A SIMPLE PROOF OF C. SIEGEL'S CENTER THEOREM

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ABSTRACT

We give an elementary proof of a particular case of C. Siegel's center theorem, based on a method of M. Herman. Even if the proof has less generality than the standard one, it is simpler and provides sharper bounds.

Introduction

Recently, M. Herman (1) has introduced a method to treat several "small demonimator" problems.

This type of difficulty appears very frequently in perturbation expansions and the standard technique to overcome it systematically has become known -- rather loosely -- as K.A.M. theory.

One of the outstanding problems for these phenomena (2) is to obtain good estimates of the strength of the perturbations that make the results of the theory no longer hold. (Of course, the mechanisms of proof break down earlier than the conclusions).

In that respect, we believe Herman's method is highly relevant. The proofs are much simpler and it is, therefore, easier to discuss optimality. Even without any attempt to optimization, the proofs yield much better constants that the ones obtained by applying straightforwardly the standard argument: (However, by slightly non-standard ways of doing the estimates, L. Chierchia has obtained good constants in a particular example (3)). The main shortcoming of the method of proof is that it seems to go through only under (much) stronger assumptions than the usual one. However, the way physicists have been looking at the problem is the analysis of particular examples. It turns out that the method applies to the ones that physicists have been considering as most relevant for the breakdown problem.

We observed that the same methods apply also to C. Siegel's center theorem and that, in that case, it was possible to make further modifications of the argument so that the proof became quite explicit and only used elementary techniques.

Even if this results in a certain loss of mathematical elegance, we have strived for elementarity and explicitness. This is consistent with the philosophy that there is a lot to learn from the analysis of concrete cases; we introduced a little bit of redundancy so that given a particular example, it should be possible to discuss the modifications that yield the best constants and compute them. The more mathematically inclined reader, interested only in qualitative theorems, without explicitly computing the constants, may advantageously substitute a *good* part of the computations in the first part by an invocation to Sobolev inequalities.

C. Siegel's Center Theorem

Suppose we were given a function $f: C \to C$ leaving the origin fixed and analytic in a neighborhood of it.

$$f(z) = az + \sum_{k \geq 2} f_k z^k$$

If we think of f as defining a dynamical evolution, it is natural to try to analyze the stability of the origin. It turns out (4) that the origin is stable (both backwards and forward in time) if and only if $|a| = 1$ and moreover, there is an analytic function conjugating f to its linear part and leaving the origin fixed.

$$(*) \qquad \begin{aligned} f \circ \phi(z) &= \phi(az) \\ \phi(0) &= 0 \\ \phi'(0) &= 1 \end{aligned}$$

The existence of such conjugating ϕ is interesting in itself and we just remark that it is very easy to show it exists if $|a|\neq 1$ (So that in that case, dynamics is determined by the linear approximation).

The following theorem is much more delicate.

<u>Theorem 1 (Siegel)</u>

If a satisfies $|a^k - 1|^{-1} < \gamma k^\nu \quad \gamma,\nu > 0$ then (*) has a unique analytic solution.

The standard proof of this theorem can be found in (4) where you can also find the reference to the original paper by Siegel, (which was based on a different idea) and other applications. There are also generalizations to several complex variables (5) (6) that, however, we will not discuss, since the methods we are going to use do not seem to carry over to several variables.

What we are going to prove in this paper is a <u>weaker version</u> of this theorem.

<u>Theorem 1'</u>

If a satisfies $|a^k - 1|^{-1} < \gamma k$ then (*) has a unique analytic solution.

We point out that uniqueness follows from identification of coefficients, so that the only non-trivial part to prove is convergence of the series thus obtained.

We also want to point out that there is no a, with modulus 1 which satisfies the inequality in Theorem 1 if $\nu<1$. (We state the theorem that way, to emphasize that the proof would -- obviously -- go through). When $\nu>1$, the numbers satisfying an inequality of this form have full measure on the unit circle. For the critical case $\nu=1$, even if it is a set of zero Lebesque measure, it is non-empty and, indeed, contains very interesting numbers; it consists precisely of the rotations with winding number having a bounded rational fraction expansion. See [7][8] and references there for a more detailed discussion of these numbers and their abundance.

Proof of theorem 1'

We first remark that, once we assume the boundary conditions, the conjugacy equation is equivalent to the one we obtain taking derivatives, namely

$$f' \circ \phi(z)\ \phi'(z) = a\phi'(az)$$

Since log f' is an analytic function in a neighborhood of the origin, we may require as well

$$\log f' \circ \phi(z) - \log a = \log \phi'(az) - \log \phi'(z).$$

Calling $\log f'(z) - \log a$, $h(z)$ we are led to the study of

$$h \circ \phi(z) = \log \phi'(az) - \log \phi'(z).$$

The second remark is that we can modify our problem so that we only have to consider h's defined on arbitrarily big neighborhoods and being there arbitrarily small with all their derivatives.

