ON THE MONGE-AMPERE EQUATION
ARISING IN THE REFLECTOR MAPPING PROBLEM

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On the Monge-Ampère Equation
Arising in the Reflector Mapping Problem

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V. OLiker

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P. WALTMAN

ABSTRACT

The problem of synthesising a reflector which illuminates a target with a prescribed intensity can be changed to a question of the existence of a certain mapping on spheres. The analytic formulation of this problem leads to an equation of Monge-Ampère type. This paper presents a new derivation of this fact using the methods of differential geometry, gives an explicit solution in the case of a radially symmetric reflector, and uses the class of radially symmetric solutions and a perturbation argument to determine nonradially symmetric solutions. The radially symmetric case leads to an eigenvalue problem for a nonlinear, highly singular, ordinary differential equation which turns out to have two distinct classes of solutions.
On the Monge-Ampère Equation
Arising in the Reflector Mapping Problem*

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0. Introduction

In several papers Brickel, Marder, Norris, and Westcott [14], [10], [3], [9], [13] considered the following problem in geometric optics. In 3-dimensional Euclidean space $\mathbb{R}^3$ fix a point $O$ and suppose that a nonisotropic light source is positioned there. Let $S$ be a unit sphere centered at $O$ and $\Omega$ some fixed domain on $S$. Denote by $F$ a smooth surface which projects radially in a one-to-one fashion onto $\Omega$. The surface $F$ is assumed to have a perfect reflection property, that is, when a light beam is issued from $O$ and reaches $F$, it is reflected from it and no loss of energy occurs. Denote by $y$ the unit vector in the direction of the reflected ray and identify it with the point on $S$, by translating $y$ parallel to itself to the point $O$. Thus we get a mapping $\gamma$ of $\Omega \subset S$ into some domain $\omega \subset S$.

The inverse problem posed in [14], roughly speaking, consists in recovering the surface $F$ from the following data: the sets $\Omega$, $\omega$, and the density $f(y)$, $y \in \omega$, of the light intensity in the reflected direction. Below we refer to this as the reflector mapping problem.

In analytic formulation the problem reduces to solving an equation of Monge-Ampère type under certain nonlinear boundary conditions. The equations of Monge-Ampère type constitute a wide class of strongly nonlinear equations and only some special cases of such equations have been studied [6], [4]. The particular equation corresponding to the above problem has not been studied before and the known methods are not readily applicable.

In [10] the authors presented results of their numerical studies of the elliptic case in the reflector mapping problem. Because of the complexity of the problem there are no rigorous results on convergence of numerical approximations and, in fact, there are no results on existence of solutions.
In [3] the problem was reformulated in the language of complex analysis, and later Marder [9] proved a uniqueness theorem in this setting for the case where elliptic solutions of the problem are considered.

In this paper, after some introductory material, the equation for the reflector mapping problem is rederived in section one using only basic concepts from differential geometry and the postulated law of reflection. Naturally the end result is the same as in [14] and [3] but the method here is within the framework of geometry and this helps to understand better the problem from geometric point of view. Some properties of the resulting operator are established in section two. The hypothesis of radial symmetry is introduced in section three and it leads to a boundary value problem for an ordinary differential equation. The principal properties of this boundary value problem are abstracted and a general system investigated in section four by a polar coordinate technique in the phase plane. It is shown that such problems are overdetermined, that nontrivial solutions can exist only for special values of the parameters, the solutions fall naturally into two classes, and that uniqueness is possible within each class up to a constant multiple.

The specific equation for the reflector mapping problem is studied in section five using the techniques outlined in section four. The phase plane technique leads to an explicit determination of the polar angle function and a determination, by quadratures, of the polar radius. Thus, in principle, radially symmetric solutions can be constructed explicitly whenever all of the parameters are appropriate. The form of the solution actually shows how the
aperture, for example, must be chosen in order to have a nontrivial radially symmetric solution. Solutions near the radially symmetric ones can be found by a perturbation argument as shown in section seven.

1. Preliminaries.

1.1. In the Euclidean space $\mathbb{R}^3$ we fix a Cartesian coordinate system with origin at a point $0$ and a sphere $S$ of unit radius with center $0$. Let $F$ be a surface of class $C^\infty$ which projects radially from $0$ in a one-to-one fashion onto some closed domain $\tilde{\mathcal{U}} \subset S$. Denote the position vector of $F$ by $r$. Then $r = \rho m: \tilde{\mathcal{U}} \to F \subset \mathbb{R}^3$, where $m \in \tilde{\mathcal{U}}$, $\rho = |r|$, and it is assumed that $\rho > 0$ in $\tilde{\mathcal{U}}$.

Fix on $F$ a unit normal vector field $n$ such that $0 < \langle n, m \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^3$. Such choice of $n$ is possible because of the following observations.

Let $u^1$, $u^2$ be some local coordinates on $S$. Put $\partial_i = \partial/\partial u^i$, $i = 1, 2$. The first fundamental form of $S$, that is, the metric on $S$ induced from $\mathbb{R}^3$ will be denoted by $e$, $e = e_{ij} du^i du^j$, where $e_{ij} = \langle \partial_i m, \partial_j m \rangle$. Here and throughout the paper the summation convention over repeated lower and upper indices is in effect. The first fundamental form of $F$ is denoted by $g$, $g_{ij} = \langle \partial_i r, \partial_j r \rangle$, and by direct calculation we find:

$$\partial_i r = (\partial_i \rho)m + \rho \partial_i m,$$

$$g_{ij} = \partial_i \rho \partial_j \rho + \rho^2 e_{ij},$$

$$\det(g_{ij}) = \rho^2 [\nabla \rho]^2 + \rho^2 \det(e_{ij})$$
where

\[ |V\rho|^2 = \epsilon^{ij}_k \partial_i \rho \partial_j \rho, \quad (\epsilon^{ij}_k) = (\epsilon^{ij}_k)^{-1}. \]

Clearly, \( \det(g_{ij}) > 0 \) in \( \tilde{\Omega} \), since it was assumed that \( \rho > 0 \) in \( \tilde{\Omega} \).

Hence, \( F \) is an immersed surface. In fact, it is an embedded surface, because it also projects in a one-to-one fashion onto \( \tilde{\Omega} \).

The normal vector field (of unit length) on \( F \) is given by

\[ n = \frac{-V\rho + \rho m}{\sqrt{|V\rho|^2 + \rho^2}} \]

where \( V\rho = \epsilon^{ij}_k (\partial_i \rho) \partial_j m \). Since \( \rho \) is a smooth positive function in \( \tilde{\Omega} \), \( n \neq 0 \).

Furthermore, \( \langle n, m \rangle = \rho/\sqrt{|V\rho|^2 + \rho^2} > 0 \).

1.2. Define a map \( \gamma: \tilde{\Omega} \rightarrow S \) via \( F \) by the formula

\[ y = m - 2\langle m, n \rangle n. \tag{1.1} \]

This is indeed a map into \( S \), since \( y^2 = m^2 = 1 \). Obviously, \( \langle y, n \rangle = -\langle m, n \rangle \), and for that reason \( F \) is called a "reflector", since the vector \( y \) satisfies the law of reflection.

Consider now the solid body defined by the map

\[ r(m, s) = sr(m): [0,1] \times \tilde{\Omega} \rightarrow \mathbb{R}^3, m \in \tilde{\Omega}. \] The volume element of this body is

\[ dV = (1/3) \sqrt{\det(g_{ij})} h \, du_1 \, du_2, \]

where \( h = \langle r, n \rangle = \rho^2/\sqrt{|V\rho|^2 + \rho^2} \).

The function \( h \) has a simple geometric meaning; it represents the oriented distance from the origin \( 0 \) to the tangent plane of \( F \) at the point with normal \( n \). It is called the support function of \( F \).
The normalized volume element is defined as \( dV_{\text{nor}} = dV/\rho^3 \). In optics this quantity is called the elementary solid angle [14].

Suppose the surface \( F \) is such that the map \( \gamma \) is a diffeomorphism of \( \tilde{\omega} \) onto some domain \( \tilde{\omega} \subset S \). In the following we denote by \( \Omega \) and \( \omega \) the corresponding domains without boundaries. Denote by \( d\sigma \) the area element of \( S \), that is,
\[
d\sigma = \sqrt{\det(e_{ij})} \, du_1 du_2.
\]
It does not depend on a particular parametrization. The corresponding elementary solid angle is \((1/3)d\sigma\).

Let \( m \in \Omega \) and \( y = \gamma(m) \in \omega \). The relative light intensity in the reflected direction \( y \) is defined as the quotient
\[
f(y(m)) = \frac{dV_{\text{nor}}(m)}{\frac{1}{3}d\sigma(y(m))}.
\]

(1.2)

1.3. In the study of the problem of constructing the reflector from the "target" area \( \tilde{\omega} \) and relative light intensity \( f(y) \), \( y \in \omega \), it is convenient to look for the surface \( F \) parametrized by points in \( \tilde{\omega} \), that is, the position vector \( r \) of \( F \) is sought as \( r = \rho m = \rho \gamma^{-1}(y) \), provided one can establish that \( \gamma \) is indeed a diffeomorphism. We will now derive such an expression for \( r \).

Previously, such an expression was derived in a different way in [14].

Since in (1.2) the numerator and denominator do not depend on the particular parametrization of \( F \) and \( S \), we can write it in the form
\[
f(y) = \frac{dV_{\text{nor}}(\gamma^{-1}(y))}{\frac{1}{3}d\sigma(y)} = \sqrt{\frac{|V\rho|^2 + \rho^2 h}{\rho^2}},
\]

(1.3)

where all the quantities involved are considered as functions of \( y \) in \( \tilde{\omega} \).
It will be always assumed that \( \tilde{\omega} \neq S \) and, therefore, without loss of
generality we may suppose that \( (u^1, u^2) \) are coordinates on \( S \) such that \( \tilde{\omega} \) lies in
one coordinate patch. The points of \( \tilde{\omega} \) we consider as a function \( y(u^1, u^2) \), and
assume that in these parameters \( y \) is an analytic function.

Thus, let the position vector \( r \) of \( F \) be given as \( \rho(y)y^{-1}(y) \). We need to
find an explicit expression for it.

