PHASE TRANSITIONS: STABILITY AND ADMISSIBILITY
IN ONE DIMENSIONAL NONLINEAR VISCOELASTICITY

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PHASE TRANSITIONS: STABILITY AND ADMISSION OF IN ONE DIMENSIONAL NONLINEAR VISCOELASTICITY

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Abstract

We study a mixed initial-boundary problem for motion of a one-dimensional viscoelastic material of rate type with nonmonotone elastic stress-strain relation. We establish strong asymptotic convergence to equilibrium as $t \to \infty$, an issue left open by Andrews and Ball (J. Diff. Eqns. 44 (1982)). We give criteria for dynamic stability and instability of equilibria. The property of strong energy minimization is shown to be not necessary for stability of equilibria. In particular, "metastable" and "stable" phases can coexist in stable, discontinuous asymptotic limits of smooth solutions. A precise description of the smoothing of solutions in time versus space is derived from a simple existence theory. Also, a viscosity criterion is proposed for the admissibility of waves in the associated elastic model, and a description of the phenomenon of propagating phase boundaries in a loaded elastic bar is put forward.

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1. **Introduction**

When considering the problem of explaining the arrangement of phases in a material system, an approach which originates with Gibbs (1906) is to identify those material states which have minimum energy, at constant entropy. Because dynamical processes dissipate energy, one may expect the system to eventually reach such a state. Identifying stable equilibria by this method elegantly circumvents dynamic analysis, and has proven to be a brilliantly successful tool for describing multiple-phase equilibria in many material systems.

But Gibbs himself was aware that there might be obstacles preventing or hindering a system from achieving a state of minimum energy. Such obstacles seem to be of considerable importance in the study of phase changes in metals and other solids. Energy minimization arguments typically predict that phases can coexist only if the energy density in each phase is the same, at the absolute minimum. But, for example, (cf. Muller and Wilmanski (1981)), austenitic and martensitic phases coexist in many metals in broad ranges of temperature. It is considered most unlikely that these phases maintain identical energy densities other than at isolated values of temperature. As a second example, twinned martensite is composed of symmetry-related phases, with equal energy densities at zero stress. When placed under a load which favors one twin, however, almost certainly the energy densities of the phases differ, yet they are observed to coexist. Explaining such phenomena seems to require a deeper understanding of dynamic processes in the approach to equilibrium. Of course, the study of dynamics also possesses intrinsic interest for describing time-dependent phenomena.

In part I of this paper we consider a simple one-dimensional model for motion of a viscoelastic bar which may undergo phase changes. This model is allied with a purely elastic one proposed by Ericksen (1975), and has been studied by Dafermos (1969), Andrews (1980) and Andrews and Ball (1982). Its principal feature is a non-monotone stress–strain relation permitting the coexistence of several states of strain at equilibrium in a range of applied stress levels. Closely related models include equations for isothermal motions of a van der Waals gas, and equations for shearing motion in polymeric fluid (Hunter and Slemrod, 1983).
The basic elastic model is an equation of mixed type, with an ill-posed initial value problem for some ranges of strain. The viscoelastic model includes a viscosity term of rate type, which serves to dissipate energy and regularize the initial value problem. In a particular initial-boundary value problem corresponding to motion of a viscoelastic bar with a prescribed "soft load" at the end, we shall establish a criterion for the dynamic stability of coexistent phases in equilibrium. (See (2.5).) This criterion does not imply that a stable equilibrium must be a Gibbsian equilibrium, a state which minimizes energy absolutely. Instead, stable equilibria are states which do minimize energy locally, in the weak sense of the calculus of variations.

Our study also concerns some other questions about dynamics in bars containing phase boundaries. In the viscoelastic model, we show how hysteresis in stress vs. strain can occur in a slow dynamic loading-unloading process, and we model a "creep" phenomenon at constant load. In part II we explore some consequences of the consideration of viscosity for dynamics in the purely elastic model. We derive an admissibility condition for propagating waves in the elastic model based on the limit of vanishing viscosity. In experiments on bars under an increasing soft load, the appearance of new phases can be associated with slowly moving, sharply defined waves. (See James (1980b) for a description of some experimental literature on polymers and metal bars.) We propose an account for the appearance of moving phase boundaries in a bar under increasing load based on an initial boundary value problem whose solution is justified by the admissibility criterion discussed.

I. Approach to equilibrium and stability of coexistent phases in a viscoelastic bar

2. Problem description and discussion

Ericksen (1975) suggested that the phenomena of phase transitions in bars might be modeled by the equation of one-dimensional elasticity,

\[ u_{tt} = \sigma(u)_x. \]
Here $u(x)$ denotes the displacement of the bar at reference point $x$. The stress $\sigma(w)$ is taken to be nonmonotonic, for example, of the form indicated in Fig. 1. Phases of the bar correspond to maximal intervals of monotonicity of $\sigma(w)$. The condition for equilibrium, $\sigma(u_x) = P$ constant, permits the existence of several equilibrium states in different phases for a given load $P$, and also equilibria with discontinuous strain $u_x^i$, having coexistent phases. We also think of (2.1) rewritten as a system of conservation laws: with $w = u_x$ the strain and $v = u_t$ the velocity, (2.1) becomes

\begin{align*}
  w_t &= v_x \\
  v_t &= \sigma(w)_x .
\end{align*}

(2.2)