In effect, if we can solve our problem for $f_\lambda(z) = \frac{1}{\lambda} f(\lambda z)$ (λ any number) in place of f we have also solved the original problem (in another neighborhood).

Taking $|\lambda|$ small enough, we may assume $\log f'_\lambda(z) - \log a$ is analytic in any ball we please, and it satisfies there any smallness conditions we want. We'll state the ones needed in the place of the proof where they are used.

The strategy of the proof is, as in Herman's method, to study the operator τ_h that sends the function ψ into the function ϕ satisfying

$$(**)\qquad \begin{aligned} h \circ \Psi &= \log \phi'(az) - \log \phi'(z) \\ \phi(0) &= 0 \\ \phi'(0) &= 1 \end{aligned}$$

and to show that, under suitable smallness assumptions on h, it maps a compact convex set into itself and is continuous on it so that, by the Schauder-Tichonov theorem) has a fixed point that solves our problem.

A convenient choice for the set on which to study τ_h is

$$A = \left\{\psi \,|\, \psi(z) = \sum_{k \geq 0} \psi_k z^k,\ \psi_o = 0,\ \psi_1 = 1,\ \sum_{k \geq 2} |\psi_k|^2 k^2 (k-1)^2 < 1/4\right\}$$

Of course, there is nothing special about 1/4; we picked it not to clutter the proof with choices.

We first prove several properties of functions in A, and afterwards, will construct τ_h step by step, keeping good track of the estimates.

Lemma 1

All ψ in A have radius of convergence bigger or equal to 1 and satisfy

$$\sup_{|z|\leq 1} |\psi(z)| \leq \frac{1}{2}\left(\sum_{k\geq 2} \frac{1}{k^2(k-1)^2}\right)^{1/2} + 1 \equiv C_1$$

$$\sup_{|z|\leq 1} |\psi'(z)| \leq \frac{1}{2}\left(\sum_{k\geq 2} \frac{1}{(k-1)^2}\right)^{1/2} + 1 \equiv C_2$$

The proof is an obvious application of Cauchy-Schwartz inequality.

From this we can deduce the helpful representation

$$\sum_{k\geq 2} |\psi_k|^2 k^2 (k-1)^2 = \sup_{r<1} \frac{1}{2\pi} \int_{|z|=r} |\psi''(z)|^2 dz$$

which is proved by expanding the integrand in powers and taking into account the orthogonality relations between them.

By a -- very slight, using H_2 theory -- abuse of notation, I will denote R.H.S. simply by $\int_{|z|=1}$

We endow A with the usual topology of sets of analytic functions. In this topology A is a compact set, since by Lemma 1 it is contained in a compact (uniformly bounded set) and is closed. (The two first conditions are obviously closed and the third one is also closed from the integral representation for it). A is also convex.

In the light of Lemma 1, $h \circ \psi$ will always be defined provided h is defined in $\{|z|<C_1\}$. That is the first smallness condition in h to be imposed.

The integral representation renders elementary the proof of (uniform for $\psi\varepsilon A$) estimates for $h\circ\psi$

$$(h\circ\psi)'' = h'' \circ\psi(\psi')^2 + h'\circ \psi \, \psi''$$

$$\sum_{k\geq 2} |(h\circ\psi)_k|^2 k^2 (k-1)^2 = 1/2\pi \int_{|z|=1} |(h\circ\psi)''|^2 ds$$

(***) $$\leq [c_2^2 \sup |h''(z)| + 1/2 \sup_{|z|\leq 1} |h'(z)|]^2 \leq \varepsilon_1^2$$

$$|(h\circ\psi)_1| = |h_1| \leq \varepsilon_2$$

$$(h\circ\psi)_0 = 0$$

The second smallness conditions to be imposed in h are those above stated. $\varepsilon_1, \varepsilon_2$ are small numbers which only depend on a ; the explicit form of the dependence will be given later.

