

On the Modified Bessel Functions of the First Kind

by

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Abstract. We consider the functions $v_\alpha(t) \equiv tI_\alpha(t)/I_{\alpha+1}(t)$ where I_α are the modified Bessel functions of the first kind of order $\alpha > 0$. We prove that v_α is strictly monotone and strictly convex on \mathbb{R}^+ . These results have application in finite elasticity¹.

1. Introduction. For $\alpha > 0$ the modified Bessel functions of the first kind², I_α , are the globally analytic solutions of the differential equation

$$t^2 \ddot{I}_\alpha(t) + t \dot{I}_\alpha(t) - (t^2 + \alpha^2) I_\alpha(t) = 0.$$

These Bessel functions have infinite series expansions

$$I_\alpha(t) = \sum_{r=0}^{\infty} \frac{(t/2)^{\alpha+2r}}{r! \Gamma(\alpha+r+1)}, \quad (1)$$

where Γ denotes the gamma function. We note that Γ satisfies

$$\begin{aligned} \Gamma(1) &= 1, & \Gamma(x+1) &= x\Gamma(x), \quad x > 0, \\ \Gamma(n+1) &= n! & & \text{for positive integers } n. \end{aligned}$$

The infinite series formula (1) can be used to show that I_α satisfy the recurrence relations

$$\begin{aligned} t \dot{I}_{\alpha+1}(t) &= t I_\alpha(t) - (\alpha+1) I_{\alpha+1}(t), \\ t \dot{I}_\alpha(t) &= t I_{\alpha+1}(t) + \alpha I_\alpha(t). \end{aligned} \quad (2)$$

The results in this paper concern the functions

$$v_\alpha(t) \equiv \frac{t I_\alpha(t)}{I_{\alpha+1}(t)}.$$

A simple consequence of the recurrence relations (2) and the infinite series (1) is that v_α satisfies

$$t \dot{v}_\alpha(t) = t^2 + 2(\alpha+1) v_\alpha(t) - v_\alpha^2(t), \quad (3)$$

$$v_\alpha(0) = 2(\alpha+1). \quad (4)$$

To simplify notation we will drop the subscript α on v_α for the remainder of this paper.

2. Results. We now prove that $v(t)$ is strictly monotone and strictly convex for all $t > 0$.

Theorem 1. $v(t)$ is strictly positive for $t \in (0, \infty)$.

Proof. If we differentiate (3) we discover that

$$t\ddot{v}(t) = 2t + (2\alpha+1)\dot{v}(t) - 2v(t)\dot{v}(t), \quad (5)$$

$$tv(t) = 2 + 2\alpha\ddot{v}(t) - 2\dot{v}^2(t) - 2v(t)\ddot{v}(t), \quad (6)$$

and hence we find, with the aid of (4), that

$$v(0) = 0, \quad \ddot{v}(0) = 1/(\alpha+2). \quad (7)$$

Since v is continuous, we conclude that \dot{v} is strictly positive on some interval $(0, s)$.

Suppose, for the sake of contradiction, that \dot{v} is not strictly positive on $(0, \infty)$ and define

$$s_* = \inf \{t \in (0, \infty) : \dot{v}(t) = 0\}.$$

We note from above that by the continuity of \dot{v} ,

$$\dot{v}(t) > 0, \quad t \in (0, s_*). \quad (8)$$

At $t = s_*$ we conclude from (5) that $\ddot{v}(s_*) = 2$ and hence that \dot{v} is strictly negative on an interval (s, s_*) . This contradicts (8). Thus \dot{v} is strictly positive on $(0, \infty)$.

□

Lemma. $v^2(t) - (2\alpha+1)v(t) - (t^2 + \alpha + 1/2) > 0 .$

Proof. Define

$$h(t) = \{(2\alpha+1) + [(2\alpha+1)^2 + 4(t^2 + \alpha + 1/2)]^{1/2}\}/2 , \quad (9)$$

the larger root of the quadratic

$$h^2 - (2\alpha+1)h - (t^2 + \alpha + 1/2) = 0 . \quad (10)$$

We will show that v is strictly greater than h to prove this lemma.

By (4) and (9)

$$\begin{aligned} v(0) - h(0) &= [(2\alpha+3) - (4\alpha^2 + 8\alpha + 3)^{1/2}]/2 \\ &= \frac{2\alpha+3}{(2\alpha+3) + (4\alpha^2 + 8\alpha + 3)^{1/2}} > 0 , \end{aligned}$$

and hence v is strictly greater than h in a neighborhood of zero.

Suppose, for the sake of contradiction, that v is not strictly greater than h on $[0, \infty)$ and define

$$s \equiv \inf \{t \in [0, \infty) : v(t) = h(t)\} .$$

It is clear (since $v(t) > h(t)$ for $t \in [0, s)$) that $\dot{v}(s) < \dot{h}(s)$.

However (putting $H = h(s)$ and using (3) and (10))

$$\begin{aligned} \dot{v}(s) - \dot{h}(s) &= \frac{1}{s} [s^2 + 2(\alpha+1)H - H^2] - \frac{s}{H - (\alpha+1/2)} \\ &= \frac{[H - (\alpha+1/2)]^2 - s^2}{s[H - (\alpha+1/2)]} . \end{aligned} \quad (11)$$

A straightforward computation, using the fact that H is a solution of the quadratic (10) shows that the numerator of the last expression is equal to

$$(\alpha+1/2)^2 + (\alpha+1/2) > 0 .$$

Finally, by (9) the denominator in (11) is strictly positive and hence so is (11). This is a contradiction. Thus no such s can exist. \square

Theorem 2. $\ddot{v}(t)$ is strictly positive on $[0, \infty)$.

Proof. By (5)

$$\ddot{v}(t) = w(t)(2v(t) - 2\alpha - 1)/t,$$

where

$$w(t) \equiv -\dot{v}(t) + 2t/(2v(t) - 2\alpha - 1) . \quad (12)$$

Since \dot{v} is positive and $v(0) = 2(\alpha+1)$, it is clear that

$$\text{sgn } w(t) = \text{sgn } \ddot{v}(t) , \quad t > 0 .$$

Now $\ddot{v}(0) = 1/(\alpha+2)$, so that both \ddot{v} and w are strictly positive on an interval $(0, s)$. Suppose, for the sake of contradiction, that \ddot{v} is not strictly positive on $[0, \infty)$ and define

$$s_* \equiv \inf\{t \in [0, \infty) : \ddot{v}(t) = 0\} .$$

We note that $\ddot{v}(s_*) = w(s_*) = 0$ and that w must be strictly positive on the interval $(0, s_*)$. Thus we conclude, with the aid of (12), that

$$0 > \dot{w}(s_*) = \frac{4v_* - 4s_* \dot{v}_* - 4\alpha - 2}{(2v_* - 2\alpha - 1)^2},$$

where $v_* \equiv v(s_*)$ and $\dot{v}_* \equiv \dot{v}(s_*)$. If we combine the last equation with (3) we find that

$$0 > v_*^2 - (2\alpha + 1)v_* - (t^2 + \alpha + 1/2).$$

This contradicts the lemma. Thus \ddot{v} is strictly positive on $[0, \infty)$.