Put \( N = -2\hbar n \). Multiplying (1.1) by \( \rho \) we get
\[
\rho y = r + N. \tag{1.4}
\]
We get by differentiating (1.4)
\[
(\partial_i \rho) y + \rho \partial_i y = \partial_i r + \partial_i N, \quad i = 1, 2. \tag{1.5}
\]
Multiplying (1.5) by \( y \) and taking into account that \( \langle \partial_i y, y \rangle = 0 \), we get
\[
\partial_i \rho = \langle \partial_i r, y \rangle + \langle \partial_i N, y \rangle, \quad i = 1, 2. \tag{1.6}
\]
Because of (1.4) \( \langle \partial_i r, y \rangle = \langle \partial_i r, r \rangle / \rho = \partial_i \rho \), and (1.6) implies that
\[
\langle \partial_i N, y \rangle = 0, \quad i = 1, 2. \tag{1.7}
\]
Put \( p = \langle N, y \rangle \). Then, \( \partial_i p = \langle N, \partial_i y \rangle \).

The vectors \( y, \partial_1 y, \partial_2 y \) form a basis in \( \mathbb{R}^3 \) and therefore the system of
equations
\[
\begin{align*}
\langle N, y \rangle &= p, \\
\langle N, \partial_i y \rangle &= \partial_i p, \quad i = 1, 2, \tag{1.8}
\end{align*}
\]
can be solved for \( N \) at any point \( y \in \tilde{\omega} \). The solution is given by the formula
\[
N = \nabla p + py, \tag{1.9}
\]
where \( \nabla p = e^{ij} \partial_j \rho \partial_i y \), which is the gradient of \( p \) in the metric of \( S \).
Now from (1.4) and (1.9) we find
\[ -r = \nabla p + (p-p)y, \]  
(1.10)
and
\[ p = \frac{|
abla p|^2 + p^2}{2p}. \]  
(1.11)

1.3.1. **Remark.** From the definition of \( p \) we obtain with the use of (1.1)
\[ p = \langle N, y \rangle = -2h\langle n, y \rangle = 2h\langle m, n \rangle. \]
Since \( \langle m, n \rangle > 0 \) and \( h = \langle r, n \rangle = p\langle m, n \rangle > 0 \) everywhere on \( F \), \( p \) must be positive in \( \tilde{\omega} \). Therefore, (1.11) makes sense.

1.3.2. **Remark.** The expression (1.10) gives a formula for the position vector \( r \) resolved in the basis \( y, \partial_1 y, \partial_2 y \) at each point \( y \in \tilde{\omega} \).

1.3.3. **Remark.** The vector function \( N \) defined above is a normal vector field on the surface \( F \). If we translate this vector field in \( \mathbb{R}^3 \) so that the initial point of all this vectors coincides with origin 0 then \( N: \tilde{\omega} \to \mathbb{R} \) defines a surface \( T \). Its tangent vectors are given by \( \partial_1 N \) and \( \partial_2 N \) and, in view of (1.7), the unit normal vector field on \( T \) is given by the vector field \( y \). The function \( p \) is then the support function of \( T \) and the equations (1.8) describe \( T \) as an envelope of the family of its tangent planes given by the equations
\[ \langle X, y \rangle = p, \text{ where } X \in \mathbb{R}^3. \]

Note also, that for any \( C^1 \) function \( p \) in \( \tilde{\omega} \) the expressions (1.9) and (1.10) define surfaces in \( \mathbb{R}^3 \), though these surfaces do not have to be immersions. A necessary and sufficient condition for \( N \) to be an immersion can be found, for example, in [8]. The question when (1.10) is an immersion can be answered in a similar but not entirely satisfactory fashion.
1.3.4. **Remark.** The map $\gamma^{-1}$ can be explicitly represented in terms of the function $p$. Namely, using (1.1), (1.4) and (1.9), we find

$$m = \frac{r}{\rho} = y - N = \frac{y - 2p(\nabla p + \rho y)}{|\nabla p|^2 + \rho^2}. \tag{1.12}$$

1.4. Introduce the operations of covariant differentiation in the metric $e$, namely, put

$$\nabla_{ij} = \partial_{ij} - r_{ijk} \partial_k,$$

where $\partial_{ij} \equiv \partial^2/\partial u_i \partial u_j$, $i,j = 1,2$, and $r_{ijk}$, $i,j,k = 1,2$, are the Christoffel symbols of the second kind of the metric $e$. In the following we use the standard rules of covariant differentiation (see, for example, [7]).

Differentiating (1.10) covariantly, we obtain

$$- \partial_{i} r = [\nabla_{ij} p + (p - \rho)e_{ij}] e^{jk} \partial_k y - (\partial_{i} \rho)y, \tag{1.13}$$

Using (1.11), we find that

$$(\partial_{i} \rho)p = [\nabla_{ij} p + (p - \rho)e_{ij}] e^{jk} \partial_k y - (\partial_{k} \rho)y.$$

Substituting this in (1.13), we obtain

$$- p \partial_{i} r = [\nabla_{ij} p + (p - \rho)e_{ij}] e^{jk}[p \partial_k y - (\partial_{k} \rho)y], i = 1,2. \tag{1.15}$$

The coefficients of the first fundamental form of $F$ are

$$g_{ij} = \langle \partial_{i} r, \partial_{j} r \rangle, i,j = 1,2.$$  Hence, from (1.15)

$$p g_{ij} = q_{ik} q_{jk} e^{ks} e^{lt} [p e_{st} + (\partial_{s} \rho)(\partial_{t} \rho)], i,j = 1,2,$$

where $q_{ik} = \nabla_{ik} p + (p - \rho)e_{ik}$.

Finally, computing the quantities in (1.2) or (1.3) we obtain an expression for the light intensity
\[ M(p) = \frac{4p^2 \det[V_i j p + (p - |V_p|^2 + p^2) e_{ij}]}{(|V_p|^2 + p^2)^2 \det(e_{ij})} = f. \quad (1.16) \]

1.4.1. The map \( m = \gamma^{-1}(y) \), defined by (1.12), gives a parametrization of the domain \( \Omega \subset S \). One can explicitly compute the Jacobian of this map, and furthermore, the area element of \( \Omega \) (as a subdomain of \( S \)) in this parametrization. Geometrically, it is obvious that the function \( f \) in (1.16) is nothing but the ratio of the area elements in \( \Omega \) and \( \omega \). Analytically, this can be shown as follows.

Put \( \varepsilon_{ij} = \langle \partial_i m, \partial_j m \rangle \), \( i, j = 1, 2 \). Since \( r = \rho m \),

\[ e_{ij} = \frac{g_{ij} - (\partial_i \rho)(\partial_j \rho)}{\rho^2}. \]

Using (1.14) we find that

\[ p^2 (\partial_i \rho)(\partial_j \rho) = q_{ik} q_{jk} e^{k s} e^{l t} (\partial_s \rho)(\partial_t \rho). \]

From this and the expression for \( p^2 g_{ij} \) we obtain

\[ e_{ij} = \frac{q_{ik} q_{jk} e^{k l}}{\rho^2}, \]

and finally

\[ \det(e_{ij}) = \frac{[\det(q_{ij})]^2}{\rho^4 \det(e_{ij})}. \]

The area element in \( \Omega \) is equal to

\[ \sqrt{\det(e_{ij})} \, du^1 du^2 = [\det(q_{ij})/\rho^2 \sqrt{\det(e_{ij})}] \, du^1 du^2 \]

and in \( \omega \) it is \( \sqrt{\det(e_{ij})} \, du^1 du^2 \). Hence, the quotient of the two is indeed \( M(p) \).
1.5. Now we are in a position to formulate analytically the inverse reflector problem. In that we follow essentially [13], p. 34.

Let \( \omega, \Omega \) be two circular domains on \( S \), that is, both \( \omega \) and \( \Omega \) are intersections with \( S \) of some cones of revolution with vertices at the center of \( S \). It is assumed that \( \omega \cap \Omega = \emptyset \). Further, let \( f \) be a continuous positive function in \( \bar{\omega} \). One needs to find a function \( p \) in \( \bar{\omega} \) such that \( p \in C^2(\omega) \cap C^1(\bar{\omega}) \), \( p > 0 \) in \( \bar{\omega} \), and satisfies the equation

\[
M(p) = f \text{ in } \omega. \tag{1.17}
\]

In addition, the vector function \( r \), constructed from \( p \), as in (1.10), must satisfy the relation

\[
\left< \frac{r}{p}, \xi \right> = \phi, \text{ on } \partial \omega, \tag{1.18}
\]

where \( \xi \) is a constant unit vector whose end point is the center of the domain \( \Omega \), and \( \phi \) a given constant between zero and one. Explicitly, the quotient \( r/p \) can be represented as

\[
m \equiv \frac{r}{p} = -\frac{2p \nabla p + (p^2 - |\nabla p|^2)\gamma}{p^2 + |\nabla p|^2}. \tag{1.19}
\]

2. Some Simple Properties of the Operator \( M \).

2.1. The operator \( M \), as well as the quotient \( r/p \), are invariant relative to the transformation \( p \rightarrow cp \), where \( c = \text{const} \neq 0 \).

2.2. Suppose \( p \) satisfies the equation \( M(p) = f \text{ in } \omega \). If \( p \) has a local maximum (minimum) at some point \( \bar{y} \in \omega (\bar{y} \in \omega) \), then at this point we have \( \partial_i p = 0, \partial_{ii} p \leq 0 (\partial_{ii} p \geq 0), i = 1,2 \). Then, it is not difficult to see that \( f(\bar{y}) \geq 1 (f(\bar{y}) \leq 1) \).
2.3. Since \( p > 0 \) in \( \omega \), we can make a substitution \( P = \ln p \). Then the operator \( M(p) \) assumes the form

\[
M(p) = L(P) = \frac{4 \det \left[ V_{ij} P_{i} P_{j} + \frac{1 - |VP|^{2}}{2} e_{ij} \right]}{(1 + |VP|^{2})^{2} \det(e_{ij})},
\]

(2.1)

where \( P_{i} = \partial_{i} P, i = 1,2 \).

2.4. In the following we restrict our attention to the class of functions \( p \) on which the operator \( M \) is uniformly elliptic. In general, it is said that \( M \) is positively (negatively) elliptic on a positive function \( V \in C^{2}(\omega) \) if the quadratic form

\[
\left[ V_{ij} V + \left( V - \frac{|VV|^{2} + V^{2}}{2V} \right) e_{ij} \right] x_{i} x_{j}
\]

is positive (negative) definite at any point of \( \omega \) and for all \( x = (x^{1}, x^{2}) \in \mathbb{R}^{2} \).

It is easy to check that \( M \) is elliptic on \( V \) if and only if \( L \) is elliptic on \( V = \ln P \), that is, if the quadratic form

\[
\left[ V_{ij} V + V_{ij} V^{2} + \frac{1 - |VV|^{2}}{2} e_{ij} \right] x_{i} x_{j}
\]

is definite in \( \omega \times \mathbb{R}^{2} \).