Lemma 2

If g is analytic in the ball of radius 1, $g(0) = 0$ there is one and only one η analytic in the ball of radius one (We will write $\eta = \Gamma g$) satisfying

$$g(z) = \eta(az) - \eta(z) \quad \forall |z|<1$$

$$\eta(0) = 0$$

Moreover, if $\sum |g_k|^2 k^2 (k-1)^2 < \varepsilon_1^2$, $|g_1| < \varepsilon_2$, then

$$\sum_{k\geq 2} |\eta_k|^2 (k-1)^2 < \gamma^2 \varepsilon_1^2$$

$$|\eta_1| < \gamma\varepsilon_2$$

Proof

Identifying coefficients we have

$$\eta_k = \frac{g_k}{a^k - 1}$$

Use the assumption on a to estimate absolute values. ∎

Therefore, in the same way as before we obtain

$$\sup_{|z|<1} |\eta(z)| < \gamma\varepsilon_1 \left(\sum_{k\geq 2} \frac{1}{(k-1)^2} \right)^{1/2} + \gamma\varepsilon_2$$

$$\frac{1}{2\pi} \int_{|z|=1} |\eta'(z)|^2 \, ds = \sum_{k\geq 1} |\eta_k|^2 k^2 < \gamma^2 \varepsilon_2^2 + 4\gamma^2 \varepsilon_1^2$$

It is clear that

$$\tau_h \psi(z) = \int_0^z \exp(\Gamma(h \circ \psi))\,(W)\, dW.$$

So that we have automatically

$$(\tau_h \psi)_0 = 0$$

$$(\tau_h \psi)_1 = 1$$

The only thing we still have to check to prove $\tau_h(A) \subset A$ is

$$\frac{1}{2\pi} \int_{|z|=1} |(\tau_h\psi)''(z)|^2 ds < 1/4 \text{ for all } \psi \varepsilon A$$

But, for such ψ

$$\begin{aligned} |(\tau_h\psi)''(z)| &= |(\tau h o \psi)'(z)| \; |\exp(\tau h o \psi)\,(z)| \\ &\leq |(\tau h o \psi)'(z)| \;\; \exp \gamma(\varepsilon_1 \tfrac{\pi}{\sqrt{6}} + \varepsilon_2) \end{aligned}$$

So that our goal is achieved whenever

$$1/4 > (\gamma^2 \varepsilon_2{}^2 + 4\gamma^2 \varepsilon_1{}^2) \exp 2\gamma[\varepsilon_2 + \tfrac{\pi}{\sqrt{6}} \varepsilon_1]$$

which can obviously be satisfied for certain ε's bigger than zero, therefore, when h satisfies (***) with these ε's in the right hand side, τ_h maps A into A.

As we remarked in the beginning, these smallness conditions for h can always be adjusted by scaling our original function f.

Once we have that τ_h maps a compact set into itself, it is easy to show it is continuous. When this is satisfied, continuity is the same as closedness of the graph, but the points in the graph are those pairs (ψ,ϕ) satisfying (**) which is obviously a closed condition.

Remark 1

There are other choices of sets A which are also acceptable for the proof. The one that was used was selected because it resembles the Sobolev spaces used in other theorems but has the integral representation which makes elementary and explicit some of the steps. There are however other possibilities e.g.

$$A = \{ \psi(z) \mid \psi_0 = 0,\ \psi_1 = 1,\ \sum_{k\geq 2} |\psi_k| k < 1/2 \}$$

Let us sketch briefly how you can adapt the steps of the proof. The only uniform estimate we are going to use is

$$\sup_{|z|<1} |\psi(z)| < 2/3 \quad \text{all } \psi \varepsilon A$$

A clearly is convex and compact with the usual topology of analytic functions of the unit disc. By the previous estimate it is equibounded and, therefore, contained in a compact set. It is also closed since

$$A = \bigcap_{n=3} \{ \psi(z) \mid \psi_0 = 0,\ \psi_1 = 1 \quad \sum_{n\geq k\geq 2} |\psi_k| k < 1/2 \}$$

which are closed conditions.

If h is an analytic function in a ball of radius bigger than 3/2 then, h is analytic in a ball of radius 1. Identifying coefficients and using the triangle inequality in all the sums, you obtain

$$n|(h\ \psi)_n| < n|h_1||\psi_n| + |h_2| \sum_{m_1+m_2=n} |\psi_{m_1}||\psi_{m_2}|(m_1+m_2) + $$
$$+ \ldots + |h_n| \sum_{m_1+\ldots+m_n=n} |\psi_{m_1}| \ldots |\psi_{m_n}| (m_1+\ldots+m_n)$$

