system

$$x_1' = 2Nx_1$$

$$x_2' = -Nx_2 + x_1^N x_3^N$$

$$x_3' = -(2N + 1)x_3$$

does not have a C^N-linearization. Nevertheless the nonresonance condition $\mathrm{Re}\,\gamma(\lambda,m) \neq 0$ is satisfied for all $\lambda \in \sum(A)$ and all m with $2 \leq |m| \leq 2N - 1$.

8. Our methods extend easily to the question of smooth linearization for diffeomorphisms in the vicinity of a fixed point. Our theory for the hyperbolic case where $N = 1$ is similar to, but not as strong as, a theorem of Bileckii (1973, 1978).

The assumption that $\gamma(\lambda,m) \neq 0$ for $2 \leq |m| \leq Q$ allows one to introduce a polynomial change of variables to eliminate the terms in Taylor series expansion of F with order between 2 and Q . The stronger assumption that $\mathrm{Re}\,\gamma(\lambda,m) \neq 0$ for $|m| = Q$ allows us to eliminate the remainder term in the Taylor series expansion of F .

The argument which we use to prove Theorems 2 and 3 is based on the theory of nonlinear perturbations of linear equations with exponential dichotomies, cf. Coppel (1965). The change of variables we introduce gives rise to a related nonlinear differential equation on a different finite dimensional Banach space. The quantities $\gamma(\lambda,m)$, for $\lambda \in \sum(A)$ and $|m| = Q$, arise as the eigenvalues of the associated linear equation, and Ineq. (10) ensures that this linear equation has an exponential dichotomy, Sell (1983b).

It is especially noteworthy that the methods we used to prove Theorems 2 and 3 extend readily to the study of smooth linearization near a compact invariant manifold M . In order to simplify the following discussion, we will assume that M, which is smoothly imbedded in W, has a trivial normal bundle. (The general problem can easily be reduced to this case.) It then follows that one can introduce smooth curvilinear local coordinates so that in the vicinity of M the vector field (1) becomes

$$\begin{aligned} x' &= A(\theta)x + F(x,\theta) \\ \theta' &= g(\theta) + G(x,\theta) , \end{aligned} \tag{11}$$

where θ represents local coordinates on M and $x \in R^k$ represents a normal vector to M. Furthermore F and G satisfy

$$(F, D_1 F, G)(0,\theta) = (0,0,0)$$

where $D_1 = \partial/\partial x$. Here $A(\theta)$ denotes the linear part of F projected in the normal x-direction at the point $\theta \in M$. The equation $\theta' = g(\theta)$ describes the flow on the manifold M.

The linearized vector field near M is defined as the vector field

$$y' = A(\phi)y \ , \ \phi' = g(\phi) \tag{12}$$

where $\phi \in M$ and $y \in R^k$. The linearized flow in the tangent bundle TM is given (in these coordinates) by

$$v' = B(\theta)v \ , \ \theta' = g(\theta) \tag{13}$$

where $B = D_2 g$, $D_2 = \partial/\partial\theta$ and $v \in R^p$ where $p = \dim M$. The normal spectrum Σ_N and the tangential spectrum Σ_T are the spectra of the linear skew-product flows generated by (12) and (13), respectively.

We seek sufficient conditions in terms of the matrices $A(\theta)$ and $B(\theta)$ in order that there exists a C^N-conjugacy H of the form

$$y = x + u(x,\theta) \quad , \quad \phi = \theta + v(x,\theta)$$

which maps Eq. (11) to Eq. (12) in the vicinity of M.

Since the dimension of the normal bundle is k, it follows from the Spectral Theorem that Σ_N is the union of q nonoverlapping compact intervals, $I_1,\ldots,I_q$, where $1 \leq q \leq k$. Let n_i = fibre-dim $\underline{V}_i$, where $\underline{V}_i$ is the spectral bundle associated with I_i. Then $n_i \geq 1$ and $n_1 + \ldots + n_q = k$. We shall say that a k-tuple $(\lambda_1,\ldots,\lambda_k)$ from the spectrum Σ_N is <u>admissible</u> provided

i) the mapping $j \to \lambda_j$ from $\{1,\ldots,k\}$ to R has its range in Σ_N, and

ii) $\mathrm{Card}\{j : \lambda_j \in I_i\} = n_i$, $1 \leq i \leq q$.

If the matrix A in (12) is independent of θ and has only real eigenvalues, then an admissible k-tuple is a listing of the eigenvalues of A repeated with their multiplicities.

<u>Theorem</u> 4. <u>Consider the equation</u> (11) <u>near</u> M <u>where the coefficients are of class</u> $4N$ <u>and</u> M <u>is normally hyperbolic of order</u> N. <u>Let</u> a <u>and</u> b <u>be defined so that</u> (6) <u>holds with</u> $r = N$. <u>Assume that one has</u>

$$\begin{aligned} &|\lambda - (m_1\lambda_1 + \ldots + m_k\lambda_k)| > Na \\ &|m_1\lambda_1 + \ldots + m_k\lambda_k| > (N+1)a \end{aligned} \tag{14}$$

<u>for all</u> $\lambda \in \Sigma_N$, <u>and all admissible</u> k-<u>tuples</u> $(\lambda_1,\ldots,\lambda_k)$ <u>and nonnegative integers</u> $m_1,\ldots,m_k$ <u>that satisfy</u>

$$2 \leq (m_1 + \ldots + m_k) \leq 2N \quad .$$

<u>Then there is a</u> C^N - <u>conjugacy between</u> (11) <u>and</u> (12).

<u>Remarks</u> 9. If, in addition, the manifold M is stable then one still obtains a C^N-conjugacy between (11) and (12) when the two inequalities in (14) hold for

$$2 < (m_1 + \ldots + m_k) < N + 1 .$$

(Compare with Theorems 2 and 3.) The homeomorphic version of the last theorem for a stable manifold appears in Pugh and Shub (1970).

10. A result of Robinson (1971) can also be used to study smooth conjugacies between (11) and (12). However his hypotheses concern the Taylor series expansion of the nonlinear terms F and G instead of the spectral properties of M.

11. Our results are somewhat stronger than similar theorems developed in Bogoljubov, Mitropolskii and Samoilenko (1976).

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