2.5. The case when \( M \) is considered on functions on which it is hyperbolic is treated by entirely different methods. The elliptic and hyperbolic solutions of (1.15) are physically interpreted in [13].

2.6. Continuing the discussion in 1.4.1 we observe that if we denote by \( d\mu \) the area element of \( S \) in domain \( \Omega \) then it follows from (1.16) that
\[
\text{area of } \Omega = \int_{\Omega} \tilde{M}(p) \sqrt{\text{det}(e_{ij})} \, du^1 du^2 \\
= \int_{\omega} \hat{f} d\sigma.
\]

Thus, a necessary condition for solvability of (1.17) is that

\[
\text{area of } \Omega = \int_{\omega} \hat{f} d\sigma. \tag{2.2}
\]

From the point of view of geometric optics this condition represents the conservation law of energy, and it is called energy compatibility equation; see [13], p. 35. Evidently, this condition does not depend on the particular shape of the domains \( \Omega \) and \( \omega \).

2.7. In [13], ch. 2, there given examples of some simple reflectors (not necessarily satisfying the requirement that the map \( \gamma: \Omega \rightarrow \omega \) is a diffeomorphism). When \( \gamma \) is singular the representation (1.10) of the reflector is not available. In [13] these examples are presented in the complex analytic representation of \( S \). However, some of them could be also recovered as particular solutions of the equation (1.17).

2.7.1. Let \( \omega \) be an arbitrary domain on \( S \) and \( p = 2 \). Then \( M(2) = 1 \), and \( r = -\gamma \). Thus, the reflector coincides with \( \bar{\Omega} \) and \( \bar{\Omega} = -\omega \).

2.7.2. Again, let \( \omega \) be an arbitrary domain on \( S \), and \( p \) a spherical harmonic of the first order restricted to \( \omega \). It is known (see, for example, [11]) that \( V_{ij} p + p e_{ij} = 0, i,j = 1,2 \), on \( S \) and \( Vp + py = Q \), where \( Q \) is a constant vector in \( \mathbb{R}^3 \). Consequently, \( |Vp|^2 + p^2 = Q^2 = \text{const} \) and we assume that \( p \) is normalized so that \( Q^2 = 1 \).
The nodal line of a spherical harmonic of the first order is an equatorial line on $S$. Thus fixing a particular spherical harmonic we can pick a hemisphere inside of which $p > 0$. We assume now that $\bar{\omega}$ lies within such hemisphere. Then (1.10) gives the position vector of a reflector $F$ with
\[ r = -Q + \frac{1}{2p} y. \]

The unit normal vector field $n$ to $F$ is proportional to $N$, $N = -2hn$, and, in view of (1.9), $n = -Q/2h$. On the other hand, in general,
\[ h = \pm \sqrt{\frac{|p|^2 + p^2}{2}}. \]
In the particular case, we consider here, $h = \pm 1/2$, and, therefore, $n = \pm Q$, which implies that $F$ must be a plane, perpendicular to the vector $Q$ at a distance $\pm 1/2$ from the origin $O$. Since it was agreed that orientation of $F$ is such that $\langle r, n \rangle = p \langle m, n \rangle > 0$ (see 1.1), one should take $n = -Q$. One can check that in this case $\tilde{\Omega} \cap \tilde{\omega} = \emptyset$.

3. Radially Symmetric Case in the Problem (1.17), (1.18).

3.1. In section 1.5 it was assumed that $\omega$ is a circular domain. If the function $f(y)$ is allowed to depend only on $\text{dist}(y, y_o)$, where $y_o$ is the center of $\omega$, then it is natural to expect that solutions of (1.17), (1.18) will be also radially symmetric (r.s.), that is, $p = p(\text{dist}(y, y_o))$.

3.2. On the sphere $S$ we introduce spherical coordinates $\alpha, \beta$ where
\[-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}, 0 \leq \beta \leq 2\pi.\]
The metric of $S$ in this coordinates has the form
\[ e_{ij}^1 du^i du^j = (d\alpha)^2 + \cos^2 \alpha (d\beta)^2. \]
Correspondingly, $\det(e_{ij}) = \cos^2 \alpha$, and
\[
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{\cos^2 \alpha}
\end{pmatrix}
\]

The Christoffel symbols in this case are

\[
\begin{align*}
\Gamma^1_{11} &= 0, & \Gamma^2_{11} &= 0, \\
\Gamma^1_{12} &= 0, & \Gamma^2_{12} &= -\tan \alpha, \\
\Gamma^1_{22} &= \frac{1}{2} \sin 2\alpha, & \Gamma^2_{22} &= 0.
\end{align*}
\]

Here the index 1 corresponds to the variable \( \alpha \) and 2 to the variable \( \beta \).

For the covariant derivatives we have the expressions:

\[
\begin{align*}
\nabla_{11} p &= \frac{\partial^2 p}{\partial \alpha^2}, & \nabla_{12} p &= \frac{\partial^2 p}{\partial \alpha \partial \beta} + \frac{\partial p}{\partial \beta} \tan \alpha, \\
\nabla_{22} p &= \frac{\partial^2 p}{\partial \beta^2} - \frac{1}{2} \frac{\partial p}{\partial \alpha} \sin 2\alpha + p \cos^2 \alpha.
\end{align*}
\]

3.3. Assume that \( p \) is radially symmetric, that is, \( p = p(\alpha) \). Then from (1.16) and above formulas it follows that

\[
M(p) = \frac{4p^2 \left[p + p - \frac{p^2 + p^2}{2p}\right] \left[-\frac{1}{2} \dot{p} \sin 2\alpha + (p - \frac{p^2 + p^2}{2p}) \cos^2 \alpha\right]}{(p^2 + p^2) \cos^2 \alpha},
\]

where \( \dot{p} = \frac{dp}{d\alpha} \), \( \ddot{p} = \frac{d^2 p}{d\alpha^2} \).

The equation \( M(p) = f \) after simple manipulations can be written in the form

\[
(p - \dot{p} \tan \alpha - \frac{p^2 + p^2}{2p}) \dot{p} + (p - \dot{p} \tan \alpha)(p - \frac{p^2 + p^2}{2p})
\]

\[
- \frac{p^2 + p^2}{2p} + \frac{(p^2 + p^2)^2}{4p^2} (1 - f) = 0.
\]
Assume further that the North Pole \((\pi/2, \beta)\) coincides with the center \(y_0\) of the domain \(\omega\), and the angle \(\alpha\) varies in the interval \([\bar{\alpha}, \pi/2]\), where \(\bar{\alpha} \in (0, \pi/2)\).

The requirement that \(p\) be a r.s. solution of class \(C^1(\omega)\) implies
\[
\dot{p} \bigg|_{\alpha = \pi/2} = 0.
\]  
(3.3)

For the boundary condition (1.18) we have
\[
m = -\frac{2pp\dot{y} + (p^2 - \dot{p}^2)y}{p^2 + \dot{p}^2},
\]
where \(\dot{y} = \partial y / \partial \alpha\), and
\[
\phi(p^2 + \dot{p}^2) = (\dot{p}^2 - p^2)\Psi - 2pp\eta,
\]  
(3.4)
where \(\Psi = \langle y, \xi \rangle\), \(\eta = \langle \dot{y}, \xi \rangle\), and everything here is evaluated at \(\alpha = \bar{\alpha}\).

It is convenient to rewrite (3.4) in a more symmetric form
\[
(\phi - \Psi)p^2 + 2pp\eta + (\phi + \Psi)p^2 \bigg|_{\alpha = \bar{\alpha}} = 0.
\]  
(3.5)

3.4. **Proposition.** Let, as before, \(\omega\) be a circular domain on \(S\) and \(p\) a positive of class \(C^2(\omega) \cap C^1(\bar{\omega})\) r.s. solution of (3.2) satisfying (3.3). Then the domain \(\bar{\Omega} = m(\bar{\omega})\), where \(m\) is defined by (1.12), is also circular.

3.4.1. **Proof.** First of all observe that the center \(y_0\) of \(\omega\) is mapped into the South Pole \(y_1 = (-\pi/2, \beta)\). This follows from (3.3) and (1.12).

Further, let \(y \in \bar{\omega}\). Then \(\langle y, y_1 \rangle = -\langle y, y_0 \rangle = -\cos \kappa, \langle \dot{y}, y_1 \rangle = -\sin \kappa\),
where \(\kappa\) is the angle between \(y_0\) and \(y\).

From this and (1.12) we get
\[
\langle m, y_1 \rangle = \langle y - \frac{2p(\ddot{y}p + py)}{p^2 + \dot{p}^2}, y_1 \rangle =
\]
\[ = - \left( 1 - \frac{2p^2}{\dot{p}^2 + \ddot{p}^2} \right) \cos \alpha + \frac{2p^2}{\dot{p}^2 + \ddot{p}^2} \sin \alpha, \]

which proves the proposition.

3.4.2. **Remark.** Similarly one shows that the surface \( r(\tilde{\omega}) \) defined by (1.10) is radially symmetric with respect to the axis of direction \( y_1 \).

3.5. It follows from the above proof that \( y_1 \) is the center of the domain \( \Omega \). Clearly, \( \Psi = \langle y, \xi \rangle \bigg| \alpha = \tilde{\alpha} = - \sin \tilde{\alpha}, \eta = \langle \dot{y}, \xi \rangle \bigg| \alpha = \tilde{\alpha} = - \cos \tilde{\alpha}, \) and (3.5) takes the form

\[ (\phi + \sin \tilde{\alpha}) \dot{\tilde{p}}^2 - 2p \tilde{p} \cos \tilde{\alpha} + (\phi - \sin \tilde{\alpha}) \ddot{p}^2 \bigg| \alpha = \tilde{\alpha} = 0. \quad (3.6) \]

4. **Existence and Uniqueness for a Related Problem.**

4.1. We want to consider the question of the existence and uniqueness of solutions of the equation

\[ M(p) = f \]

subject to the boundary conditions (3.3), (3.5), where \( M \) is given by (3.1). It is more convenient however to abstract the key features of the equation and the boundary conditions and work in a more general setting. This has the advantage of making clear exactly which properties are important in reaching our conclusions. It is also important since the procedure may work in other cases as well. Later in section 5 we will show that the specific problem of interest falls under this scheme.
4.2. Consider the boundary value problem

\[ \ddot{z} = \Phi(t, z, z'), \quad \dot{z}(a) = 0, \quad \dot{z}(b) = 0, \quad A\dot{z}(b) + Bz(b)\ddot{z}(b) + C\dot{z}(b) = 0, \quad A, B, C \in \mathbb{R}. \]  

We shall assume that \( \Phi(t, z, z') \)

(i) is continuous on \([a, b) \times \mathbb{R}^2\)

(ii) satisfies a local Lipschitz condition on \( \mathbb{R}^2 \)

(iii) has the property \( \Phi(t, cz, cz') = c\Phi(t, z, z') \)

for \( c \in \mathbb{R} \).