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REFERENCES

- [1] Cheng, S. Y., Ariaratnam, S.T., and R. N. Dubey, Axisymmetric bifurcation in an elastic-plastic cylinder under axial load and lateral hydrostatic pressure, Q. Appl. Math. 29 (1971), 41-51.
- [2] Green, A.E. and A.J.M. Spencer, The stability of a circular cylinder under finite extension and torsion, J. Math. and Phys., 37 (1959), 316-338.
- [3] Miles, J. P., Bifurcation in plastic flow under uniaxial tension, J. Mech. Phys. Solids 19 (1971), 89-102.
- [4] Sensenig, Chester B., Instability of thick elastic solids. Comm. Pure Appl. Math. 17 (1964), 451-491.
- [5] Simpson, H. C. and S. J. Spector, On barrelling for a special material in finite elasticity. Institute for Mathematics and its Applications Preprint Series.
- [6] Watson, G. N., A Treatise on the Theory of Bessel Functions, Cambridge University Press (1922).
- [7] Wilkes, E.W., On the stability of a circular tube under end thrust, Q. J. Mech. Appl. Math., 8 (1955), 88-100.

FOOTNOTES

- 1 SIMPSON & SPECTOR [5] use these results, with $\alpha = 0$, to prove that a non-linearly elastic cylinder eventually becomes unstable in uniaxial compression. We anticipate that for general $\alpha > 0$ these results will have application in the buckling and necking of such cylinders, cf., e.g., CHENG, ARIARATNAM & DUBEY [1], GREEN & SPENCER [2], MILES [3], SENSENIG [4], and WILKES [7].
- 2 cf., e.g., WATSON [6], chapter 3.71 for a discussion of these functions.

On Barrelling for a Special Material in Finite Elasticity

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Introduction

In this paper we investigate the question of stability for a solid circular cylinder, composed of a particular homogeneous isotropic (compressible) nonlinearly elastic material, that is subjected to compressive end forces in the direction of its axis (so as to give fixed axial displacements at the ends)¹.

The particular material used in this analysis has stored energy function σ given by ²

$$\sigma(\underline{F}) = \frac{1}{2} \underline{F} \cdot \underline{F} + \frac{1}{m} (\det \underline{F})^{-m}$$

for some $m > 0$ and all 3×3 matrices \underline{F} (BLATZ & KO [3] have shown that, with $m = 13.3$, this material can be matched to experimental data of BRIDGMAN [4].).

We begin our analysis with a formulation of the relevant non-linear boundary-value problem and a proof of the existence of a trivial homogeneous solution³ to this problem. We then consider the superimposed linear (infinitesimal) problem⁴ about a given solution of the nonlinear problem.

We prove that all solutions⁵ of the linear problem, sans the traction free boundary condition on the curved surface of the cylinder, can be obtained by separation of variables. We therefore reduce the question of stability to that of proving that a certain determinant, that corresponds to the traction free condition, is zero at some critical value of the loading.

We then show that for each eigenmode $n = 1, 2, 3, \dots$, there exists a unique loading value $\lambda(n)$ that makes the above mentioned determinant zero and hence conclude that the cylinder will eventually become unstable when compressed in an axial direction.

In addition, we obtain information concerning the dependence of the loading value λ (where instability occurs) on the parameter n . These results allow us to conclude that:

- (i) for a given value of λ at most two eigenmodes can occur;
- (ii) for sufficiently thick cylinders $n = 1$ is the eigenmode that occurs first⁶;
- (iii) for sufficiently thin cylinders $n = 1$ is not the eigenmode that occurs first;
- (iv) there exist cylinders whose first point of instability occurs at two distinct eigenmodes simultaneously.

Finally, we use numerical techniques to obtain a global picture of the dependence of λ upon n .

1. Notation

We let

$\text{Lin} \equiv$ space of all linear transformations from \mathbb{R}^3 into \mathbb{R}^3
with inner product

$$\underline{G} \cdot \underline{H} = \text{trace}(\underline{G} \underline{H}^T),$$

where \underline{H}^T is the transpose of \underline{H} and denote

$$\text{Lin}^+ = \{\underline{H} \in \text{Lin} : \det \underline{H} > 0\},$$

where \det is the determinant.

We write ∇ and div for the gradient and divergence operators in \mathbb{R}^3 :
for a vector field \underline{u} , $\nabla \underline{u}$ is the tensor field with components $(\nabla \underline{u})_{ij} = \partial u_i / \partial x_j$;
for a tensor field \underline{S} , $\text{div} \underline{S}$ is the vector field with components $\sum_j \partial S_{ij} / \partial x_j$.
Given any function $\phi(\underline{F})$, we denote the Frechet derivative of ϕ by,

$$\frac{d}{d\underline{F}} \phi(\underline{F}).$$

Finally, we denote by $I_n(t)$ the modified Bessel functions of the first kind, so that I_n are globally analytic solutions of the differential equation

$$t^2 \ddot{I}_n(t) + t \dot{I}_n(t) - (t^2 + n^2) I_n(t) = 0,$$

and are given by

$$I_n(t) = \sum_{r=0}^{\infty} \frac{(t/2)^{n+2r}}{r!(n+r)!}.$$

2. The constitutive relation. The non-linear problem

We consider a homogeneous body B and identify B with the region (of \mathbb{R}^3) it occupies in a fixed homogeneous reference configuration. A deformation \underline{f} of the body is a member of the space

$$\text{Def} = \{ \underline{f} \in C^2(B, \mathbb{R}^3) : \det \nabla \underline{f} > 0 \}.$$

We assume that the body in the reference configuration is in the shape of a right circular cylinder of height L and radius R . In rectangular coordinates we take

$$B = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 < R^2, x_3 \in [0, L] \}$$

with lateral surface

$$S = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 = R^2, x_3 \in [0, L] \}.$$

The remaining two pieces of ∂B , the top and bottom of the cylinder, we denote by C_T and C_B , respectively.

We assume that B is hyperelastic with response function $\sigma : \text{Lin}^+ \rightarrow \mathbb{R}$. σ gives the internal energy

$$\sigma(\nabla \underline{f}(\underline{x}))$$

at any point $\underline{x} \in B$ when the body is deformed by \underline{f} . Writing \underline{F} for $\nabla \underline{f}(\underline{x})$ we assume that

$$\sigma(\underline{F}) = \frac{1}{2} \underline{F} \cdot \underline{F} + \frac{1}{m} (\det \underline{F})^{-m} \tag{2.1}^7$$

for all $\underline{F} \in \text{Lin}^+$ and for some $m \in \mathbb{R}^+$.

The (Piola-Kirchhoff) stress $\underline{S} : \text{Lin}^+ \rightarrow \text{Lin}$ is given by

$$\underline{S}(\underline{F}) \equiv \frac{d}{d\underline{F}} \sigma(\underline{F}) = \underline{F} - (\det \underline{F})^{-m} \underline{F}^{-T}, \tag{2.2}$$

while the linear transformation $\underline{A}(\underline{F}): \text{Lin} \rightarrow \text{Lin}$ defined by

$$\begin{aligned} \underline{A}(\underline{F})\underline{H} &\equiv \left(\frac{d}{d\underline{F}}\underline{S}(\underline{F})\right)\underline{H} \\ &= \underline{H} + (\det \underline{F})^{-m} [m(\underline{F}^{-T}\underline{H})\underline{F}^{-T} + \underline{F}^{-T}\underline{H}\underline{F}^{-T}] \end{aligned}$$

is called the elasticity tensor.