Since

$$\sum_{n\geq 1} \sum_{m_1+\ldots+m_j=n} |\psi_{m_1}| \ldots |\psi_{m_j}| (m_1+\ldots+m_j) =$$

$$= j\left(\sum_m m|\psi_m|\right)\left(\sum_m |\psi_m|\right)^{j-1}$$

We clearly have

$$\sum_{n\geq 1} n|(h\circ\psi)_n| < \sum_{n\geq 1} n|h_n| (3/2)^n$$

From that, we easily obtain

$$\sum_{n\geq 1} |\eta_n| < \gamma \sum_{n\geq 1} n|h_n| \ (3/2)^n$$

By the same method we obtain also

$$\sum_{n\geq 0} |(\exp\eta)_n| < \exp \gamma \sum_{n\geq 1} n|h_n| \ (3/2)^n$$

Since

$$\sum_{n\geq 0} |(\exp \ \eta)_n| = \sum_{n\geq 1} n|\phi_n|$$

It suffices to impose the following smallness condition to h

$$\exp \gamma \sum_{n\geq 1} n|h_n|(3/2)^n < 3/2$$

The argument for continuity is, obviously, the same.

Let me discuss briefly how to treat a simple case, namely

$$f(z) = az+z^2$$

with $a = \exp 2\pi i \ 1/2 \ (\sqrt{5}-1)$

It is known that $\zeta = 1/2(\sqrt{5}-1)$ (the golden mean) satisfies

$$|k\zeta-p| < \sqrt{5} \ k^{-1}$$

Since $|\exp(2\pi ix) - \exp(2\pi iy)| > 4|x-y|$ if $|x-y|<1/2$

It is then easy to see that it suffices to take

$$\gamma = \sqrt{5}/4 \approx 0.56$$

(It is not sharp.)

In this case everything can be computed.

$$f'_\lambda(z) = a+2\lambda z$$

$$h(z) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \ (2\lambda/a)^n \ Z^n$$

The condition given at the end becomes

$$\exp(3\lambda/(1-3\lambda)) < 3/2$$

suffices to take

$$\lambda < \frac{\log 3/2}{2\cdot 3/2(\gamma+\log 3/2)} \approx 0.088$$

Of course, we cold have chosen other numbers instead of 3/2 in the definition of A. Call them e^u. Then, the condition becomes

$$\lambda = \frac{u\ e^{-u}}{2\ (\gamma+u)}$$

It is not very difficult to find the optimal choice of u. It gives that.

$\lambda = 0.143$ suffices.

Of course, this can be improved even further by a more sophisticated tailoring of the proof like using the estimates - more refined than the one used here - in [8].

The reason why it is interesting to choose λ as big as possible is because it gives us information about the size of the domain of stability. This domain (Siegel domain) for f_λ is the range of ϕ. But since ϕ is injective (this is true in general from results in [4], but for our case it suffices to remark that is a perturbation of the identity with Lipschitz constant less than 1), the area of the domain can be computed as follows:

$$\text{meas(Range } \phi) \geq \int_{|z|<1} |\phi'|^2 dxdy = \Pi \sum_k k|\phi_k|^2 \geq 1$$

So that the area of the Siegel domain for f is bigger or equal than $\Pi\lambda^2$.

We can also obtain other pieces of information about Siegel domain. For example if the set A is chosen as in the proof given in remark 1

$$A = \{\psi(z) \mid \psi_0 = 0,\ \psi_1 = 1,\ \sum_{k\geq 2} |\psi_k| k < e^u - 1\}$$

It can be readily proved that all the functions in this set are such that their range contains a ball around the origin of radius $r = 1 - 1/2(e^u - 1) = 3/2 - e^u/2$. Therefore, the Siegel domain for f should contain a ball around the origin of radius

$$\lambda r = \frac{(3-e^u)u\ e^{-u}}{4(\gamma+u)}$$

Taking $u = 0.33$, we obtain that, for the particular case we are discussing, the Siegel domain contains a ball of radius

$$\lambda r = 0.107$$

On the other hand, since the Siegel domain cannot contain in its interior any point in the orbit of the critical point, we can see that the Siegel domain does not contain any ball contered in the origin and with radius bigger than 1/4 (the image of the critical point is $-a^2/4$.). Actually, taking a few iterations with the help of a pocket calculator (9 suffice), this upper bound can be improved to .22.

Remark 2

After the preceding proof was completed, I received a letter from M. Herman pointing out that he was aware of the existence of such simple proofs and he had even written notes for a Seminar. He also mentioned that he and R. Douady observed that it is not necessary to use the Schauder-Tychonov Theorem; using stronger smallness conditions in h, it suffices to use the contraction mapping principle. This observation also applies to other proofs.

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This work would have not been possible if I had not attended the Les Houches session of July 1981. I profited very much from conversations with M. Herman, J. Mather and D. Goroff.

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