For the moment we take \( G = \mathbb{R}^2 \) and discuss the modifications for a less extensive domain in 4.6. (i), (ii) guarantee local existence and uniqueness for initial value problems [5]. The question of existence and uniqueness of (4.1) - (4.3) will be investigated using a polar coordinate technique [5], [1], [12]. For these purposes it is convenient to change to a system before making the polar coordinate transformation. Rewrite (4.1) - (4.3) as

\[ \begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \Phi(t, z_1, z_2), \\
z_2(a) &= 0,
\end{align*} \]

\[ A\dot{z}_1(b) + Bz_1(b)z_2(b) + Cz_2(b) = 0. \]
4.3. Now make the change to polar coordinates by putting

\[ R^2(t) = z_1^2(t) + z_2^2(t), \quad R(t) \geq 0, \]

\[
\theta(t) = \begin{cases} 
\tan^{-1}\left(\frac{z_2(t)}{z_1(t)}\right), & z_1(t) \neq 0, \\
\text{by continuity elsewhere.} & 
\end{cases}
\]

Since \( z_1(t) = R(t)\cos\theta(t), \ z_2(t) = R(t)\sin\theta(t) \) one easily obtains the differential equations for \( R(t) \) and \( \theta(t) \) as

\[
\dot{R} = R \sin\theta \cos\theta + \Phi(t, R \cos\theta, R \sin\theta) \sin\theta,
\]

\[
\dot{\theta} = -R \sin^2\theta + \Phi(t, R \cos\theta, R \sin\theta) \cos\theta.
\]

The power of hypothesis (iii) is now apparent, for if \( R(t) \neq 0 \), \( (R(t) \neq 0 \) is always a solution) the above takes the form

\[
\dot{R} = R[\sin\theta \cos\theta + \Phi(t, \cos\theta, \sin\theta) \sin\theta], \quad (4.7)
\]

\[
\dot{\theta} = -\sin^2\theta + \Phi(t, \cos\theta, \sin\theta) \cos\theta. \quad (4.8)
\]

It is the form of the right-hand side of (4.7) and (4.8) which plays an important role in the existence and uniqueness questions. The boundary condition (4.5) can be expressed as

\[
\theta(a) = k\pi, \ k = 0, \pm 1, \pm 2, \ldots \quad (4.9)
\]

(4.8), (4.9) form a (well posed) initial value problem, so the function \( \theta = \theta(t) \) is uniquely determined (and may be regarded as a known function on \((a, b)\)) as long as \((z_1, z_2) \in G\.) \ (Note that \( \dot{\theta}(t) \) is bounded on \([a, b'] \times \{[0,1] \times [0,1] \cap G\}, b' < b, \ so \ solutions \ can \ be \ continued.) \ Any
solution of the (4.1) (4.2) will yield this same function \( \theta(t) \) -- thus, generally, (4.1)-(4.3) is an overdetermined problem. Moreover, the righthand side of (4.7) contains \( R \) as a factor and, given \( \theta(t) \), \( R(t) \) may be determined uniquely with \( R(a) = R_0 \) by solving (4.7). From the form of (4.7) all solutions of it can be continued to any interval where \( \theta(t) \) is defined. In particular, if a solution of (4.1)-(4.3) does exist, any constant times this solution is again a solution. Uniqueness is possible only within the class of constant multiples. (This could also have been observed directly from (4.1)-(4.3).)

4.4. The next task is to determine whether or not any nontrivial solutions exist. The boundary condition (4.6) is rewritten as

\[
R(b)[A \cos^2(\theta(b)) + B \sin(\theta(b)) \cos(\theta(b)) + C \sin^2(\theta(b))] = 0
\]
or

\[
R(b)[A + B \tan(\theta(b)) + C \tan^2(\theta(b))] = 0. \tag{4.10}
\]

It is convenient at this point to think of \((R(b), \theta(b))\) as a point on the phase plane. We see that the locus of points given by (4.10) is a point, a ray, or two rays as \( A = 0, \quad B^2 - 4AC = 0, \quad B^2 - 4AC > 0, \quad (B^2 - 4AC < 0 \text{ forces } R = 0) \).

Let \( \theta_1, \theta_2 \) denote the (possible) two real roots of (4.10), i.e.

\[
\theta_i(b) = \tan^{-1} \left( \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right), \quad i = 1, 2, \quad -\pi < \theta_i \leq \pi.
\]

Then one can write an explicit condition for a nontrivial solution of (4.1)-(4.2) to satisfy, namely that for \( \theta_i \) \( (i = 1 \text{ or } 2) \)

\[
\theta_i + 2k\pi = \int_a^b \{-\sin^2(\theta(t)) + \Phi(t, \cos(\theta(t)), \sin(\theta(t)))\cos(\theta(t))\} \, dt, \quad k = 0, 1, \ldots \tag{4.11}
\]
where $\theta(t)$ is uniquely determined by (4.8) (4.9). Conversely, if $\phi$ is such that the solution of (4.8), (4.9) or (4.6) on $[a,b)$ satisfies (4.11) (where the integral may be improper) then there exists a nontrivial solution of (4.1) - (4.3). If $b$ is regarded as a parameter, then $b$ must be chosen so that (4.11) is satisfied.

4.5. Finally we note that the principal difficulty comes from the second boundary condition. If, for example, (4.3) were of the form

$$\dot{z}(b) - g(z(b)) = 0 \text{ where } \ddot{g} > 0 \text{ or } \ddot{g} < 0 \text{ then (see figure 4.1)}$$

![Figure 4.1](image)

there would be a unique nontrivial solution for an interval of values of the parameter $b$. To make this clear, suppose we seek to solve (4.1) (4.2) and

$$\dot{z}(b) - g(z(b)) = 0$$

(4.12)

where $\phi$ satisfies (i), (ii), and (iii), $g$ satisfies $g(0) = 0$, and the curve $z_2 = g(z_1)$ intersects the ray $\theta = \text{constant}$ in exactly one point other than the origin. A convex or a concave curve is sufficient ($\ddot{g} > 0$ or $\ddot{g} < 0$) for this geometric property to hold. Let

$$\Gamma = \{\lambda | \theta = \delta \text{ and } z_2 = g(z_1) \text{ have an intersection } \neq \{0,0\}\}.$$
\( \Gamma \) is an open set and it is these values of \( \lambda \) for which a unique nontrivial solution may be possible. We give an algorithm for constructing a solution.

Convert (4.1) to the polar coordinate system (4.7) and (4.8).

Determine the set \( \Gamma \) by solving the equations \( \theta = \delta \left( \text{i.e. } \tan^{-1} \frac{z_2}{z_1} = \delta \right) \) and \( z_2 = g(z_1) \). Equivalently intersect the line \( z_2 = (\tan \delta)z_1 \) and \( z_2 = g(z_1) \).

One must take care to assign the proper angle to \( \delta \) (the upper half plane corresponds to \( \delta > 0 \) and the lower to \( \delta < 0 \)). Let \( R_1 \) denote the polar radius of the intersection point. Solve the initial value problem (4.8) (4.9) determining a real number \( \theta_1 = \theta(b) \). If \( \theta_1 \notin \Gamma \) a solution is possible (in fact, for all \( \theta \) in a neighborhood of \( \theta_1 \)). Given \( \theta(t) \) solve (4.7) with the initial condition \( R(a) = 1 \) determining a function \( R(t) \). A unique nontrivial solution of the boundary value problem (4.1) (4.2) and (4.10) is given by

\[
 z(t) = \frac{R_1}{R(b)} R(t) \cos \theta(t). \]

4.6. The roles of \( a \) and \( b \) may be interchanged. Putting \( \dot{z}(a) = 0 \) is the natural condition for radially symmetric solutions of the partial differential equation but if there is difficulty defining the function \( \Phi \) at that point, reversing the order of the conditions may help. If continuity fails on the closed interval \([a,b]\) one can use \([a,b]\) and a limiting process. Moreover, it is clear from the technique that the domain \( G \) need not be all of \( \mathbb{R}^2 \). Indeed, in some problems (the reflector problem among them) only positive solutions are of interest. Let \( \theta = \theta, \theta = \pi/2 \) denote two
rays in $\mathbb{R}^2$ and $\Sigma$ the closed section between them. If $\bar{G} = \Sigma$ and if solutions which begin in $\Sigma$ at $t = a$ remain in $\Sigma$ until $t = b$ or $\theta \in \partial \Sigma$ then everything sketched above will work. For example, let $\phi$ be continuous on

$[a, b) \times \mathbb{R}_+ \times \mathbb{R}$. Then since $\dot{\theta} < 0$ at $\pi/2$ and $-\pi/2$ solutions cross "into" the right half plane at values of $t$ such that $\theta(t) = \pi/2$ and leave at $\theta(t) = 3\pi/2$. Thus $G$ can be chosen as $\mathbb{R}_+ \times \mathbb{R}$.

Since $\theta(t)$ is determined by (4.8), (4.9), $R(t)$ can be written

$$R(t) = R(a) \exp \{ \int_a^t \left[ \sin \theta(\tau) \cos \theta(\tau) + \phi(\tau, \cos \theta(\tau), \sin \theta(\tau) \sin \theta(\tau)) \right] d\tau \}.$$  


5.1. We now translate the reflector problem into the general framework of section 4. There are several technical difficulties to be considered.

Equation (3.1) may be written as

$$\frac{[2p\dot{p} + p^2 - \dot{p}^2][-2p\dot{p} \tan \alpha + p^2 - \dot{p}^2]}{(p^2 + \dot{p}^2)^2} = f(\alpha), \quad \alpha \in (\bar{\alpha}, \frac{\pi}{2}).$$  

At this point we assume that $f \in C([\bar{\alpha}, \pi/2])$ and positive. Since $f$ is positive, it is implicit in the derivation that each of the square brackets is never zero if $p^2 + \dot{p}^2 \neq 0$. Let
\( \Delta(\alpha) = \{(p, \dot{p})| p^2 - \dot{p}^2 - 2p\dot{p}\tan\alpha = 0\} \). We will examine the set \( \Delta(\alpha) \) below.