Let $\lambda \in (0,1]$ and consider the problem of finding a deformation \underline{f} that satisfies

$$\begin{aligned} \text{div } \underline{S}(\nabla \underline{f}) &= \underline{0} && \text{in } B, \\ \underline{f}_3 &= 0 && \text{on } C_B, \\ \underline{f}_3 &= \lambda L && \text{on } C_T, \\ S_{13}(\nabla \underline{f}) = S_{23}(\nabla \underline{f}) &= 0 && \text{on } C_B \text{ and } C_T, \\ \underline{S}(\nabla \underline{f})\underline{n} &= \underline{0} && \text{on } S, \end{aligned} \tag{2.3}$$

where \underline{n} is the outward unit normal to the lateral surface and the parameter λ is the ratio of the final to the initial height of the cylinder.

We note that if a deformation \underline{f} satisfies (2.3) then so does $\underline{g} \circ \underline{f}$ if \underline{g} is either a translation perpendicular to the axis of the cylinder or any rotation about this axis. In order to eliminate this trivial non-uniqueness of solutions we impose additional constraints upon the deformation \underline{f} . We require

$$\begin{aligned} \int_B ((\underline{f}-\underline{1})_1) &= \int_B ((\underline{f}-\underline{1})_2) = 0, \\ \int_B (\underline{f}_{1,2} - \underline{f}_{2,1}) &= 0. \end{aligned} \tag{2.4}$$

A deformation \underline{f} that satisfies (2.3) and (2.4) will be called a solution of the non-linear problem or simply a non-linear solution.

Proposition 2.1. For any $\lambda \in (0,1]$

$$f_{\lambda}(x) = \begin{pmatrix} \mu^{1/2} \\ \mu^{1/2} \\ \lambda \end{pmatrix} x \quad (2.5)$$

is a solution to (2.3) and (2.4) if μ is given by

$$\mu = \lambda^{\frac{-m}{m+1}}.$$

Proof. It is clear that f_{λ} satisfies (2.3)₁₋₄ and (2.4) for any μ . By (2.3)₅ and (2.2) we want to determine μ so that

$$\mu^{1/2} - (\mu\lambda)^{-m} \mu^{-1/2} = 0.$$

The desired result clearly follows. □

Remark. We have assumed that the cylinder will frictionlessly adhere to the top and bottom surfaces for $\lambda \in (0,1)$. Physically, this will occur only if the solution puts the body in compression in the x_3 -direction⁸. We verify that this indeed occurs. By (2.2)

$$S_{33}(\nabla f_{\lambda}) = \lambda - (\mu\lambda)^{-m} \lambda^{-1},$$

and hence

$$S_{33}(\nabla f_{\lambda}) = \lambda - \lambda^{\frac{-m}{m+1}} \lambda^{-1}$$

which is clearly negative for $\lambda \in (0,1)$.

3. The linear problem

We now investigate the stability of the non-linear solution by linearizing the boundary-value problem. For a given $\lambda \in (0,1]$ and f_{λ} we linearize (2.3) and (2.4) to arrive at the problem of finding a C^2 function \underline{u} that satisfies

$$\begin{aligned} \operatorname{div} \underline{A}[\underline{\nabla} \underline{u}] &= \underline{0} && \text{in } B, \\ u_3 &= 0 && \text{on } C_B \text{ and } C_T, \\ \underline{A}[\underline{\nabla} \underline{u}]_{13} &= \underline{A}[\underline{\nabla} \underline{u}]_{23} = 0 && \text{on } C_B \text{ and } C_T, \\ \underline{A}[\underline{\nabla} \underline{u}]_{\underline{n}} &= \underline{0} && \text{on } S, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \int_B u_1 &= \int_B u_2 = 0, \\ \int_B (u_{1,2} - u_{2,1}) &= 0, \end{aligned} \tag{3.2}$$

where $\underline{A} = \underline{A}(\underline{\nabla} f_{\lambda})$ and once again \underline{n} is the outward unit normal to the lateral surface.

A C^2 function \underline{u} that satisfies (3.1) and (3.2) will be called a solution of the linear problem corresponding to f_{λ} or simply a linear solution.

We now observe that by Proposition 2.1 the tensor

$$\underline{B} = \underline{A}(\underline{\nabla} f_{\lambda})[\underline{U}] \tag{3.3}$$

is given by

$$\begin{aligned}
 B_{11} &= 2U_{11} + m(U_{11} + U_{22}) + mt^{1/2}U_{33}, \\
 B_{33} &= [1 + t(m+1)] U_{33} + mt^{1/2}(U_{11} + U_{22}), \\
 B_{12} &= B_{21} = U_{12} + U_{21}, \\
 B_{13} &= U_{13} + t^{1/2}U_{31}, \\
 B_{31} &= U_{31} + t^{1/2}U_{13},
 \end{aligned} \tag{3.4}$$

for $i = 1, 2$ and for any tensor \underline{U} , where

$$t \equiv \mu\lambda^{-2} = \lambda^{-\frac{(3m+2)}{m+1}}. \tag{3.5}$$

For the remainder of this paper we will only be interested in considering specific types of instabilities, those that are axisymmetric. We shall therefore consider linear solutions of the form

$$\underline{u}(x_1, x_2, x_3) = \begin{pmatrix} \phi(r, z)x_1 \\ \phi(r, z)x_2 \\ \ell(r, z) \end{pmatrix}, \tag{3.6}$$

where $r^2 = x_1^2 + x_2^2$ and $z = x_3$.

A straightforward computation, using (3.4), (3.5), and (3.6), shows that (3.1)₁ reduces to

$$\begin{aligned}
 0 &= (\text{div } \underline{B})_i = x_i [(m+2)(3\phi_r/r + \phi_{rr}) + \phi_{zz} + t^{1/2}(m+1)\ell_{rz}/r], \\
 0 &= (\text{div } \underline{B})_3 = \ell_r/r + \ell_{rr} + t^{1/2}(m+1)(2\phi_z + r\phi_{rz}) + [1 + t(m+1)]\ell_{zz},
 \end{aligned}$$

for $i = 1, 2$. If we let $\theta(r, z) = r^2\phi(r, z)$ we find that the linear boundary-value problem reduces to the problem of finding a pair of C^2 functions (θ, ℓ) that satisfy

$$(m+2)(\theta_r/r)_r + \theta_{zz}/r + t^{1/2}(m+1)\ell_{rz} = 0$$

on $(0,R] \times [0,L]$, (3.7)

$$(r\ell_r)_r + [1+t(m+1)]r\ell_{zz} + t^{1/2}(m+1)\theta_{rz} = 0$$

$$\ell(r,0) = \ell(r,L) = 0$$

for $r \in (0,R]$, (3.8)

$$\theta_z(r,0) = \theta_z(r,L) = 0$$

$$R\ell_r(R,z) + t^{1/2}\theta_z(R,z) = 0$$

for $z \in [0,L]$. (3.9)

$$(m+2)\theta_r(R,z) + t^{1/2}mR\ell_z(R,z) = 2\theta(R,z)/R$$

In addition the requirement $u \in C^2$ yields

$$\theta(r,z)/r \rightarrow 0 \text{ as } r \rightarrow 0$$

(3.10)

uniformly in z .

Before proceeding further, we first recall a result concerning Fourier series of a function of two variables.