For strain in ranges where $\sigma(w)$ is decreasing, equation (2.1) is elliptic, making the initial-value problem ill-posed. There is some hope that by considering weak solutions of (2.2) with strain restricted to lie outside the elliptic range, the initial value problem might be rendered well-posed; our results of part II have some relevance for this issue. In any case, the problem of the stability of equilibria in (2.1) is at present intractable. We consider instead a physically relevant regularization of (2.1), namely an equation of viscoelasticity of rate type,

\begin{equation}
  u_{tt} = \left( \sigma(u_x) + \mu u_{xt} \right)_x , \quad \mu > 0 .
\end{equation}

(2.3)

Note that (2.1) and (2.3) have the same equilibrium solutions, with possibly discontinuous strain. In this part I we study the long-time behavior of solutions of a particular initial boundary value problem corresponding to a bar in a soft loading device. In such a device, one end of the bar is fixed, the other subjected to a prescribed load. Fixing $\mu = 1$ for convenience, we consider the problem

\begin{equation}
  \begin{align*}
    u_{tt} &= \left( \sigma(u_x) + u_{xt} \right)_x , \quad \text{for } 0 < x < 1, \ t > 0 , \\
    u(0,t) &= 0 , \\
    (\sigma(u_x) + u_{xt})(1,t) &= P \quad \text{for } t > 0 , \\
    u_x(x,0) &= u_0(x) , \\
    u_t(x,0) &= u_1(x) \quad \text{for } 0 \leq x \leq 1 .
  \end{align*}
\end{equation}

(2.4)
These equations were first considered with monotone \( \sigma \) by Greenberg, MacCamy and Mizel (1968) (see also Greenberg (1969) and Greenberg and MacCamy (1970)) who showed that given smooth initial data, a smooth solution exists globally in time and decays to equilibrium at an exponential rate. Dafermos (1969) considered the more general equation \( u_{tt} = \sigma(u_x, u_x) \) with a parabolicity assumption and no monotonicity assumption in the first argument of \( \sigma \), but with a rather restrictive growth condition. He showed that smooth solutions exist globally in appropriate Hölder classes (with \( u(\cdot, t), u_t(\cdot, t) \) in \( C^{2+\alpha}(0, 1) \)) and found that the velocity \( u \) and the total stress \( \sigma(u_x, u_x) \) decay to zero as \( t \to \infty \) in the Sobolev space \( W^{1,\infty}(0, 1) \). Dafermos also observed that the asymptotic behavior of the strain was an interesting issue, arguing that \( u(x, t) \) in general need not approach a continuous function as \( t \to \infty \).

Andrews and Ball (1982), following Andrews (1980), established global existence of weak solutions to the problem (2.4) for initial data with \( u_x(x, 0) \) in \( L^\infty(0, 1) \), \( u_t(x, 0) \) in \( L^2(0, 1) \), in particular admitting equilibria with discontinuous strain as data. Due to an a priori bound of Andrews for the strain \( u_x \), the growth of \( \sigma \) was hardly restricted, and physically relevant \( \sigma \) satisfying \( \sigma(w) \to -\infty \) as \( w \downarrow 0 \) could be treated for data with limited energy. These authors showed that as \( t \to \infty \), the velocity \( u_t \to 0 \) in \( L^2 \) and the elastic part of the stress \( \sigma(u_x) \to \mathcal{P} \) in \( L^2 \). They established weak-* convergence of the strain \( u_x(x, t) \to w_\infty(x) \) in \( L^\infty \), but this left open the issue of whether in fact \( \sigma(w_\infty(x)) = \mathcal{P} \), that is, whether the strain converged to equilibrium.

The present treatment of the problem (2.4) will be largely self-contained for the convenience of the reader, though it owes a sizeable debt to the works above. In the first place, in §3 we present a simplified local existence and regularity theory for solutions of (2.4), based on the theory of abstract semilinear parabolic equations as presented in Henry (1981). As a consequence, we find that given initial strain \( u_x(x, 0) \) in \( L^\infty \) and velocity \( u_t(x, 0) \) in \( L^2 \), the equation (2.4) is satisfied in an "almost classical" sense for \( t > 0 \): The total stress \( \sigma(u_x) + u_{xt} \) is \( C^1 \) in \( x \) and \( t \),
and both sides of (2.4.) represent continuous functions. However, the strain $u_x^1$ and elastic stress $\sigma(u_x)$ have no better regularity than $L^\infty$ in $x$. Initial discontinuities in strain must persist for all time without moving. (This fact has been discovered by Hoff and Smoller (to appear) in the context of isothermal gas dynamics with viscosity.) In $t$ however, solutions are smooth if $\sigma$ is smooth. In the same abstract framework, we can also establish existence of classical solutions (with $C^1$ strain $u_x$) provided $u_x(x,0) \in C^1[0,1]$ and $u_t(x,0) \in W^{1,2}[0,1]$ with $u_t(0,0) = 0$.

Next, in §4 we begin to investigate the stability of equilibria having possibly discontinuous strain $u_x = w(x)$, and introduce the following dynamic stability criterion:

$$\sigma'(\tilde{w}(x)) \geq \sigma_0 > 0 \quad \text{a.e. where } \sigma(\tilde{w}(x)) = P.$$  

Based on a spectral study, we establish global existence and exponential decay to zero of small perturbations of equilibria satisfying (2.5). Perturbations in strain are required to be small in $L^\infty$, however, so discontinuities in the asymptotic strain must also be present in the initial data. (This requirement will be relaxed in §6, at the expense of the exponential decay rate.)