As noted in 1.3.1, \( p \) must be positive in \( \bar{\omega} \). Therefore the solution we seek will not have the denominator on the left side of (5.1) zero. Thus (5.1) may be rewritten as

\[
2p\ddot{p} + p^2 - \dot{p}^2 = \frac{f(\alpha)(p^2 + \dot{p}^2)^2}{p^2 - \dot{p}^2 - 2p\dot{p}\tan\alpha}
\]

Since, again, only \( p > 0 \) is of interest, (5.1) can be written

\[
\ddot{p} = \Phi(\alpha, p, \dot{p}),
\]

where

\[
\Phi(\alpha, p, \dot{p}) = \frac{p^2 - \dot{p}^2}{2p} + \frac{f(\alpha)(p^2 + \dot{p}^2)^2}{2p(p^2 - \dot{p}^2 - 2p\dot{p}\tan\alpha)}.
\]

Clearly \( \Phi \) satisfies the homogeneity property (iii) and is continuously differentiable with respect to \( p \) and \( \dot{p} \) at every point of

\( [\bar{\alpha}, \pi/2) \times (\mathbb{R}^2 - \Delta(\alpha)) \). As noted at the end of section 5, the origin can be avoided in our procedure. The set \( \Delta(\alpha) \) must also be avoided and we deal with this below. The boundary conditions are

\[
(\phi - \psi)\dot{p}^2 + 2n\ddot{p} + (\phi + \psi)p^2\bigg|_{\alpha=\bar{\alpha}} = 0, \text{ (from (3.6))}
\]

and reproducing here (3.3) for easy reference we have

\[
\dot{p}(\pi/2) = 0.
\]

In system form the equation (5.2) is

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \Phi(\alpha, z_1, z_2),
\end{align*}
\]

which corresponds to (4.4). The equation of the polar angle then is
\[ \dot{\theta} = -\frac{1}{2} \frac{f(\alpha)}{2(\sin 2\theta \tan \alpha - \cos \theta)} . \]

5.2. The last equation can be transformed to the following form

\[ \dot{D} \cos D = f \cos \alpha, \quad (5.6) \]

where \( D(\alpha) = 2\theta(\alpha) + \alpha \). (Note that (5.6) can be obtained directly from (5.1) by means of the substitutions \( D = 2\theta + \alpha \) and

\[ \tan \theta = \frac{D}{p}, \quad \alpha \in [0, \frac{\pi}{2}), \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}). \]

This was done for a general equation in section 4.)

The boundary condition (5.4) becomes

\[ \theta(\frac{\pi}{2}) = 0, \quad (5.7) \]

and the equation for the polar radius is

\[ \dot{R} = R(\sin \theta \cos \theta - \frac{\cos 2\theta}{2 \cos \theta} + \frac{f \cos \alpha}{2 \cos \theta \cos (2\theta + \alpha)}). \quad (5.8) \]

5.2. Now we examine the set \( \Delta(\alpha) \). In terms of variables \( \alpha, \theta \) the locus \( \Delta(\alpha) \) is the set

\[ \tan^2 \theta + 2 \tan \theta \tan \alpha - 1 = 0. \]

We are interested in the solutions of this equation only in the rectangle \( Q = [0, \pi/2) \times (-\pi/2, \pi/2) \). These solutions are given by the segments

\[ \theta = -\frac{\pi}{4} - \frac{\alpha}{2}; \quad \theta = \frac{\pi}{4} - \frac{\alpha}{2}. \]

Consequently the rectangle \( Q \) splits into three regions:
\[ Q_1 = \left\{ (\alpha, \theta) \mid 0 \leq \alpha < \frac{\pi}{2}, -\frac{\pi}{2} < \theta < -\frac{\pi}{4} - \frac{\alpha}{2} \right\}, \]

\[ Q_2 = \left\{ (\alpha, \theta) \mid 0 \leq \alpha < \frac{\pi}{2}, -\frac{\pi}{4} - \frac{\alpha}{2} < \theta < \frac{\pi}{4} - \frac{\alpha}{2} \right\}, \]

\[ Q_3 = \left\{ (\alpha, \theta) \mid 0 \leq \alpha < \frac{\pi}{2}, \frac{\pi}{4} - \frac{\alpha}{2} < \theta < \frac{\pi}{2} \right\}. \]

Within each of these regions the standard theorems from the theory of ordinary differential equations guarantee existence and uniqueness of a solution of an initial value problem for (5.6).

5.3. The equation (5.6) is easily integrated and the solution through a point \((\bar{\alpha}, \theta(\bar{\alpha}))\) is implicitly given by

\[
\sin D(\alpha) = \sin D(\bar{\alpha}) + \int_{\bar{\alpha}}^{\alpha} f(\tau) \cos \tau \, d\tau, \tag{5.9}
\]

for \(\alpha\) such that the right hand side has absolute value less than or equal to one. Assume now that the integral \(S(\bar{\alpha}, \alpha)\), where

\[
S(\alpha_1, \alpha_2) = \int_{\alpha_1}^{\alpha_2} f(\tau) \cos \tau \, d\tau, \quad \alpha_1, \alpha_2 \in [\bar{\alpha}, \frac{\pi}{2}],
\]

exists (perhaps as an improper integral) on the interval \([\bar{\alpha}, \pi/2]\).

Suppose the following relation

\[
1 - \sin[2\theta(\bar{\alpha}) + \bar{\alpha}] = S(\bar{\alpha}, \pi/2) \tag{5.10}
\]

is satisfied. Then, since \(S(\bar{\alpha}, \alpha)\) on \([\bar{\alpha}, \pi/2]\) is a monotone function of \(\alpha\) (the integrand is positive),
\[ |\sin D(\alpha) + S(\alpha, \alpha)| \leq 1, \quad \alpha \in [\alpha, \pi/2], \]

and (5.9) defines a solution on \([\alpha, \pi/2]\). Furthermore,

\[ \sin D(\pi/2) = \cos 2\theta(\pi/2) = 1, \]

which implies \(\theta(\pi/2) = 0\) in \(\tilde{Q}\). Thus, (5.7) is also satisfied.

(5.10) can be viewed as a compatibility condition to be satisfied if \(\theta(\alpha)\) is to be chosen, while the initial value of the solution is given at \(\alpha = \pi/2\).

Since \((\pi/2, 0) \notin \tilde{Q}_1\), it cannot support a solution satisfying (5.7).

From this it follows that we have two solutions of (5.6) (5.7), - one in \(Q_2\) and one in \(Q_3\). The choice is determined by \(\theta(\alpha)\).

Note that the function \(f\cos \alpha/cos D\) is discontinuous at \((\pi/2, 0)\) and though the two solutions in \(\tilde{Q}_2 \cup \tilde{Q}_3\) represent two branches of a continuous curve in \((\alpha, \theta)\) - plane, this curve is not differentiable at \((\pi/2, 0)\), even if the

\[ \lim_{\alpha \to \pi/2} f(\alpha) \]

exists. This fact has important consequences for numerical solution of (5.6), (5.7). In particular, if the numerical solution is initiated at \((\pi/2, 0)\) additional information should be given specifying which of the two solutions is to be found.

5.4. As in section 4 we now need to consider the boundary condition (5.3).

Using (3.6) we rewrite (5.3) in the form

\[ (\phi + \sin \alpha) \tan^2 \theta(\alpha) - 2 \tan \theta(\alpha) \cos \alpha + (\phi - \sin \alpha) = 0. \quad (5.3)' \]

From this

\[ \tan \theta(\alpha) = \frac{\cos \alpha \pm \sqrt{1 - \phi^2}}{\phi + \sin \alpha}. \]
Using proposition 3.5 we find the equations of \( \theta \Omega \), namely,

\[ \theta \Omega = \{ \tilde{\alpha} = -\pi/2 + \arccos \phi, \ 0 \leq \beta \leq 2\pi \}. \]

Then

\[
\tan \theta(\tilde{\alpha}) = \frac{\cos \tilde{\alpha} \pm \sin(\pi/2 + \tilde{\alpha})}{\cos(\pi/2 + \tilde{\alpha}) + \sin \tilde{\alpha}}
\]

\[
= \begin{cases} 
\cot \frac{\tilde{\alpha} - \tilde{\alpha}}{2} & \text{if the sign "+" is taken,} \\
-\tan \frac{\tilde{\alpha} + \tilde{\alpha}}{2} & \text{if the sign "-" is taken.}
\end{cases}
\]

Since \( \tilde{\alpha} \in (0, \pi/2) \), \( \tilde{\alpha} \in (-\pi/2, 0) \), and \( \theta(\tilde{\alpha}) \) must be the ordinate of a point in \( Q_2 \cup Q_3 \), we conclude with the use of (5.10) that for a fixed \( \tilde{\alpha} \) the value of \( \tilde{\alpha} \) should be chosen so that

\[ 1 + \sin \tilde{\alpha} = S(\bar{\alpha}, \pi/2). \]

Then for \( \theta(\tilde{\alpha}) \) we should take either

\[ \theta(\bar{\alpha}) = -\frac{\bar{\alpha}}{2} - \frac{\tilde{\alpha}}{2} \text{ for } Q_2 \]

(5.11)

or

\[ \theta(\bar{\alpha}) = \frac{\pi}{2} - \frac{\bar{\alpha}}{2} + \frac{\tilde{\alpha}}{2} \text{ for } Q_3. \]

(5.12)

We summarize the results of this section in the following theorem.

5.5. **Theorem.** Let \( f(\alpha) \) be positive and continuous on \([0, \pi/2]\) and the integral \( S(0, \pi/2) \) exists. Further, let \( \tilde{\alpha} \) be any number in the interval \((-\pi/2, 0)\) and \( \bar{\alpha} \) be the solution of the equation

\[ 1 + \sin \bar{\alpha} = S(\bar{\alpha}, \pi/2), \ \bar{\alpha} \in (0, \pi/2). \]
Then for each choice of \( \theta(\tilde{a}) \) according to (5.11) or (5.12) there exists a unique solution of class \( C^2([\tilde{a}, \pi/2]) \cap C([\tilde{a}, \pi/2]) \) of the boundary value problem (5.6), (5.7), and (5.3)'.

5.5.1. Remark. An obvious sufficient condition on \( f \) for the equation
\[
1 + \sin \tilde{a} = S(\tilde{a}, \pi/2)
\]
to have a solution \( \tilde{a} \in (0, \pi/2) \) is \( S(0, \pi/2) \geq 1 \).