Proposition 3.1. Let θ and ℓ be C^2 on $[0,R] \times [0,L]$

with

$$\ell(r,0) = \ell(r,L) = \theta_z(r,0) = \theta_z(r,L) = 0.$$

Then θ and λ have Fourier series

$$\begin{aligned}\theta(r,z) &= \sum_{n=1}^{\infty} \theta_n(r) \cos(\rho_n z), \\ \lambda(r,z) &= \sum_{n=1}^{\infty} \lambda_n(r) \sin(\rho_n z),\end{aligned}\tag{3.11}$$

where $\rho_n = n\pi/L$. Moreover, the series are uniformly and absolutely convergent, θ_n and λ_n are C^2 on $[0,R]$ and are given by

$$\begin{aligned}\theta_n(r) &= \int_0^L \theta(r,z) \cos(\rho_n z) dz, \\ \lambda_n(r) &= \int_0^L \lambda(r,z) \sin(\rho_n z) dz.\end{aligned}\tag{3.12}$$

Proof. The only non-trivial part of the proof is the uniform and absolute convergence of the series. If we integrate by parts twice we find that

$$\theta_n(r) = - \int_0^L \theta_{zz}(r,z) \cos(\rho_n z) / \rho_n^2 dz$$

and since θ is C^2 that

$$|\theta_n(r)| < k/n^2,$$

where k is independent of z , r , and n . The desired result follows from the Weierstrauss M-test.



Theorem 3.2.⁹ Let $\lambda \in (0,1)$. Then any C^2 solution (θ, ℓ) of (3.7), (3.8), and (3.10) satisfies the conclusions of the previous proposition with θ_n and ℓ_n a linear combination of

$$\theta_{ni}(r) = r I_1(\rho_n r e_i),$$
(3.13)

$$\ell_{ni}(r) = D_i I_0(\rho_n r e_i),$$

$i = 1, 2$, where

$$e_1 = 1, \quad e_2^2 = 1 + (t-1) \frac{m+1}{m+2},$$
(3.14)

$$D_1 = -t^{-1/2}, \quad D_2 = -t^{1/2}/e_2.$$

Proof. Let (θ, ℓ) be a C^2 solution of (3.7) and (3.8). If we multiply (3.7)₁ by $\cos(\rho_n z)$ and (3.7)₂ by $\sin(\rho_n z)$, integrate over $[0, L]$ with respect to z , and then integrate by parts, we conclude, with the aid of (3.8), (3.11), and (3.12), that

$$(m+2) (\theta'_n/r)' + \rho_n t^{1/2} (m+1) \ell'_n = \rho_n^2 \theta_n/r$$
(3.15)

$$(r \ell'_n)' - \rho_n t^{1/2} (m+1) \theta'_n = \rho_n^2 [1 + t(m+1)] r \ell_n.$$

A straightforward computation shows that (θ_{ni}, ℓ_{ni}) as given in (3.13) and (3.14) satisfies (3.15). It is also clear that the replacement of the modified Bessel functions of the first kind I_0 and I_1 , by those of the second kind K_0 and K_1 gives two additional solutions. Hence, we have constructed four linearly independent solutions to (3.15) and any other solution can therefore be written as a linear combination of these solutions.

To complete the proof we must, of course, eliminate the functions K_0 and K_1 . By (3.10) and (3.11) we conclude that

$$\theta_n(r)/r \rightarrow 0 \text{ as } r \rightarrow 0,$$

from which the desired results follows. □

We next want to show that there is a $\lambda \in (0,1)$ such that our linear problem, in the form (3.7) - (3.10), has a non-trivial solution. Clearly, Theorem 3.2 reduces this problem to that of finding a λ such that one of the solutions (θ, λ) to (3.7), (3.8), and (3.10) satisfies the boundary condition on the lateral surface (3.9). We note that (3.11) and (3.12) give necessary and sufficient conditions for (θ, λ) to satisfy (3.9) namely

$$B_n \begin{pmatrix} \theta_n \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ at } r = R,$$

where

$$B_n = \begin{bmatrix} (m+2) \frac{d}{dr} - \frac{2}{r} & \rho_n t^{1/2} m r \\ - \rho_n t^{1/2} & r \frac{d}{dr} \end{bmatrix}$$

and (θ_n, λ_n) is a linear combination of $(\theta_{ni}, \lambda_{ni})$, $i = 1, 2$.

Since we desire a non-trivial solution, (θ_n, λ_n) must be non-zero for at least one n . We therefore conclude, with the aid of Theorem 3.2, that a necessary and sufficient condition for the existence of a solution to (3.7) - (3.10) is that

$$\det \left\{ B_n \begin{pmatrix} \theta_{n1} \\ \xi_{n1} \end{pmatrix} \Big|_{r=R} ; B_n \begin{pmatrix} \theta_{n2} \\ \xi_{n2} \end{pmatrix} \Big|_{r=R} \right\} = 0 \quad (3.16)$$

for some n .

If we substitute (3.13) into the last we conclude, with the aid of (3.14) and (A.2), that

$$0 = -4t^{1/2} v(\rho_n R) + 2(t^{1/2} - t^{-1/2}) - (t^{1/2} + t^{-1/2})(m+2 - \frac{tm}{e_2}) v(\rho_n R e_2), \quad (3.17)$$

where

$$v(r) \equiv r I_0(r) / I_1(r).$$

We now prove that for any given value of

$$\rho = \rho_n R = n\pi R/L$$

equation (3.17) is satisfied by a unique $\lambda \in (0,1)$.

Theorem 3.3. For each $\rho > 0$ there exists a unique $\lambda(\rho)$ such that

$$H(\rho, \lambda(\rho)) = 0, \quad (3.18)$$

where

$$H(\rho, \lambda) \equiv v(\rho f) - R(\rho, \lambda),$$

$$R(\rho, \lambda) \equiv \frac{4t}{(1+t)^2} f^2 v(\rho) - \frac{2(t-1)}{(1+t)^2} f^2, \quad (3.19)$$

$$f^2 \equiv e_2^2 = 1 + (t-1) \frac{m+1}{m+2}, \quad t \equiv \lambda \frac{-(3m+2)}{m+1}.$$

Moreover $\lambda: (0, \infty) \rightarrow (0, 1)$ and is continuously differentiable in ρ .

Proof. Existence. Fix $\rho > 0$ and consider H as a function of λ . We prove $H = 0$ has a root λ in the interval $(0, 1)$ by showing

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(\rho, 1) &> 0, \\ \lim_{\lambda \rightarrow 0^+} H(\rho, \lambda) &= +\infty \end{aligned} \tag{3.20}$$

(existence clearly follows from (3.20), $H(\rho, 1) = 0$, and the continuity of H).

To prove (3.20)₁ we first note that H is a function of λ thru the variable t and hence

$$\frac{\partial H}{\partial \lambda}(\rho, 1) = \left(\frac{\partial H}{\partial t} \frac{dt}{d\lambda} \right) \Big|_{\lambda=1} = -\frac{(3m+2)}{m+1} \left(\frac{\partial H}{\partial t} \right) \Big|_{t=1}.$$

It thus suffices to show that

$$\frac{\partial H}{\partial t} \Big|_{t=1} < 0. \tag{3.21}$$

A straightforward computation, involving (3.19), yields (at $t = 1$)

$$\frac{\partial H}{\partial t} = \frac{1}{2} \frac{m+1}{m+2} \rho \dot{v}(\rho) - \frac{m+1}{m+2} v(\rho) + \frac{1}{2},$$

and hence by (A.5)₁ we find that (at $t = 1$)

$$\frac{\partial H}{\partial t} = \frac{1}{2} \frac{m+1}{m+2} \left[\rho^2 + \frac{m+2}{m+1} - v^2(\rho) \right]. \tag{3.22}$$

Equation (3.21) now follows from (3.22) and the first proposition in the Appendix.

To prove (3.20)₂ we note that as $\lambda \rightarrow 0^+$, $t \rightarrow \infty$, and hence $\frac{f^2}{t} \rightarrow \frac{m+1}{m+2}$. It then follows from (3.19)₂ that R is bounded as $\lambda \rightarrow 0^+$. But by Theorems 1 and 2 of [10], v and \ddot{v} are strictly positive. Thus $v(\rho f) \rightarrow +\infty$ and hence by (3.19)₁ we conclude (3.20)₂. This completes the existence portion of the proof.