In §5 we investigate the asymptotic behavior of solutions for a large class of initial data. (If $\sigma(w)$ is defined for all $w$ and satisfies the mild condition (5.1), the initial data are unrestricted.) We show that solutions converge strongly to equilibrium, establishing the following asymptotic properties for the solution: As $t \to \infty$,

$$u_t \to 0 \quad \text{in } W^{1,2}(0,1)$$

$$\sigma(u_x) + u_{xt} \to P \quad \text{in } W^{2,2}(0,1)$$

$$u_x(x,t) \to w_\infty(x) \quad \text{boundedly a.e., where } \sigma(w_\infty(x)) = P \text{ a.e.}$$

Asymptotic states for smooth solutions are investigated in §6. Here equilibria $\tilde{u}(x)$ with possibly discontinuous strain $\tilde{w}(x) = \tilde{u}_x$ satisfying the stability criterion (2.5) are shown to be stable in a rather strong sense: If the perturbation in strain is small except on a set of small measure $\epsilon$, then the solution will approach
an equilibrium with strain equal to the unperturbed strain except perhaps on the same set of measure $\epsilon$. It follows that a large class of smooth solutions have asymptotic limits with discontinuous strain.

It is important to relate the stability criterion obtained here to traditional energy criteria for stability of equilibria involving phase mixtures. (In the context of the present problem, such criteria have been discussed in Ericksen (1975) and James (1980a).)

Define a stored energy function

\[(2.7) \quad W(w) = \int_{\omega} \sigma(s) \, ds.\]

In the soft loading device of \((2.4)\), the total energy

\[\int_{0}^{1} \left[ \frac{1}{2} u_x^2 + W(u_x) - P u_x \right](y, t) \, dy\]

is a decreasing function of time. One expects the total stored energy functional

\[(2.8) \quad I(u) = \int_{0}^{1} \left[ W(u_x) - P u_x \right](y, t) \, dy\]

to approach a minimum. In the present model, a typical function $W(w) - Pw$ as obtained from Fig. 1 might be as in Fig. 2, having two local minima $\alpha_-$ and $\beta_-$. The problem of minimizing $I(u)$ in \((2.8)\) is a standard one in the calculus of variations. Much attention has traditionally been focussed on absolute minima, which may be achieved only at a strong minimizer $\bar{u}(x)$. A strong minimizer satisfies $I(\bar{u} + v) \geq I(\bar{u})$ for all absolutely continuous $v(x)$ with $\|v\|_{L^\infty} \leq \epsilon$ small. At a strain discontinuity the Weierstrass-Erdmann corner conditions for a strong minimizer imply that $W(u_x) - P u_x$ is continuous across the discontinuity. Thus in the situation of Fig. 2, an equilibrium containing a phase discontinuity from $\alpha_-$ to $\beta_-$ cannot be
a strong minimizer unless the stored energy density is the same in the two phases, that is, $W(\alpha_-) - P\alpha_- = W(\beta_-) - P\beta_-$. This implies that the stress level $P$ in Fig. 1 is at the Maxwell line, meaning that the two bounded regions in Fig. 1 delineated by the curves $\sigma = P$ and $\sigma = \sigma(w)$ have equal area.

Regardless of these facts, it is clear that an equilibrium $\tilde{u}(x)$ for (2.4), whose strain $\tilde{u}_x$ takes both values $\alpha_-$ and $\beta_-$, can easily satisfy the condition (2.5) for dynamic stability even if the stored-energy densities at $\alpha_-$ and $\beta_-$ differ. Indeed, the condition (2.5) is the condition in the calculus of variations that the second variation $\delta^2 I(u)$ of the functional $I(u)$ be strictly positive, and it implies that $\tilde{u}(x)$ is a weak relative minimizer of $I(u)$, satisfying $I(\tilde{u} + v) \geq I(\tilde{u})$ whenever both $\|v\|_{L^\infty}$ and $\|v_x\|_{L^\infty}$ are small.

To paraphrase our results, then: Positivity of the second variation implies dynamic stability in the viscoelastic problem (2.4). The property of being a strong minimizer, on the other hand, is not necessary for dynamic stability in (2.4). It is common to call a state of strain metastable (resp. stable) if it is at a local (resp. global) minimum of the stored-energy density $W(w) - Pw$. Thus for (2.4) metastable states can coexist stably with stable states in asymptotic limits of smooth solutions.

We finish part I with an application of the stability analysis of §6 to a hypothetical load-deformation experiment. In §7, we consider a bar placed in a soft loading device, in which the end load $P$ is to be raised or lowered in increments, and "fluctuations" are to be imposed on the displacement and velocity, at discrete time intervals. (The fluctuations are regarded as a model accounting for unknown physical influences on the bar.) We illustrate how hysteresis occurs dynamically under suitable restrictions on the load increments and fluctuations. The process must proceed slowly, but we do not assume it is "quasistatic"; inertial effects are fully taken into account. Holding the load constant near a level of transition between phases and making certain assumptions about the fluctuations, we also illustrate how a "creep" phenomenon can occur, as a slow change in the overall length of the bar due to inhomogeneities induced by localized fluctuations of small energy.
We conclude this section by introducing our main tools for studying (2.4). The following transformation of the equations will be very useful throughout:

Suppose we have a (smooth) solution of (2.4), and introduce quantities

\[
p(x, t) = \int_{1}^{x} u_t(y, t) \, dy
\]

\[q(x, t) = u_x(x, t) - p(x, t) \quad .\]

The solution \( u \) may be recovered from

\[
u(x, t) = \int_{0}^{x} (p + q)(y, t) \, dy \quad .\]