5.5.2. Remark. Observe that construction of the map \( m \) given by (1.19) requires the knowledge of \( p/p \) only, and, therefore, \( m \) can be constructed from \( \tan \theta \). Furthermore, the two solutions correspond to maps for reflectors without real caustics or with two real caustics respectively. Details on the physics of this situation can be found in [13], Ch. 3.

5.6. Our ultimate objective in this section is to construct a function \( p \) satisfying (5.1), (5.3), and (5.4) and then show that a reflector surface indeed can be described in terms of the function \( \tilde{p}(\alpha, \beta) = p(\alpha), \alpha \in [\tilde{a}, \pi/2], \beta \in [0, 2\pi] \). For that it is necessary to show that \( p \) can be recovered from \( \theta \), that
\[
p > 0 \text{ in } \tilde{\omega} = \{(\alpha, \beta) | \tilde{a} \leq \alpha \leq \pi/2, 0 \leq \beta \leq 2\pi\}, \quad \tilde{p} \in C^2(\omega) \cap C^1(\omega),
\]
and that the map \( m \) is a diffeomorphism. We begin by establishing further smoothness of \( \theta \).

5.7. Lemma. Suppose the conditions of the Theorem 5.5 are satisfied and \( \theta(\alpha), \alpha \in [\tilde{a}, \pi/2], \) is the function defined by (5.9) and either (5.11) or (5.12). Further, let \( f \in C([\tilde{a}, \pi/2]) \). Then \( \theta \in C^1([\tilde{a}, \pi/2]) \).
5.7.1. **Proof.** From standard results on ordinary differential equations it follows that \( \theta \in C^1[\bar{\alpha}, \pi/2] \). Thus, we need to consider only the point \( \alpha = \pi/2 \).

Since \( \theta(\bar{\alpha}) \) is given by (5.11) or (5.12), \( \sin D(\bar{\alpha}) = 1 - S(\bar{\alpha}, \pi/2) \).

Then from (5.9)

\[
\sin D(\alpha) = 1 - S(\alpha, \frac{\pi}{2}).
\]

By the mean value theorem

\[
S(\alpha, \frac{\pi}{2}) = f(\tau(\alpha))(1 - \sin \alpha),
\]

where \( \tau(\alpha) \in [\alpha, \pi/2] \). Then evidently

\[
\sin D(\alpha) = 1 - f(\tau(\alpha))(1 - \sin \alpha).
\]

In a left hand side neighborhood of \( \pi/2 \) we have

\[
1 - \sin \alpha = \frac{1}{2} \left( \frac{\pi}{2} - \alpha \right)^2 + o(A),
\]

\[
\sin D(\alpha) = 1 - \frac{1}{2} f(\tau(\alpha)) \left( \frac{\pi}{2} - \alpha \right)^2 + o(A),
\]

and

\[
1 - \sin^2 D(\alpha) = f(\tau(\alpha)) \left( \frac{\pi}{2} - \alpha \right)^2 + o(A),
\]

where

\[
A = o(\left[ \frac{\pi}{2} - \alpha \right]^2).
\]

From the equation (5.6), satisfied by \( D(\alpha) \), we have for \( \alpha < \pi/2 \)

\[
[D(\alpha)]^2 = f^2 \frac{\cos^2 \alpha}{\cos^2 D(\alpha)} = f^2 \frac{\left( \frac{\pi}{2} - \alpha \right)^2 + o(A)}{f(\tau(\alpha)) \left( \frac{\pi}{2} - \alpha \right)^2 + o(A)}.
\]

Thus, \( \dot{\theta}(\pi/2) \) exists, and evidently,
\[ \dot{\theta}(\pi/2) = -\frac{1 \pm \sqrt{f(\pi/2)}}{2}, \]

which corresponds to two possible solutions according to the Theorem 5.5.

5.8. To recover \( p(\alpha) \) and \( \dot{p}(\alpha) \) it is necessary to determine \( R(\alpha) \) (see section 4). Let \( \theta(\alpha) \) be determined as above. Then from (5.8) \( R(\alpha) \) is given by

\[ R(\alpha) = R(\overline{\alpha}) \exp \left[ \int_{\alpha}^{\overline{\alpha}} \left[ \frac{\cos \theta(\tau) - \cos 2\theta(\tau)}{2\cos \theta(\tau)} \right] d\tau \right], \]

provided the integrand is defined and bounded.

Since \( \theta(\tau) \) is continuous on \([\overline{\alpha}, \pi/2]\), the first term in the square bracket is continuous. The only possible difficulty with the second term occurs if

\[ \theta(\alpha) = \pm \pi/2 \text{ for some } \alpha \in (\overline{\alpha}, \pi/2). \]

The solution \( \theta(\alpha) \) lies in one of the regions \( Q_2 \) or \( Q_3 \). In \( Q_3 \) \( \dot{\theta}(\alpha) < 0 \), so \( \theta(\alpha) \) is decreasing there and therefore cannot take on the value \( \pi/2 \). If \( \theta(\overline{\alpha}) \in Q_2 \) then it is bounded away from \(-\pi/2\) by \( Q_1 \) and from \( \pi/2 \) by \( Q_3 \). Thus, along a solution \( \theta(\alpha) \neq \pm \pi/2 \), and the second term in square brackets is continuous.

The remarks in the preceding paragraph apply to the third term as well. Difficulties then center potentially about \( \cos(2\theta(\alpha) + \alpha) = 0 \). This cannot occur in \([\overline{\alpha}, \pi/2]\) since \( 2\theta(\alpha) + \alpha = \pm \pi/2 \) represent the boundaries of the regions \( Q_1 \) and we have shown that if \( (\overline{\alpha}, \theta(\overline{\alpha})) \) is chosen properly the solution remains in \( Q_2 \) or \( Q_3 \). Finally as \( \alpha \to \pi/2 \), we have shown that

\[ \lim_{\alpha \to \pi/2} \cos \alpha / \cos(2\theta(\alpha) + \alpha) \]
exists. Hence the quantity under the exp sign is continuous on \([\tilde{\alpha}, \pi/2]\), and
\[ R(\alpha) > 0 \text{ if } R(\tilde{\alpha}) > 0. \]

Since
\[ \dot{\tilde{\alpha}} = (R(\alpha) \sin \theta(\alpha))' = \tilde{R}(\alpha) \cos \theta(\alpha) + R(\alpha) \sin \theta(\alpha) \dot{\theta}(\alpha), \]
it follows from the above discussion that \(\dot{\tilde{\alpha}}\) is continuous on \([\tilde{\alpha}, \pi/2]\). It is important to emphasize that the regularity is established only for the solution where \(\theta(\alpha)\) is chosen as outlined in sections (5.2) - (5.5). Other choice of the initial conditions may lead to solutions of (5.6) which have an unbounded second derivative or which do not satisfy the boundary condition at \(\alpha = \pi/2\).

5.9. **Lemma.** Let the function \(p\) be as in 5.8, \(p(\alpha, \beta) = p(\alpha)\), \(r : \tilde{\omega} \to \mathbb{R}^3\) the map defined by (1.10), and \(m : \tilde{\omega} \to \tilde{\Omega}\) the map defined by (1.19). Then \(r\) defines an imbedded surface which projects radially one-to-one onto \(\tilde{\Omega}\) and the map \(m\) is a diffeomorphism. In addition, \(r(\tilde{\omega})\) is a surface of revolution.

5.9.1. **Proof.** The Jacobian of the map \(r\) will have rank equal to two if
\[
\frac{\det(g_{i j})}{\det(e_{i j})} = \frac{\det < \partial_i r, \partial_j r >}{\det(e_{i j})} \neq 0 \text{ in } \tilde{\omega}.
\]
In the notation of section 1.4 we have
\[
\frac{\det(g_{i j})}{\det(e_{i j})} = \frac{\det[V_{i j} p + (p - \rho)e_{i j}]}{p^2 |\nabla \rho|^2} \frac{p^2 + |\nabla \rho|^2}{p^2 \det(e_{i j})^2} = f^2 \frac{(p^2 + |\nabla \rho|^2)^2}{16 p^6} \neq 0.
\]
The same computation, in essence, shows that the Jacobian of the map \(m\) also has rank 2 (see section 1.4.1).
Let us show now that \( r(\bar{\omega}) \) is a surface which projects univalently on \( \bar{\Omega} \). This will imply that \( m \) is a \( C^1 \) diffeomorphism. Suppose on the contrary that a ray of direction \( d \in \bar{\Omega} \) intersects \( r(\bar{\omega}) \) at two or more points. Then, evidently, there exists a point \( \bar{u} \in \bar{\omega} \) such that \( \langle r(\bar{u}), n(\bar{u}) \rangle = 0 \), where \( n(\bar{u}) \) is the normal vector to the surface \( r \) at the point \( \bar{u} \). But from (1.9), (1.10) and (1.11) we have

\[
- \langle r, n \rangle = \frac{N^2 - pp}{\sqrt{|\nabla p|^2 + p^2}} = \frac{\sqrt{|\nabla p|^2 + p^2}}{2} \neq 0 \text{ in } \bar{\omega}.
\]

The last statement follows from 3.4.2. The lemma is proved.

5.10. Combining the assertions of Theorem 5.5 and Lemmas 5.6, 5.9 we have the following

**Theorem 5.10.** Let \( f(\alpha) \) be a positive continuous function on \([0, \pi/2]\), and \( \bar{\alpha} \) any number in \((-\pi/2, 0)\). Assume that there exists \( \bar{\alpha} \) satisfying the equation

\[
1 + \sin \bar{\alpha} = S(\bar{\alpha}, \pi/2), \quad \bar{\alpha} \in (0, \pi/2).
\]

Then there exist two reflector surfaces described by the maps \( r \) constructed from functions \( p_1 \) and \( p_2 \) defined as in section 5.8. These reflector surfaces solve the inverse reflector problem stated in 1.5 for the prescribed aperture \( \partial \Omega \), target domain \( \bar{\omega} \), and intensity \( f \). The function \( p \) (\( p_1 \) or \( p_2 \)) of which the reflector surface \( r: \bar{\omega} \to \mathbb{R}^3 \) is constructed is of class \( C^2(\bar{\omega}) \).