Uniqueness. By (3.19)₃ we see that f is strictly monotone in λ so that we could just as well consider (3.17) as a relation between ρ and f and seek roots f of H . Thus we rewrite (3.18) and (3.19) as

$$\hat{H}(\rho, f) = 0, \quad (3.23)$$

where

$$\hat{H}(\rho, f) = v(\rho f) - \tilde{R}(\rho, f), \quad (3.24)$$

$$\tilde{R}(\rho, f) = \frac{2(m+1)f^2}{[(m+2)f^2+m]^2} \{ (m+2)(2v-1)f^2 + (m+2 - 2v) \},$$

and $v \equiv v(\rho)$.

We will show that \hat{H} is a strictly convex function of f for $f > 1$. This allows us to conclude that there is at most one f satisfying (3.23) (since $\hat{H}(\rho, 1) = 0$) and hence at most one λ satisfying (3.18).

Since \ddot{v} is strictly positive (cf. [10]), it suffices to show that

$$\frac{\partial^2 \tilde{R}}{\partial f^2} < 0 \quad \text{for all } f > 1. \quad (3.25)$$

By direct calculation we find that

$$\frac{\partial^2 \tilde{R}}{\partial f^2} = \frac{4(m+1)}{[(m+2)f^2+m]^4} \{ a(m)f^4 + b(m)f^2 + c(m) \},$$

where

$$a(m) = -3(m+2)^2 [(4m+2)v - 3m - 2],$$

$$b(m) = m(m+2) [(12m+16)v - 14m - 16],$$

$$c(m) = m^2(m+2 - 2v).$$

We show that the quantity in braces is strictly negative for all $f > 1$.

Put $x = f^2$ and define $Q(x)$ to be the quadratic in braces. It clearly suffices to show that both $Q(1)$ and $\dot{Q}(1)$ are strictly negative.

Clearly

$$Q(1) = -v(16m^2 + 40m + 24) - (4m^3 - 28m - 24)$$

and

$$\dot{Q}(1) = -(m+2)[(12m^2 + 44m + 24)v - (4m^2 + 32m + 24)].$$

The desired result now follows from $v > 2$ (cf. [10]). This establishes the uniqueness of the root λ of (3.18).

Differentiability. By the implicit function theorem the continuous differentiability of $f(\rho)$, and hence $\lambda(\rho)$, follows provided we know that

$$\frac{\partial \hat{H}}{\partial f}(\rho, f(\rho)) \neq 0.$$

Since \hat{H} is zero at $f=1$ and $f=f(\rho)$, we can apply Rolle's Theorem to conclude that there exists an $f^* \in (1, f(\rho))$ such that $\hat{H}_f(\rho, f^*) = 0$. But \hat{H} is strictly convex and hence \hat{H}_f is a strictly increasing function of f , thus

$$\frac{\partial \hat{H}}{\partial f}(\rho, f(\rho)) > 0. \tag{3.26}$$

This completes the proof. □

4. Discussion. Further results

In this section we derive further information concerning the dependence of λ on ρ ($=n\pi R/L$). We show that the function $\lambda = \lambda(\rho)$ is strictly decreasing for large ρ and strictly increasing for small ρ , and that it has no local minima. These results along with those of the previous section lead us to the following conclusions.

For a given geometry (ratio R/L) and a given mode number n there is a (unique) corresponding $\lambda_n \in (0, 1)$ such that the linear problem (3.7) - (3.10) has a nontrivial solution. Letting

$$\lambda_{cr} = \sup \{ \lambda_n : n = 1, 2, 3, \dots \},$$

and noting that $\lambda_{cr} = \lambda_n$ for some finite value of n , we find that for $\lambda \in (\lambda_{cr}, 1]$ the nonlinear homogeneous solution f_{λ} (cf. (2.5)) is stable with respect to axisymmetric (barrelling) perturbations and that at $\lambda = \lambda_{cr}$ this stability breaks down.

In addition, the absence of local minima in the function $\lambda(\rho)$ implies that at most two modes can occur simultaneously as nontrivial solutions of the linear problem. Clearly, it would be of interest to know precisely which value of n yields the largest λ_n . To date we have been unable to determine this analytically. However, the results of this section do tell us that for sufficiently thick cylinders ($R/L \gg 1$) $n = 1$ yields the largest λ while for sufficiently thin cylinders $n = 1$ does not yield the largest λ .

We first investigate the asymptotic behavior of $\lambda(\rho)$ near zero and infinity.

Proposition 4.1.

- (i) $\liminf_{\rho \rightarrow \infty} f(\rho) > 1$;
- (ii) $f(\rho)$ is bounded as $\rho \rightarrow \infty$;
- (iii) $\lim_{\rho \rightarrow 0} f(\rho) = +\infty$;
- (iv) $\liminf_{\rho \rightarrow 0} \rho f(\rho) > 0$.

Proof. Consider the function $h(f) \equiv v(\rho f)$. If we apply the mean value theorem to h we find that

$$v(\rho f) = v(\rho) + \rho \dot{v}(\rho f^*)(f-1), \tag{4.1}$$

where

$$1 < f^* < f. \tag{4.2}$$

If we then substitute (4.1) into (3.19) and divide the result by $\rho(t-1)$ we conclude, with the aid of (3.19)₃, that

$$\frac{v(\rho)}{\rho} \frac{(m+2)(t-1) - 4t(m+1)}{(m+2)(1+t)^2} + \frac{\dot{v}(\rho f^*)}{(f+1)} \frac{(m+1)}{(m+2)} = -\frac{2f^2}{\rho(1+t)^2}. \quad (4.3)$$

To prove (i) we assume, for the sake of contradiction, that there is a sequence $\rho_i \rightarrow \infty$ such that $f(\rho_i) \rightarrow 1$. If we let $\rho = \rho_i \rightarrow \infty$ in (4.3) we find, with the aid of (3.19)₃, (4.2), and (A.6), that (4.3) reduces to

$$\frac{-4(m+1)}{(m+2) 2^2} + \frac{1}{2} \frac{(m+1)}{(m+2)} = 0.$$

This is clearly impossible for any $m > 0$. Thus we arrive at (i).

To prove (ii) we assume, again for the sake of contradiction, that there is a sequence $\rho_i \rightarrow \infty$ such that $f_i \equiv f(\rho_i) \rightarrow \infty$. If we divide (3.24) by ρf we find that

$$\frac{v(\rho f)}{\rho f} = \frac{2(m+1)}{[(m+2)f^2 + m]^2} \left\{ (m+2) \left(2 \frac{v}{\rho} - \frac{1}{\rho} \right) f^3 + \left(\frac{m+2}{\rho} - \frac{2v}{\rho} \right) f \right\},$$

where $v \equiv v(\rho)$. Letting $\rho = \rho_i \rightarrow \infty$ we find, by (A.6)₁, that the lefthand side of the last equation approaches one while the righthand side goes to zero. Thus no such sequence can exist. This proves (ii).