Then \( p \) and \( q \) form a solution to the problem

\[
p_t = p_{xx} + \sigma(p+q) - P \quad \text{for } 0 < x < 1, \quad t > 0
\]

\[q_t = -\sigma(p+q) + P \]

\[p_x(0, t) = 0 \quad , \quad p(1, t) = 0 \quad \text{for } t > 0
\]

\[p(x, 0) = p_0(x) \quad , \quad q(x, 0) = q_0(x) \quad \text{for } 0 \leq x \leq 1
\]

I note that the "modified strain" \( q \) is the same (up to sign) as that used by Andrews (1980) to establish global existence of solutions to (2.4) based on the ODE that \( q \) satisfies. In equilibrium \( q \) is equal to the strain \( u_x \). The "velocity potential" \( p \) also has an interesting property. Write the viscoelastic equation in (2.4) as a system in the standard way: with the total stress

\[
S = \sigma(u_x) + u_{xt} - P
\]

we have

\[
w_t = v_x
\]

\[v_t = S_x
\]

where
\[ w = u_x, \quad v = u_t. \]

Then
\[ v = p_x, \quad S = p_t. \]

The fact that \( S \) is the time derivative of \( p \) will be important for establishing its regularity. We remark that the structure of (2.10), a parabolic PDE coupled to an ODE, illustrates why the viscoelastic equation may be viewed as an abstract parabolic equation (even though the viscosity in (2.12) is "degenerate"!), and exhibits explicitly a zero-speed characteristic in equation (2.3), associated with the ODE for \( q \) in (2.10).

The transformation above is our main tool for the existence and regularity theory. Two other ingredients are important in the study of asymptotic behavior and stability: the energy identity, and a lemma on invariant intervals for the ODE in (2.10). With \( W(w) \) the stored energy function from (2.7), the energy identity for (2.4) reads

\[
(2.13) \quad \int_0^1 \left( \frac{1}{2} u_t^2 + W(u_x) - Pu_x \right) (y, t) dy + \int_0^t \int_0^1 u_{x t}^2 (y, \tau) dy d\tau
\]

\[ = \int_0^1 \left( \frac{1}{2} u_t^2 + W(u_x) - Pu_x \right) (y, 0) dy. \]

In terms of \( p \) and \( q \) this reads

\[
(2.14) \quad \int_0^1 \frac{1}{2} p_x^2 (x, t) dy + \int_0^1 \left( W(p+q) - P(p+q) \right) (y, t) dy + \int_0^t \int_0^1 \frac{1}{2} p_{xx}^2 (y, \tau) dy d\tau
\]

\[ = \int_0^1 \frac{1}{2} p_x^2 (y, 0) dy + \int_0^1 \left( W(p+q) - P(p+q) \right) (y, 0) dy. \]

Provided \( W(w) - Pw \) is bounded below, the energy identity yields an \textit{a priori} bound
on the kinetic energy, hence on \( p \) pointwise, by the estimate

\[
|p(x, t)|^2 \leq (1 - x) \int_0^1 p_x^2(y, t) \, dy \leq \int_0^1 u_t^2(y, t) \, dy
\]

valid since \( p(1, t) = 0 \). Control on \( q \) is to be achieved pointwise in \( x \) via the following easily proved lemma:

**Lemma 2.1 (Invariant intervals).** Consider the ODE \( q'(t) = -\sigma(p(t) + q(t)) \) where \( \sigma(w) \) is a locally Lipschitz function on an interval in \( \mathbb{R} \) and \( p(t) \) is locally Lipschitz in \( t \). Assume that \( \epsilon > 0 \) and \( w_- < w_+ \) exist such that

i) \( |p(t)| < \epsilon \) for \( 0 \leq t \leq T \leq \infty \)

ii) \( \sigma(w_- + w) < 0 < \sigma(w_+ + w) \)

whenever \( |w| < \epsilon \).

Then the interval \([w_-, w_+] \) is positively invariant for the ODE on the time interval \([0, T]\). That is, if \( q(t_0) \in [w_-, w_+] \) and \( 0 \leq t_0 \leq T \), then \( q(t) \in [w_-, w_+] \) whenever \( t_0 \leq t \leq T \) (Fig. 3).

The meaning of this lemma is that the strain in (2.4) cannot change phase at any point, provided only that the total kinetic energy of the bar remains small. The stability results of §6 then depend on controlling the conversion of potential energy to kinetic energy in a situation where there may be smooth transition layers joining "metastable" and "stable" phases.

3. **Local existence and regularity**

We consider the problem (2.4) (or equivalently, (2.10), see 3.2, 3.3 below). We assume that the stress \( \sigma(w) \) is defined on an open interval \( J \subseteq \mathbb{R} \), and will consider cases where a) \( \sigma(w) \) is locally Lipschitz, and b) \( \sigma(w) \) is \( C^1 \) with \( \sigma'(w) \) locally Lipschitz. In the latter case we obtain greater regularity and existence of classical solutions for (2.4). In §5 below, we will impose an additional condition (e.g., see (5.1)) on \( \sigma(w) \) to establish global existence for initial data of unrestricted energy.
Our basic existence theorem concerns the transformed problem (2.10):