6.1 The preceding discussion concludes the development of a radially symmetric reflector. To emphasize the constructive nature of the solution we reiterate the steps. The aperture is fixed by specifying $\bar{a}$ and the intensity is given by a function $f(\alpha)$. The constant $\bar{a}$ is fixed by solving

$$1 + \sin \bar{a} = S(\bar{a}, \pi/2).$$

Recall that this is merely choosing the lower limit of an integral where the integrand is a known function, so this part of solution is not difficult. The choice of $\bar{a}$ restricts the target size to be illuminated. An initial condition for the function $\theta$ is given by (5.11) or (5.12), depending on the solution desired. With this initial condition the equation (5.6) can be solved. More directly, one can use $D(\alpha)$ given by (5.9) and compute $\theta(\alpha)$ from $D(\alpha) = 2\theta(\alpha) + \alpha$. $\theta(\alpha)$ is in $C^1([\bar{a}, \pi/2])$ and $R(\alpha) \in C^1([\bar{a}, \pi/2])$ and can be obtained by a quadrature. The solution to the original problem is given by

$$p(\alpha) = R(\alpha)\cos \theta(\alpha),$$

$$\dot{p}(\alpha) = R(\alpha)\sin \theta(\alpha),$$

and $p$ has been shown to be in $C^2([\bar{a}, \pi/2])$. Given $p(\alpha)$ the reflector is described by first extending $p$ to a function of two variables in a radially symmetric way, and then writing out the expression (1.10) for the position vector $r$ of the reflector surface.

7.1. In order to show existence of nonradially symmetric solutions for intensities varying in latitudinal and longitudinal directions we need first to establish further smoothness of the solutions of the radially symmetric problem.

7.2. Lemma. Suppose \( f \) is positive and continuous on \([0,\pi/2]\), \( \tilde{\alpha} \) any number on \((-\pi/2,0)\) and \( \tilde{\alpha} \in (0,\pi/2) \) a solution to the equation

\[
1 + \sin \tilde{\alpha} = S(\tilde{\alpha}, \pi/2)
\]

(\( \tilde{\alpha} \) is assumed to exist). Assume further that \( f \in C^1([\tilde{\alpha}, \pi/2]) \). Then the function \( \theta(\alpha), \alpha \in [\tilde{\alpha}, \pi/2] \), defined by (5.9) and either (5.11) or (5.12) is of class \( C^2([\tilde{\alpha}, \pi/2]) \).

7.2.1. Proof. We preserve here all notation from 5.7 and 5.7.1. The function \( D(\alpha) = 2\theta(\alpha) + \alpha \) satisfies the equation (5.6) from which by differentiation at \( \alpha < \pi/2 \) we obtain

\[
\ddot{D}\cos D = \dot{D}^2 \sin D \cdot f \sin \alpha + f \cos \alpha.
\]

We rewrite this as

\[
\ddot{D} = \frac{\dot{D}^2 (\sin D - \sin \alpha)}{\cos D} + \frac{(D^2 - f) \sin \alpha + f \cos \alpha}{\cos D}.
\]  

(7.1)

On the interval \([\tilde{\alpha}, \pi/2]\) \( \ddot{D} \) exists, as it follows from standard results on ordinary differential equations. Thus, we have to consider the point \( \alpha = \pi/2 \).
Repeating the argument in the beginning of 5.7.1 we find that

\[ \sin D(\alpha) = 1 - f(\tau(\alpha))(1 - \sin \alpha), \quad \tau(\alpha) \in [\tilde{a}, \pi/2]. \]

In a left hand side neighborhood of \(\pi/2\) we have

\[ \sin \alpha = 1 - \frac{1}{2} (\alpha - \frac{\pi}{2})^2 + o(A) \]

and

\[ \sin D(\alpha) - \sin \alpha = \frac{1}{2} [1 - f(\tau(\alpha)](\alpha - \frac{\pi}{2})^2 + o(A). \]

By Cauchy's theorem

\[ \cos D(\alpha) = \dot{D}(\alpha')\sin D(\alpha')(\frac{\pi}{2} - \alpha), \quad \alpha' \in (\alpha, \pi/2). \]

By Lemma 5.7 \(\dot{D}^2(\alpha) \to f(\pi/2)\) when \(\alpha \to \pi/2\). Thus,

\[ \lim_{\alpha \to \pi/2} \frac{\dot{D}^2(\sin D - \sin \alpha)}{\cos D} = 0. \]

Now we need to consider the second term on the right of (7.1).

We record some expansions at \(\pi/2\) that will be used later:

\[\cos \alpha = - (\alpha - \frac{\pi}{2}) + \frac{1}{3!} (\alpha - \frac{\pi}{2})^3 + o(|\alpha - \frac{\pi}{2}|^4);\]

\[\sin \alpha = 1 - \frac{1}{2} (\alpha - \frac{\pi}{2})^2 + \frac{1}{4!} (\alpha - \frac{\pi}{2})^4 + o(|\alpha - \frac{\pi}{2}|^5);\]

\[\sin D(\alpha) = 1 - f(\tau(\alpha))(1 - \sin \alpha)\]

\[= 1 - f(\tau(\alpha)) \left[ \frac{1}{2} (\alpha - \frac{\pi}{2})^2 - \frac{1}{4!} (\alpha - \frac{\pi}{2})^4 + o(|\alpha - \frac{\pi}{2}|^5) \right];\]

\[1 - \sin^2 D(\alpha) = f(\tau(\alpha))(\alpha - \frac{\pi}{2})^2 + O(|\alpha - \frac{\pi}{2}|^4);\]

\[f(\alpha) = f\left(\frac{\pi}{2}\right) + \tilde{f}(c)(\alpha - \frac{\pi}{2}), \quad c \in (\alpha, \pi/2).\]
From the equation (5.6) we have for \( \alpha < \pi/2 \)

\[
\hat{D}^2 = f^2 \frac{\cos^2 \alpha}{\cos^2 D} = f^2 \frac{\cos^2 \alpha}{1 - \sin^2 \alpha} = f^2 \frac{(\alpha - \frac{\pi}{2})^2 - \frac{1}{3}(\alpha - \frac{\pi}{2})^4 + o(|\alpha - \frac{\pi}{2}|^5)}{f(\tau(\alpha))(\alpha - \frac{\pi}{2})^2 + O(|\alpha - \frac{\pi}{2}|^4)}
\]

Then

\[
(\hat{D}^2 - f)\sin \alpha + \hat{\cos} \alpha = \left( \frac{f^2}{f(\tau(\alpha))} - f \right) \sin \alpha + \hat{\cos} \alpha + O(|\alpha - \frac{\pi}{2}|^2).
\]

Further,

\[
\frac{f^2}{f(\tau(\alpha))} - f = \frac{f^2 - f(\tau(\alpha))f}{f(\tau(\alpha))} =
\]

\[
\frac{f^2(\frac{\pi}{2}) + 2f(\frac{\pi}{2})\hat{\xi}(c)(\alpha - \frac{\pi}{2}) + \hat{\xi}^2(c)(\alpha - \frac{\pi}{2})^2 - f(\tau(\alpha))[f(\frac{\pi}{2}) + \hat{\xi}(c)(\alpha - \frac{\pi}{2})]}{f(\tau(\alpha))}
\]

\[
= \frac{f^2(\frac{\pi}{2}) - f(\tau(\alpha))f(\frac{\pi}{2})}{f(\tau(\alpha))} + \frac{2f(\frac{\pi}{2})\hat{\xi}(c) - f(\tau(\alpha))\hat{\xi}(c)}{f(\tau(\alpha))}(\alpha - \frac{\pi}{2})^2,
\]

and

\[
(\hat{D}^2 - f)\sin \alpha + \hat{\cos} \alpha = \frac{f(\frac{\pi}{2})}{f(\tau(\alpha))} \left[ f(\frac{\pi}{2}) - f(\tau(\alpha)) \right] 1 - \frac{1}{2}(\alpha - \frac{\pi}{2})^2 +
\]

\[
\frac{2f(\frac{\pi}{2})\hat{\xi}(c) - f(\tau(\alpha))\hat{\xi}(c) - f(\tau(\alpha))\hat{\xi}(\alpha)}{f(\tau(\alpha))} (\alpha - \frac{\pi}{2})^2 + A.
\]

Then

\[
\frac{(\hat{D}^2 - f)\sin \alpha + \hat{\cos} \alpha}{\cos D} = \frac{f(\frac{\pi}{2})[f(\frac{\pi}{2}) - f(\tau(\alpha))] + 2f(\frac{\pi}{2})\hat{\xi}(c) - f(\tau(\alpha))\hat{\xi}(c) - f(\tau(\alpha))\hat{\xi}(\alpha)}{f(\tau(\alpha))\hat{D}(\alpha') \sin D(\alpha')} + O(|\alpha - \frac{\pi}{2}|)
\]
Note that the limit of the second term on the right exists and equal to zero when $\alpha \to \pi/2$.

We have, taking into account that

$$S(\alpha, \pi/2) = f(\tau(\alpha))(1 - \sin \alpha),$$

$$f(\frac{\pi}{2}) - f(\tau(\alpha)) = \frac{(f(\frac{\pi}{2}) - f(\tau(\alpha)))(1 - \sin \alpha)}{1 - \sin \alpha}$$

$$\int_\alpha^{\pi/2} \frac{[f(\frac{\pi}{2}) - f(\tau)] \cos \tau d\tau}{1 - \sin \alpha} = \frac{\pi}{2} \frac{\dot{f}(c) \int_\alpha^{\pi/2} \frac{\pi}{2} - \tau \cos \tau d\tau}{1 - \sin \alpha} = -\frac{2}{3} \frac{\dot{f}(c)(\alpha - \pi/2) + o(A)}{1 - \sin \alpha}.$$

Finally,

$$\lim_{\alpha \to \pi/2} \frac{(\dot{D}^2 - f) \sin \alpha + \dot{f} \cos \alpha}{\cos D} = \frac{2}{3} \frac{\dot{f}(\pi/2)}{D(\pi/2)},$$

and, applying Lemma 5.7, we conclude that

$$\lim_{\alpha \to \pi/2} \dot{b} = \pm \frac{2}{3} \frac{\dot{f}(\pi/2)}{\sqrt{f(\pi/2)}}.$$

Recalling that $D(\alpha) = 2\theta(\alpha) + \alpha$, we find that

$$\theta(\pi/2) = \pm \frac{1}{3} \frac{\dot{f}(\pi/2)}{\sqrt{f(\pi/2)}}.$$

7.2.2. Corollary. The function $p(\alpha)$ constructed from $\theta(\alpha)$ as in 5.8 is of class $C^3([\bar{\alpha}, \pi/2])$ provided the conditions of the Lemma 7.2 are satisfied.