To prove (iii) we multiply (4.3) by ρ and let $\rho \rightarrow 0$ to conclude, with the aid of (3.19)₃ and (A.6)₂, that (4.3) reduces to

$$\lim_{\rho \rightarrow 0} \frac{t(2m+1) + (m+1)}{(m+2)(1+t)^2} = 0, \quad (4.4)$$

where $t \equiv t(\rho)$. Now Theorem 3.3 implies that $t(\rho)$ is a C^1 function for $\rho \in (0, \infty)$ and moreover that $t(\rho) \in (1, \infty)$ for the same ρ . Clearly, the only way (4.4) can be satisfied is for $t(\rho)$, and hence $f(\rho)$, to become infinite. This proves (iii).

To prove (iv) we assume, once more for the sake of contradiction, that there is a sequence $\rho_j \rightarrow 0$ such that $\rho_j f(\rho_j) \rightarrow 0$. If we let $\rho = \rho_j \rightarrow 0$ in (3.19) we find, with the aid of (A.5)₂, and part (iii) of this proposition, that

$$2 - 8 \frac{m+1}{m+2} + 2 \frac{m+1}{m+2} = 0.$$

This is clearly impossible for any $m > 0$. Thus no such sequence can exist. This concludes the proof of the proposition. \square

To analyze the behavior of $\frac{d\lambda}{d\rho}$ near zero and infinity, we differentiate (3.18) with respect to ρ to conclude

$$\frac{d\lambda}{d\rho} = - \frac{\partial H}{\partial \rho}(\rho, \lambda(\rho)) / \frac{\partial H}{\partial \lambda}(\rho, \lambda(\rho)).$$

In view of (3.19)₃ and (3.26) we find that $H_\lambda(\rho, \lambda(\rho))$ is strictly negative and hence that

$$\text{sgn } \frac{d\lambda}{d\rho} = \text{sgn } \frac{\partial H}{\partial \rho}(\rho, \lambda(\rho)). \quad (4.5)$$

Theorem 4.2. There exists a $\rho_0 > 0$ such that

$$\frac{d\lambda}{d\rho} > 0 \text{ for } \rho \in (0, \rho_0). \quad (4.6)$$

Proof. In view of (4.5) we prove that $H_\rho(\rho, \lambda(\rho))$ is strictly positive for $\rho \in (0, \rho_0)$. If we differentiate (3.19) with respect to ρ we find that

$$\frac{1}{f(\rho)} \frac{\partial H}{\partial \rho}(\rho, \lambda(\rho)) = \dot{v}(\rho f(\rho)) - \frac{4t(\rho)f(\rho)}{(1+t(\rho))^2} \dot{v}(\rho). \quad (4.7)$$

We note that by Theorem 1 of [10], \dot{v} is strictly positive on $(0, \infty)$. Thus if we let $\rho \rightarrow 0$ in (4.7) we discover, with the aid of (3.19)₃, and Proposition 4.1(iv), that the righthand side of (4.7) is strictly positive for sufficiently small ρ . This gives the desired result.

□

We next get more precise information on the asymptotic behavior of $\lambda(\rho)$ near infinity.

Proposition 4.3. As $\rho \rightarrow \infty$ $\lambda(\rho)$ tends to the value $\tau^{-\frac{(m+1)}{3m+2}}$ where τ is the unique root (greater than one) of the cubic polynomial

$$\tau^3 - \frac{11m+6}{m+2} \tau^2 - 5\tau - 1.$$

Proof. For brevity we put

$$T_1 = \frac{4t}{(1+t)^2}, \quad T_2 = \frac{2(t-1)}{(1+t)^2}, \quad (4.8)$$

and substitute the asymptotic estimate (A.6)₁ into (3.19) to get

$$f[1 - fT_1] = -\frac{1}{2\rho} [1 - T_1f^2 + 2T_2f^2] - \frac{1}{\rho} [a(\rho f) - f^2T_1a(\rho)]. \quad (4.9)$$

If we let $\rho \rightarrow \infty$ in (4.9) we find, with the aid of (3.19)₃, (A.6)₁, and Proposition 4.1 (ii), that

$$\lim_{\rho \rightarrow \infty} f(\rho)T_1(\rho) = 1. \quad (4.10)$$

From the last and the definitions of f and T_1 , we discover that $t(\rho)$ tends to a nonnegative root of the quartic polynomial

$$(\tau - 1) \left(\tau^3 - \frac{11m+6}{m+2} \tau^2 - 5\tau - 1 \right).$$

We note that a simple computation shows that both the cubic polynomial and its first derivative are negative at one and hence that the polynomial has a unique root greater than one. Proposition 4.1(i) and (3.19)₃ then allow us to conclude that $t(\rho)$ tends toward this root. This concludes the proof. □

We next analyze the behavior of $\frac{d\lambda}{d\rho}$ near infinity.

Theorem 4.4. There exists a $\rho_1 > 0$ such that

Proof. In view of (4.5) we prove that $H_\rho(\rho, \lambda(\rho))$ is strictly negative for $\rho \in (\rho_1, \infty)$. If we differentiate (3.19) with respect to ρ we find, with the aid of (4.8), that

$$\rho \frac{\partial H}{\partial \rho} = \rho f \dot{v}(\rho f) - T_1 f^2 \rho \dot{v}(\rho).$$

Substituting the asymptotic expansion (A.6)₂ into the last expression yields

$$\rho \frac{\partial H}{\partial \rho} = \rho f [1 - fT_1] + [\rho f b(\rho f) - f^2 T_1 \rho b(\rho)].$$

If we combine the last expression and (4.9) we find that

$$\rho \frac{\partial H}{\partial \rho}(\rho, \lambda(\rho)) = -\frac{1}{2} [1 - T_1(\rho)f(\rho)^2 + 2T_2(\rho)f(\rho)^2] + c(\rho), \quad (4.11)$$

where

$$c(\rho) = (\rho f b(\rho f) - \rho f^2 T_1 b(\rho)) - (a(\rho f) - f^2 T_1 a(\rho)).$$

It is clear from Proposition 4.3 and (A.6) that $c(\rho)$ tends to zero as ρ becomes infinite. To evaluate the expression in square brackets on the right hand side of (4.11)₁ we use (4.10), (4.8), and Proposition 4.3 to conclude that

$$\lim_{\rho \rightarrow \infty} \rho \frac{\partial H}{\partial \rho}(\rho, \lambda(\rho)) = \frac{-1}{8\tau} (3\tau+1)(\tau-1),$$

where τ is the unique root (greater than one) of the cubic polynomial constructed in Proposition 4.3. This yields the desired result. □

Finally, we show that the function $\lambda(\rho)$ has no local minima.

Theorem 4.5 If $\dot{\lambda}(\rho) = 0$ for some $\rho > 0$ then $\ddot{\lambda}(\rho) < 0$ at the same ρ .

Proof. We first note that by (3.26) and the implicit function theorem, the function $\lambda(\rho)$ is C^2 . If we differentiate $H(\rho, \lambda(\rho)) = 0$ twice with respect to ρ and let $\dot{\lambda} = 0$ we find that

$$H_\rho = 0, \quad \ddot{\lambda} = -H_{\rho\rho} / H_{\lambda\lambda}.$$

By (3.19)₃ and (3.26) it therefore suffices to prove that $H_{\rho\rho} < 0$ whenever $H_\rho = 0$.