**Theorem 3.1.**

(1) **(Local existence)** Suppose \( \sigma(w) \) is locally Lipschitz. Assume that \( q_0 \in L^\infty(0, 1) \) and \( p_0 \in W^{1,2}(0, 1) \) with \( p_0(1) = 0 \), and suppose \( p_0(x) + q_0(x) \in \mathcal{J}_0 \) a.e. where \( \mathcal{J}_0 \subset \mathcal{J} \). Then there exists \( T > 0 \) so that a unique (strong) solution to (2.10) exists for \( 0 \leq t \leq T \), with

\[
p \in C([0, T], W^{1,2}) \cap C^1([0, T], L^2) \cap C([0, T], W^{2,2})
\]

and \( p_x(0, t) = 0 = p(1, t) \) for \( t > 0 \),

\[
q \in C([0, T], L^\infty).
\]

(2) **(Regularity)** For any \( \delta > 0 \) and \( \nu < 1/2 \), the solution above has, for some \( \beta > 0 \),

\[
p_t = s \in C^\beta([\delta, T], C^1, \nu) \quad .
\]

with \( p_x(0, t) = 0 = p(1, t) \) for \( t > 0 \),

\[
q_t, p_{xx} \in C^\beta([\delta, T], L^\infty) \quad .
\]

(Here \( C^\beta \) is a space of Hölder continuous functions, and \( C^1, \nu \) is the space of \( C^1 \) functions on \([0, 1]\) with Hölder continuous derivative, with exponent \( \nu \).) If \( \sigma(w) \) is \( C^1 \) with \( \sigma'(w) \) locally Lipschitz, then also

\[
p_{tt} = s_t \in C^\beta([\delta, T], C^1, \nu)
\]

\[
p_{txx} = s_{xx} \in C^\beta([\delta, T], L^\infty)
\]

Furthermore, if \( \sigma(w) \) is \( C^r \), \( r \geq 2 \) (resp. analytic), then \( p \) and \( q \) are \( C^r \) in \( t \) (resp. analytic in \( t \)) with values in \( W^{1,2} \) and \( L^\infty \) respectively, for \( t > 0 \).

The result above yields "almost classical" solutions to the problem (2.4), obtained from \( u(x, t) = \int_0^x (p + q)(y, t) \, dy \):

**Corollary 3.2.** Suppose \( \sigma(w) \) is locally Lipschitz. Assume \( u_0 \in L^\infty(0, 1) \) with \( u_0(x) \in \mathcal{J}_0 \) a.e., \( \mathcal{J}_0 \subset \mathcal{J} \), and \( u_1 \in L^2(0, 1) \). Then a unique solution \( u(x, t) \) to (2.4)
exists for $t$ in some interval $[0, T)$, $T > 0$, with (for any $\delta > 0$, $\nu < 1/2$, and some $\beta > 0$)

$$u \in C([0, T], W^{1, \infty}) \quad u(0, t) = 0$$

$$u_{xt} \in C^\beta([\delta, T], L^\infty)$$

$$u_{tt} \in C^\beta([\delta, T], C^\nu)$$

$$S = u_{xt} + \sigma(u_x) - p \quad \text{for a.e. } x.$$ 

If $\sigma(w)$ is $C^1$ with $\sigma'(w)$ locally Lipschitz, then also

$$u_{tt} \in C^\beta([\delta, T], W^{1, \infty}).$$

Thus both $u_{tt}$ and $S_x$ are at least Hölder continuous for $t > 0$, with $u_{tt} = S_x$, $u(0, t) = 0 = S(1, t)$.

**Remark 3.3.** The weak solutions of (2.4) constructed by Andrews and Ball with initial data as in (3) yield, by the transformation (2.9), weak solutions of (2.10) which are solutions pointwise a.e., with $p, q \in C([\delta, T], L^\infty)$ for any $\delta > 0$. By uniqueness, the solutions agree with those of (3.1). So the weak solutions of Andrews and Ball in fact have the regularity indicated in (3.2) above. (To establish the correspondence, write the weak form of the equation (2.4) in terms of test functions $\psi$ of the form $\psi = \phi_x$. It follows that (2.10) holds in the sense of distributions, for $L^\infty$ functions.)

At the end of this section we will discuss the local existence of classical solutions of (2.4) (having $u_x \in C^1([0, 1])$) by modifying a bit the framework we employ below. We will also make a remark concerning the persistence of initial discontinuities in $u_x$ in (2.4). Now, we present a fact that will make the study of solutions of the ODE (2.10) technically a bit easier:

**Corollary 3.4.** Let $(p, q)$ be a solution of (2.10) as given by Theorem 3.1. Then there is a fixed subset of $[0, 1]$, depending on the solution, such that, for those $x$ in the subset, $q(x, t)$ satisfies (2.10) for all $t, 0 < t < T$. Thus $q$ is a classical solution of the ODE (2.10) for $t > 0$, for a.e. $x$ in $(0, 1)$. 

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Proof of 3.1: Our approach is to apply standard results in Henry (1981) concerning the abstract initial value problem in a Banach space $X$,

\begin{align*}
  z_t + A z &= f(z) \quad t > 0 \\
  z(0) &= z_0
\end{align*}

and deduce pointwise properties of the solution from embedding theorems. In our application, $z = (p,q)$ and the space $X = (L^2_t, L^\infty_x)$. The operator $A = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$ where $\Delta = \frac{\partial^2}{\partial x^2}$. For the boundary conditions in (2.10) the domain of $A$ is taken as $D(A) = (D(\Delta), L^\infty)$ where

$$D(\Delta) = \left\{ p \in W^{2,2}_b(0,1) \mid p_x(0) = 0 = p(1) \right\}.$$

We also write $D(\Delta) = W^{2,2}_b$. Now $-\Delta$ is a self-adjoint, densely defined, positive operator on $L^2(0,1)$, so $A$ is sectorial on $X$. For the sequel, we need to identify the domain of the square root of $A$, $X^{1/2} = D(A^{1/2})$:

Lemma

$$X^{1/2} = (W^{1,2}_b, L^\infty_x),$$

where

$$W^{1,2}_b = \left\{ p \in W^{1,2}(0,1) \mid p(1) = 0 \right\}.$$

Proof: It is clear that $X^{1/2} = (D((-\Delta)^{1/2}), L^\infty)$. Denote $D((-\Delta)^{1/2})$ by $Y$. The space $Y$ is the closure of $D(\Delta)$ in the norm given for $u \in D(\Delta)$ by

$$\|u\|_{1/2}^2 = \|(-\Delta)^{1/2} u\|_2^2 = (-\Delta u, u)$$

$$= \int_0^1 -u_{xx} u = \int_0^1 u_x^2.$$

Therefore $Y$ is the closure of $D(\Delta)$ in the $W^{1,2}$ norm, so $Y \subset W^{1,2}_b$ and $u(1) = 0$ for $u \in Y$. So $Y \subset W^{1,2}_b$. Conversely, if $u \in W^{1,2}_b$ we may approximate $u$ by $u_n \in D(\Delta)$ as follows: $u(x) = \int_1^x u_x$, so let $v_n = u_{xx}$ approximated by smooth $v_n$ in $L^2$ with $v_n(0) = 0$. With $u_n = \int_1^x v_n$ we have $u_n \in D(\Delta)$, $u_n \to u$ in $W^{1,2}$.  

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The last thing to check before applying the abstract theory is the smoothness of the map

\[(p, q) \mapsto (\sigma(p+q) - P, -\sigma(p+q) + P)\]

from \(W^{1, 2}_b, L^\infty\) to \(L^2, L^\infty\). Since the inclusions

\[W^{1, 2}_b \hookrightarrow L^\infty \hookrightarrow L^2\]

are analytic (linear), the smoothness of the map above is the same as the smoothness of the map \(w(x) \mapsto \sigma(w(x))\) on \(L^\infty\). But it is clear that if \(\sigma\) is Lipschitz, \(C^r\) or analytic on a closed interval \([a, b]\), this map is Lipschitz, \(C^r\) or analytic on \(\{w \in L^\infty(0, 1) \mid w(x) \in [a, b], \text{ a.e.}\}\).

We may now invoke Henry 3.3.3 (1981) to deduce existence of a unique local solution of (3.1), given \(z_0 = (p_0, q_0) \in X^{1/2}\) such that \((p_0 + q_0)(x) \in J\) a.e. This solution has the following regularity at this point (see Henry 3.2.1 and the proof of 3.3.3):

\[z(t) \in C\left([0, T], X^{1/2}\right) \cap C^1\left([0, T], X\right) \cap C\left((0, T], D(A)\right).\]

(Continuity into \(D(A)\) follows because \(z\) satisfies (3.1) for \(t > 0\).) Taking components of \(z\), part (1) of Theorem 3.1 follows. That \(q_t \in C\left([0, T], L^\infty\right)\) follows from the equation (2.10). Note at this point that \(p_t\) exists as a distribution with \(p_{tt}\) in \(L^2\), and (2.10) holds for each \(t > 0\) in the sense of \(L^2\) functions. Later in §5 we will need more precise bounds for \(z(t)\) in the spaces above.

The next step is to deduce greater regularity of the solution for \(t > 0\) from Henry 3.5.2, which implies

\(z_t\) is locally Hölder continuous from \((0, T]\) to \(X^\gamma\), for any \(\gamma < 1\).

Now clearly \(X^\gamma = (D((-\triangle)^\gamma), L^\infty)\), and it is true that

\[D((-\triangle)^\gamma) \subset C^{1, \nu}\]

if \(2\gamma > \nu + 3/2\)

(This follows from Henry 1.6.1.) Since \(p_{xx} = p_{tt} + q_t\), the first part of (2) follows by taking components.
Suppose now that $\sigma(w)$ is $C^1$ with $\sigma'(w)$ locally Lipschitz. We claim that then $S = p^t$ is a strong solution of

$$
S_t = S_{xx} + \sigma(p+q)_t \quad \text{for } t > 0 \\
S_x(0,t) = 0 = S(1,t).
$$

(3.2)

The reason this is true is that from Henry 3.4.4 and 3.4.6 we know $S$ is a mild solution of this equation, and we can verify that the map $t \mapsto \sigma(p+q)_t$ is locally Holder continuous for $t > 0$ into $L^\infty \subset L^2$, so from Henry 3.2.2, $S$ is a strong solution on $[\delta, T]$ for any $\delta > 0$. We conclude that

$$
S = p^t \in C^1((\delta, T], L^2) \cap C((\delta, T], W^2, b^2).
$$

But now the map $(t, S) \mapsto \sigma'(p+q)(S+q)_t$ is clearly locally Lipschitz from $\mathbb{R} \times W^{1,2}$ to $L^\infty$, so by applying Henry 3.5.2 again we may conclude that $S_t$ is locally Hölder continuous into $D((-\Delta)^\gamma)$ for any $\gamma < 1$. Then from equation (3.2) it follows that $S_{xx}$ is locally Hölder continuous into $L^\infty$.

The remarks concerning the case when $\sigma(w)$ is $C^r$ or analytic follow from Henry 3.4.4 and 3.4.6. This finishes the proof of Theorem 3.1.