7.3. Let $\omega$ and $\Omega$ be circular domains as in sections 3 and 5 and assume that conditions of the Theorem 5.10 are satisfied with additional assumption that $f \in C^1([\bar{\alpha}, \pi/2])$. Fix any one of the two solutions $\theta$ whose existence is guaranteed by Theorem 5.5 and denote by $\bar{p}$ the corresponding solution of the
equation (5.1) satisfying (5.3), (5.4). By the above corrolary \( \tilde{p} \in C^3([\alpha, \pi/2]) \) and consequently, \( \tilde{p}(\alpha, \beta) \equiv \tilde{p}(\alpha), \quad \alpha \leq \alpha \leq \pi/2, \ 0 \leq \beta \leq 2\pi, \) is of class \( C^3(\bar{\omega}) \).

It will be convenient to denote by \( \tilde{f} \) the light intensity of the r.s. solution \( \tilde{p} \), that is, \( M(\tilde{p}) = \tilde{f} \).

7.4. **Theorem.** Suppose \( \omega, \Omega, \) and \( \tilde{f} \) are as in Theorem 5.10 and, in addition, \( \tilde{f} \in C^1([\bar{\alpha}, \pi/2]) \). For \( \delta \in (0,1) \) put

\[
H = \{ f \in C^0,\delta(\omega) \mid \int_{\omega} f d\sigma = 2\pi \int_{\bar{\alpha}}^{\pi/2} \tilde{f}(\tau) \cos \tau d\tau \}.
\]

Assume further that the radius (geodesic) of the domain \( \Omega \) is such that

\[
\phi = - \sin \bar{\alpha}.
\]  

(7.2)

Then there exists an \( \epsilon > 0 \) such that for any \( f \in H \), \( \| f - \tilde{f} \|_{C^0,\delta(\omega)} < \epsilon \) the equation

\[
M(p) = f \text{ in } \omega,
\]  

(7.3)

with the boundary condition

\[
\langle m, y_1 \rangle \bigg|_{\partial \omega} = \phi
\]  

(7.4)

admits two classes of solutions of class \( C^2,\delta(\bar{\omega}) \). Within each of the classes the solution is determined uniquely up to a multiplicative constant. In (7.4), as before, \( m \) is defined by (1.19) and \( y_1 \) is the South Pole of the sphere \( S \) - the center of \( \Omega \).

7.4.1. **Proof.** It involves several steps. First of all we fix (up to a multiplicative constant) one of the r.s. solutions corresponding to \( \tilde{f} \) and denote it, as in 7.3, by \( \bar{p} \). Everything below works the same way for the other class of r.s. solutions.
7.4.2. Let
\[ \zeta = \frac{\partial \bar{p}}{\partial \nu} \bigg|_{\partial \omega} - \frac{\bar{p}}{\bar{p}} \bigg|_{\alpha = \bar{\alpha}}, \]
where \( \nu \) is the interior normal to the boundary \( \partial \omega \), and
\[ \tilde{C}^2, \delta(\bar{\omega}) = \{ p \in C^2, \delta(\bar{\omega}) \mid \frac{\partial p}{\partial \nu} - \zeta p = 0 \}. \]

Obviously, \( \tilde{C}^2, \delta \) is a linear Banach space relative to the norm of \( C^2, \delta(\bar{\omega}) \).

Further, let \( E : \tilde{C}^2, \delta(\bar{\omega}) \to \mathbb{R}, 0 < \delta < 1 \), be a function defined by the formula
\[ E(p) = \int_{\omega} p^2 d\sigma, \quad p \in \tilde{C}^2, \delta(\bar{\omega}), \]
and
\[ M = \{ p \in \tilde{C}^2, \delta(\bar{\omega}) \mid p > 0, E(p) = 1 \}. \]

7.5. Lemma. The set \( M \) is a submanifold of the Banach space \( \tilde{C}^2, \delta(\bar{\omega}) \) modelled on a closed hyperplane of \( \tilde{C}^2, \delta(\bar{\omega}) \).

7.5.1. The proof is standard and we omit it.

7.6. Proof of the Theorem 7.4. By Theorem 5.10 and Corollary 7.2.2 the r.s. solution \( \bar{p} \in C^2, \delta(\bar{\omega}) \), and, since it is defined up to a multiplicative constant, we can normalize it so that \( \bar{p} \in M \).

Let us show now that for any \( p \in \tilde{C}^2, \delta(\bar{\omega}) \) the corresponding map \( m = m_p \), constructed as in (1.19), satisfies (7.4). Indeed, in polar coordinates \( \alpha, \beta \) on \( S \)
\[ m = \frac{2p \left( \frac{\partial p}{\partial \alpha} \frac{\partial y}{\partial \alpha} + \frac{1}{\cos^2 \alpha} \frac{\partial p}{\partial \beta} \frac{\partial y}{\partial \beta} \right) + \left( p^2 - \left( \frac{\partial p}{\partial \alpha} \right)^2 - \frac{1}{\cos^2 \alpha} \left( \frac{\partial p}{\partial \beta} \right)^2 \right)y}{p^2 + \left( \frac{\partial p}{\partial \alpha} \right)^2 + \frac{1}{\cos^2 \alpha} \left( \frac{\partial p}{\partial \beta} \right)^2}. \]
The domain $\omega$ is circular with center at $(0,0,1)$ – the North Pole of $S$. Then clearly, $\langle \partial y/\partial \beta, y_1 \rangle \bigg|_{\partial \omega} = 0$ and, taking into account (7.2), we have

$$\langle m, y \rangle \bigg|_{\partial \omega} = - \frac{2p \frac{\partial r}{\partial \alpha} \langle \frac{\partial y}{\partial \alpha}, y_1 \rangle + \frac{1}{2} \frac{\partial r}{\partial \beta}^2 - \frac{1}{2} \frac{\partial r}{\partial \beta}^2}{p^2 + \frac{\partial r}{\partial \alpha}^2 + \frac{1}{2} \frac{\partial r}{\partial \beta}^2} \langle y, y_1 \rangle$$

$$= - \frac{2p \frac{\partial r}{\partial \alpha} \langle \frac{\partial y}{\partial \alpha}, y_1 \rangle + \frac{1}{2} \frac{\partial r}{\partial \beta}^2 - \frac{1}{2} \frac{\partial r}{\partial \beta}^2}{p^2 + \frac{\partial r}{\partial \alpha}^2 + \frac{1}{2} \frac{\partial r}{\partial \beta}^2} \langle y, y_1 \rangle$$

$$\phi \left[ \frac{-2}{p^2 + \frac{\partial r}{\partial \alpha}^2 + \frac{1}{2} \frac{\partial r}{\partial \beta}^2} \right] = \phi.$$

On the other hand the Jacobian of the map $m_p : \omega \to \Omega$ is given by $M(p)$ (see 1.4.1). Therefore, there exists a neighborhood $U$ of $\vec{p}$ in $\mathcal{M}$ such that for all $p \in U$ $M(p)$ is not zero and $m_p$ covers $\tilde{U}$ univalently. Consequently,

$$\int_{\omega} M(p) d\sigma = \text{area of } \Omega = 2\pi \int_{-\alpha}^{\alpha} \tilde{f}(t) \cos t dt.$$

Then the operator $M$ maps $U \to H$. Clearly this map is continuous and bounded.

Our objective is to apply the Implicit Function Theorem and for that we need to know the kernel of the derivative map $dM : T_{-\mathcal{M}} \to H$, where $T_{-\mathcal{M}}$ is the tangent space to $\mathcal{M}$ at $\vec{p}$. For an arbitrary element $\eta \in T_{-\mathcal{M}}$ and $t \in \mathbb{R}$ we have

$$dM(\vec{p}) \frac{d}{dt} M(p + \eta t) \bigg|_{t = 0} = M(\vec{p}) \frac{d}{dt} \log M(p + \eta t) \bigg|_{t = 0}$$
\[
\begin{align*}
\tilde{\pi} &= \tilde{\pi}^2 \left( \frac{2 \eta}{\eta} + \frac{\text{cof}[\nabla_{ij} \tilde{\pi} + (\tilde{\pi} - \frac{|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij}]}{2\tilde{\pi}} \right) \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \}
\end{align*}
\]

where \( \text{cof}[ \quad ] \) denotes the operation of taking cofactors of the matrix and

\[
\nabla(\tilde{\pi}, \eta) = \hat{e}_{ij} \tilde{\pi} \hat{e}_{ij} \eta.
\]

Since \( \tilde{\pi} \neq 0 \) in \( \tilde{\omega} \), we may introduce a function \( \eta = \frac{\eta}{\tilde{\pi}} \). Then

\[
\begin{align*}
\nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \} \\
&= \chi \{ \nabla_{ij} \eta + \{ \eta - \frac{2\tilde{\pi} \nabla(\tilde{\pi}, \eta)}{2\tilde{\pi}} + \frac{\eta^2}{2\tilde{\pi}} - (|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij} \}
\end{align*}
\]

Now the equation \( dM(\tilde{\pi}) = 0 \) on \( T_{\tilde{\omega}} \) is equivalent to

\[
\begin{align*}
\frac{\text{cof}[\nabla_{ij} \tilde{\pi} + (\tilde{\pi} - \frac{|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}) \hat{e}_{ij}]}{2\tilde{\pi}} \\
&= \frac{\nabla(\tilde{\pi}, \eta)}{|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}} - \frac{\eta}{|\nabla \tilde{\pi}|^2 + \frac{\eta^2}{\eta}}
\end{align*}
\]
\[- \frac{V(p, v)}{|vp|^2 + p^2} = 0 \text{ in } \omega, \quad (7.5)\]
and
\[\frac{\partial \eta}{\partial v} - \zeta \eta = -p \frac{\partial v}{\partial v} = 0 \text{ on } \partial \omega. \quad (7.6)\]

Since \( \tilde{p} \in C^2(\overline{\omega}), \tilde{p} > 0, \) and \( M(\tilde{p}) = \tilde{\varepsilon} > 0 \) in \( \tilde{\omega}, \) the equation (7.5) is uniformly elliptic and the maximum principle is applicable. This together with (7.6) implies that \( v = \text{const} = c \) and therefore \( \eta = c \tilde{p}. \) On the other hand, because \( \eta \in T_p \mathcal{M} \) we should have
\[\int_{\omega} \tilde{p} \eta \sigma = 0,\]
from which we conclude that \( c \neq 0. \)

Under these circumstances the Implicit Function Theorem is applicable [2], p. 113, to the map \( M : U \rightarrow H \) and the conclusion of the theorem follows.

7.7. **Remark.** From the function \( p \) solving (7.3), (7.4) the reflector surface \( r : \tilde{\omega} \rightarrow \mathbb{R}^3 \) is constructed with the use of (1.10). The details are similar to those in 5.9.1 and we do not repeat them here.
BIBLIOGRAPHY