If we differentiate (3.19) with respect to ρ we find that

$$H_\rho = f\dot{v}(\rho f) - \frac{4t}{(1+t)^2} f^2 \dot{v}(\rho),$$

$$H_{\rho\rho} = f^2 \ddot{v}(\rho f) - \frac{4t}{(1+t)^2} f^2 \ddot{v}(\rho).$$

Thus, using $H_\rho = 0$, we conclude that

$$H_{\rho\rho} = \rho^{-1} f \dot{v}(\rho f) [w(\rho f) - w(\rho)] ,$$

where

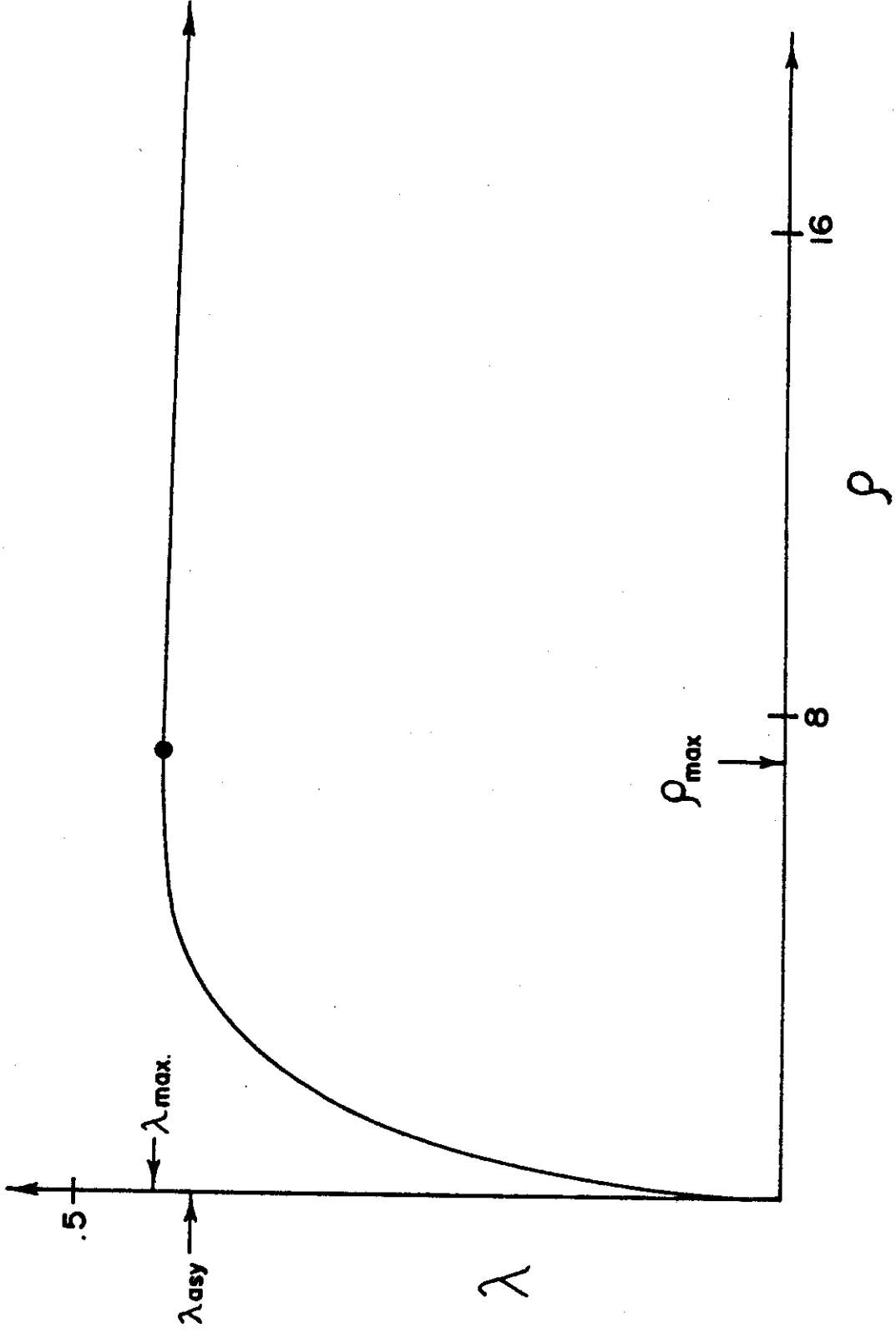
$$w(t) \equiv t \ddot{v}(t) / \dot{v}(t) .$$

Now, by Theorem 1 of [10], \dot{v} is strictly positive, while by the second proposition in the Appendix w is strictly decreasing. Thus, since $f > 1$, we conclude that $H_{\rho\rho} < 0$. This concludes the proof. \square

5. Numerical Results

In figure 1 we plot the curve $\lambda(\rho)$ that solves $H(\rho, \lambda(\rho)) = 0$ when $m = 13.3$. Other values of m between .1 and 1000 gave essentially the same picture.

FIGURE 1



If one wants to find the mode and loading value at which instability first occurs, first fix values of R and L and plot the points $\rho_n = n\pi R/L$ (for $n = 1, 2, \dots$) on the ρ axis. Then plot the points $(\rho_n, \lambda(\rho_n))$ using the graph. The highest point $\lambda(\rho_n)$ determines both the eigenmode n and the first λ . We note that if $\pi R/L > .59$ then $n = 1$ is the first eigenmode to bifurcate. If $\pi R/L \in (.32, .59)$ then $n = 2$ comes first and if $\pi R/L \in (.23, .32)$ then $n = 3$ comes first, etc.

Remark. It follows from Theorem 4.2 and Theorem 4.4 (or the above graph) that there exists geometries (ratios R/L) such that the largest λ , at which instabilities occur, can have two eigenmodes associated with it. Since the maximum value of λ , λ_{\max} , and the asymptotic value of λ , λ_{asy} , are so close to .45 for any R/L the corresponding value of λ_{cr} will always be near .45, i.e. instability always occurs for λ near .45.

Remark. We note that our numerical results are not consistent with the experiments of BEATTY & HOOK [2] and BEATTY & DADRAS [1]. While our largest value of λ_{cr} never exceeded .45, their smallest value was never less than .70. The most likely reason for this is that our choice of constitutive relation ((2.8)) is not appropriate for solid rubbers.

Appendix.

We briefly discuss the properties of the Bessel functions that we have used in this paper. The functions $I_n(t)$, the modified Bessel functions of the first kind, are globally analytic solutions of the differential equation

$$t^2 \ddot{w}(t) + t \dot{w}(t) - (t^2 + n^2)w(t) = 0,$$

and are given by

$$I_n(t) = \sum_{r=0}^{\infty} \frac{(t/2)^{n+2r}}{r!(n+r)!} . \tag{A.1}$$

This infinite series formula can be used to prove that I_0 and I_1 satisfy the recurrence relations

$$I_0'(t) = I_1(t) , \quad t I_1'(t) = t I_0(t) - I_1(t) \tag{A.2}$$

It can be shown¹⁰ that I_0 and I_1 have asymptotic expansions

$$\sqrt{2\pi t} e^{-t} I_0(t) = 1 + \frac{1}{8t} + \frac{9}{128t^2} + O\left(\frac{1}{t^3}\right), \tag{A.3}$$

$$\sqrt{2\pi t} e^{-t} I_1(t) = 1 - \frac{3}{8t} - \frac{15}{128t^2} + O\left(\frac{1}{t^3}\right)$$

as $t \rightarrow \infty$.

The results in the paper specifically deal with the function

$$v(t) \equiv t I_0(t)/I_1(t) . \tag{A.4}$$

A simple consequence of the recurrence relations (A.2) and the infinite series (A.1) is that v satisfies

$$\begin{aligned} t\dot{v}(t) &= t^2 + 2v(t) - v^2(t), \\ v(0) &= 2. \end{aligned} \tag{A.5}$$

One can also use (A.3) and (A.5) to establish the asymptotic estimates

$$\begin{aligned} v(t) &= t + \frac{1}{2} a(t), & \lim_{t \rightarrow \infty} ta(t) &= \frac{3}{8}, \\ \dot{v}(t) &= 1 + b(t), & \lim_{t \rightarrow \infty} tb(t) &= 0. \end{aligned} \tag{A.6}$$

Next, we prove a result concerning the global behavior of v .