**Proof of Corollary 3.2:** Given the solution $z = (p, q)$ from Theorem 3.1, let

$$
u(x, t) = \int_0^x (p+q)(y, t) dy.
$$

Then $u_{xt} = p^t + q^t = p_{xx}$ is locally Hölder continuous into $L^\infty$ and is $C^1$ into $L^2$ for $t > 0$, so $u = \int_0^x p_{xx} = p_x$, whence $u_{tt} = p_{xt} = S_x$ as distributions. The regularity of $u_{tt}$ then follows from the regularity of $S$.

**Proof of Corollary 3.4:** Because $q(t)$ satisfies (2.10.) in $L^\infty$, $q \in C^1([0, T], L^\infty)$. Then, in $L^\infty$

$$
q(t) = q_0 + \int_0^t q_t(s) ds = q_0 - \int_0^t (\sigma(p+q)(s) - p) ds
$$

so long as the solution exists. The integral converges as a Riemann integral in $C([0, T], L^\infty)$. Fixing a sequence of partitions of $(0, \infty)$ with norm approaching
zero, we may delete a countable union of sets of measure zero in \((0,1)\) to guarantee that on a set of full measure in \(x\), the integral converges in sup norm. For \(x\) in this set, then,

\[
q(x,t) = q(x,0) + \int_0^t \sigma(p+q)(x,s)ds
\]

so \(q(x,t)\) is Lipschitz in \(t\).

We have established that the solution of the problem (2.4) enjoys some limited spatial smoothing: The velocity \(u_t\) improves from \(L^2\) at \(t = 0\) to \(W^{1,\infty}\) for \(t > 0\), and the total stress \(S\) gains from \(W^{-1,2}\) (roughly) at \(t = 0\) to \(W^{2,\infty}\) for \(t > 0\), if \(\sigma(w)\) is smooth. But whereas the solution is smooth in time if \(\sigma(w)\) is smooth, we observe that the strain \(u_x\) is not smoothed in space:

Proposition 3.5 (Persistence of strain discontinuities). Suppose \(x_0 \in (0,1)\) is a point of discontinuity of the initial strain \(u_0(x)\) in (2.4). Then \(x_0\) is a point of discontinuity of \(u(x,t)\) for the solution \(u(x,t)\) from Corollary 3.2, for any \(t > 0\) fixed, so long as the solution exists.

(An observation of this type has been made by Hoff and Smoller (to appear) for the system of isothermal gas dynamics.)

For the proof, recall that \(u_x = p+q\). Now \(p(x,t)\) is absolutely continuous in \((x,t)\) for \(t > 0\), so \(x_0\) is a point of discontinuity of \(q_0(x)\). The ODE (2.10) now implies \(x_0\) remains a point of discontinuity of \(q(x,t)\) for any fixed \(t > 0\). For if not, if \(q(x,t_0)\) were continuous at \(x_0\) for some \(t_0 > 0\), then running the ODE backwards from \(t = t_0\) to \(t = 0\), continuous dependence on parameters in ODEs implies \(q(x,0)\) is continuous at \(x_0\), a contradiction.

Solutions of (2.4) also preserve their initial smoothness, as established in Dafermos (1969) and Greenberg et al. (1968). Thus sufficiently smooth data yield classical solutions, for which every term in (2.4) is continuous. By a simple variant of the argument for Theorem 3.1, we can obtain such classical solutions:

Theorem 3.6 (Classical solutions). Suppose \(\sigma(w)\) is \(C^1\), with \(\sigma'(w)\) locally Lipschitz. Assume \(u_0 \in C^1[0,1]\), \(u_0(x) \in J_0\) a.e., \(J_0 \subset J\), and \(u_1 \in W^{1,2}[0,1]\)
with \( u_1(0) = 0 \). Then for some \( T > 0 \), a classical solution \( u(x,t) \) exists for (2.4) for \( 0 \leq t \leq T \), having

\[ u \in C([0,T], C^2) \text{,} \]

\( u_{tt}^* u_{tx} \) continuous in \((x,t)\) for \( t > 0 \).

Proof of 3.6. Again we consider (2.10) with initial data \( p_0(x) = \int_0^x u_1(y) dy \), \( q_0(x) = u_0(x) - p_0(x) \), so that \( p_0 \in W_b^{2,2} \), \( q_0 \in C^1 \). As in the proof of 3.1, we seek to apply Henry 3.3.3 to obtain local existence. Redefine \( X = (L^2, C^1) \). The operator \( A \) generates an analytic semigroup on \( X \). We seek to obtain solutions in \( C([0,T], X^\alpha) \) for \( 3/4 < \alpha < 1 \). Since \( X^\alpha = (D((-\Delta)^\alpha), C^1) \) and \( W_b^{2,2} = D((-\Delta)^1) \subset D((-\Delta)^\alpha) \), we have \( (p_0, q_0) \in X^\alpha \), so all we must show is that

\[ f(p,q) = (\sigma(p+q) - P, -\sigma(p+q) + P) \]

is locally Lipschitz from \( X^\alpha \) to \( X \). But for \( \alpha > 3/4 \),

\[ D((-\Delta)^1) \subset C^1 \subset L^2 \text{.} \]

So the desired fact follows because the map \( w(x) \rightarrow \sigma(w(x)) \) is locally Lipschitz from \( C^1 \) to \( C^1 \). Now from Henry 3.5.2 the local solution

\[ (p, q) \in C([0,T], X^\alpha) \cap C^1([0,T], X^\gamma) \]

for any \( \gamma < 1 \). In particular, the functions \( p_{tx}, q_{tx} \) are continuous in \((x,t)\) for \( t > 0 \) and so with \( u(x,t) = \int_0^x (p+q)(y,t) dy \), we find that the equation

\[ u_{tt} = \sigma'(u_x u_{xx} + u_{xx}^{\text{tx}}) \]

is satisfied, with each term continuous in \((x,t)\) for \( t > 0 \).