Proposition. $v^2(t) - (t^2 + \frac{m+2}{m+1}) > 0$.

Proof. Define

$$h(t) = (t^2 + \frac{m+2}{m+1})^{1/2}, \tag{A.7}$$

the positive root of the quadratic

$$h^2(t) - (t^2 + \frac{m+2}{m+1}) = 0. \tag{A.8}$$

We will show that v is strictly greater than h to prove this proposition.

By (A.5)₂ and (A.7)

$$v(0) - h(0) = 2 - (\frac{m+2}{m+1})^{1/2} > 0 \text{ for } m > 0,$$

and hence v is strictly greater than h in a neighborhood of zero.

Suppose, for the sake of contradiction, that v is not strictly greater than h on $[0, \infty)$ and define

$$s \equiv \inf \{t \in [0, \infty): v(t) = h(t)\}.$$

It is clear (since $v(t) > h(t)$ for $t \in [0, s)$) that

$$\dot{v}(s) < \dot{h}(s).$$

However (putting $H = h(s)$ and using (A.5)₁ and (A.8))

$$\begin{aligned} \dot{v}(s) - \dot{h}(s) &= \frac{1}{s} [s^2 + 2H - H^2] - \frac{s}{H} \\ &= \frac{1}{sH} \left[\left(2H - \frac{m+2}{m+1}\right)H - s^2 \right] \\ &= \frac{1}{sH} \left[\left(H - \frac{m+2}{2m+2}\right)^2 + \frac{1}{4} \frac{m+2}{m+1} \left(4 - \frac{m+2}{m+1}\right) \right] > 0, \end{aligned}$$

for $m > 0$. This is a contradiction and thus no such s can exist.

□

Finally, we prove a result concerning the function

$$w(t) \equiv \ddot{v}(t)/\dot{v}(t).$$

Proposition. $\dot{w}(t)$ is strictly negative for $t \in (0, \infty)$.

Proof. If we differentiate (A.5)₁ we find that

$$\frac{1}{2} w(t) = \frac{t}{v(t)} - v(t) + \frac{1}{2},$$

and hence that

$$\frac{1}{2} \dot{w}(t) = -2[u(t) - v(t)]/\dot{v}(t),$$

where

$$u(t) \equiv [2t + \dot{v}^3(t)]/2\dot{v}(t).$$

To finish the proof we will show that

$$u(t) > v(t) \quad \text{for all } t > 0. \quad (\text{A.9})$$

To show that (A.9) is true near $t = 0$ define

$$q(t) \equiv v(t) [u(t) - v(t)]. \quad (\text{A.10})$$

Then by differentiating (A.5)₁ and (A.10) a sufficient number of times one can show that

$$\dot{v}(0) = 0, \quad \ddot{v}(0) = \frac{1}{2}, \quad \dddot{v}(0) = 0, \quad v^{(4)}(0) = -\frac{1}{4}$$

and hence that

$$q(0) = 0, \quad \dot{q}(0) = 0, \quad \ddot{q}(0) = 0, \quad \dddot{q}(0) = \frac{1}{8}.$$

We conclude that (A.9) holds in a neighborhood of zero.

Next, assume for the sake of contradiction that (A.9) is not true on $(0, \infty)$ and define

$$s \equiv \inf \{t \in (0, \infty): u(t) = v(t)\}.$$

It is clear that $v(s) > u(s)$. However

$$\begin{aligned} \dot{u}(s) - \dot{v}(s) &= [\dot{v}^3 \ddot{v} + (\dot{v} - s\ddot{v}) - \dot{v}^3] / \dot{v}^2 \\ &= [\dot{v}^3 \ddot{v} + (2v\dot{v} - 2s) - \dot{v}^3] / \dot{v}^2 \\ &= \dot{v}(s)\ddot{v}(s) > 0, \end{aligned}$$

a contradiction. Thus no such s can exist. □

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REFERENCES

- [1] Beatty, M. F. and P. Dadrus, Some experiments on the elastic stability of highly elastic bodies, *Int. J. of Eng. Sci.*, 14 (1976), 233-238.
- [2] Beatty, M. F. and D. E. Hook, Some experiments in the stability of circular rubber bars under end thrust *Int. J. Solids Structures* 4 (1968), 623-635.
- [3] Blatz, P. J. and W. L. Ko, Application of finite elasticity theory to the deformation of rubbery materials, *Trans. Soc. Rheology* 6 (1962), 223-251.
- [4] Bridgman, P., The compression of sixty-one solid substances to 25,000 kg/cm², determined by a new rapid method, *Proc. Am. Acad. Arts Sci.*, 76 (1945), 9-24.
- [5] Burgess, I. W. and M. Levinson, The instability of slightly compressible rectangular solids under biaxial loadings, *Int. J. Solids Struc.* 8 (1972) 133-148.
- [6] Copson, E. T., *Asymptotic Expansions*, Cambridge University Press: Cambridge (1965).
- [7] Sawyers, K. N. and R. S. Rivlin, Stability of a thick elastic plate under thrust, *J. Elasticity*, 12 (1982) 101-125.
- [8] Sawyers, K. N., Stability of a thick neo-Hookean plate, *J. Elasticity*. Forthcoming.
- [9] Simpson, H. C. and S. J. Spector, On barrelling instabilities in elasticity, *J. Elasticity*. Forthcoming.
- [10] Simpson, H.C. and S. J. Spector, On the modified Bessel functions of the first kind. Institute for Mathematics and its Applications Preprint Series.
- [11] Wilkes, E. W., On the stability of a circular tube under end thrust, *Q. J. Mech. Appl. Math.*, 8 (1955) 88-100.

FOOTNOTES

- 1 Experiments of BEATTY & HOOK [2] and BEATTY & DADRAS [1] have shown that such cylinders retain their cylindrical shape until a certain critical value of the loading is reached. Further loading of the cylinder results in bucking for thin cylinders and axisymmetric bulging (barrelling) for thick cylinders.
- 2 BURGESS & LEVINSON [5] have used this material in their analysis of biaxial compression of rectangular rods. Some of the analysis in this paper has already been done for a Hadamard - Green material, cf. SIMPSON & SPECTOR [9].
- 3 A solution in which the cylinder is deformed homogeneously and retains its cylindrical shape.
- 4 The first such analysis was done by WILKES [11], who assumed that the cylinder was infinite and composed of an incompressible material.
- 5 This is a crucial point that (to our knowledge) has not been addressed in the literature. If an additional solution, not obtainable by separation of variables, exists this solution might satisfy the remaining boundary condition at a smaller value of the loading than the separation of variables solution.
- 6 This contrasts with results of SAWYERS & RIVLIN [7] and SAWYERS [8] for incompressible materials. Their analysis indicates that λ increases with n for barrelling and hence the first mode is $n = \infty$. This is thought to be exhibited physically as surface wrinkling.
- 7 We note that we only use this constitutive relation for linear transformations whose determinant is less than or equal to one.
- 8 This observation is due to Professor E. Sternberg.
- 9 cf. SIMPSON & SPECTOR [9], Theorem 4.3.
- 10 cf. COPSON [6], pp. 72-75 where asymptotic expansions for J_0 and J_1 are rigorously derived using Watson's Lemma.