The proof above suggests that in Theorem 3.1, where \( X = (L^2, L^\infty) \), we can find local solutions of (2.10) whenever \((p_0, q_0) \in X^\alpha \) for \( 1/4 < \alpha < 1 \). For \( \alpha < 1/2 \), this would admit initial data with infinite kinetic energy \( \frac{1}{2} \int p_x^2 \). (The kinetic energy would be finite at any positive time, however, since one would have \( p \in C([0,T], W_b^{1,2*}) \).
4. **Exponentially stable states**

In this section, by means of linearized stability analysis, we identify equilibria of the problem (2.10) which are stable with exponential decay rate for perturbations small in the space $X^{1/2} = (W^{1,2}_b, L^\infty)$. The criterion for stability, (2.5), is positivity of the second variation of the stored-energy functional (2.8). Equilibria with negative second variation are unstable (see 4.2 below).

**Theorem 4.1 (Exponential stability).** Suppose $\sigma(w)$ is $C^1$ and $\sigma'(w)$ is locally Lipschitz. Let $(p, q) = (0, \tilde{w}(x))$ be an equilibrium solution of (2.10), and assume

$$\sigma(\tilde{w}(x)) = P, \quad \sigma'(\tilde{w}(x)) \geq \sigma_0 > 0 \text{ a.e.}$$

Then for any $\delta < \min\left\{ \frac{\pi^2}{8}, \sigma_0 \right\}$, a unique solution $(p, q)$ of (2.10) exists globally for $t > 0$ and satisfies, for some $C_0 > 0$,

$$\|p(t)\|_{W^{1,2}} \leq C_0 e^{-\delta t}, \quad \|q(t) - \tilde{w}\|_{L^\infty} \leq C_0 e^{-\delta t},$$

provided that $\|p_0\|_{W^{1,2}}$ and $\|q_0 - \tilde{w}\|_{L^\infty}$ are sufficiently small, with $p_0(1) = 0$. Here $\frac{\pi^2}{8} = \beta_0 / 2$ where $\beta_0$ is the first eigenvalue of $-\Delta$ on $(0, 1)$ with boundary conditions (2.10).\textsuperscript{3}

**Proof:** We shall apply Henry 5.1.1 in the abstract framework of the proof of 3.1. Let $\tilde{x} = (0, \tilde{w})$, $z = (p, q)$. For the abstract equation (3.1) we have

$$f(z + \tilde{x}) = f(\tilde{x}) + Bz + g(z)$$

where $f(\tilde{x}) = 0,$

$$Bz = B(p, q) = \sigma'(\tilde{w}(x)) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

and

$$g(z) = (\sigma(p + q + \tilde{w}) - \sigma(\tilde{w}) - \sigma'(\tilde{w})(p + q)) (1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
small in $X^{1/2} = (W^1_b, L^\infty)$, then for some $C$,

$$\| g(z) \|_X \leq C \| p + q \|_L^\infty^2.$$  

The linear map $B$ is bounded on $(L^\infty, L^\infty)$, so is bounded from $X^{1/2}$ to $X$.

Our main work is to check the last hypothesis of Henry 5.1.1: to show that the spectrum of the operator $A - B$ on $X$ lies in a half plane

$$\{ \lambda \in \mathcal{C} \mid \text{Re} \lambda \geq \min(\sigma_0', \beta_0'/2) \}.$$  

This shall be accomplished in two stages. First, we will show that the essential spectrum of $A - B$ (the spectrum with discrete eigenvalues of finite multiplicity deleted) is concentrated on the essential range of $\sigma'(\tilde{\omega}(x))$. Second, we show that no eigenvalue $\lambda$ can satisfy $\text{Re} \lambda < \min(\sigma_0', \beta_0'/2)$. The first step will be achieved using the invariance of the essential spectrum under relatively compact perturbation (see Henry, ch. 5, A.1). The second step employs an energy method.

We proceed with step one. Decompose $B$ as $B = \frac{p}{q} + \frac{1}{q} B$ where

$$B_p = \sigma'(\tilde{\omega}) \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad B_q = \sigma'(\tilde{\omega}) \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$  

Then whenever $\lambda \in \mathcal{C}$ is not contained in the spectrum $\text{sp}(-\Delta)$ or the essential range of $\sigma'(\tilde{\omega}(x))$, an explicit representation for the resolvent of the operator

$$A - B_q = \begin{pmatrix} -\Delta & -\sigma' \\ 0 & \sigma' \end{pmatrix}$$  

is

$$(\lambda - A + B_q)^{-1} = \begin{bmatrix} (\lambda + \Delta)^{-1} & -(\lambda + \Delta)^{-1} \sigma'(\lambda - \sigma')^{-1} \\ 0 & (\lambda - \sigma')^{-1} \end{bmatrix}.$$  

On the interval $(0, 1)$ with boundary conditions $(2.10, 3)$, $\text{sp}(-\Delta)$ consists of discrete positive eigenvalues, so the essential spectrum of $A - B_q$ is the essential range of $\sigma'(\tilde{\omega}(x))$. 

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Now we claim that $B_p (A+B)^q$ is compact on $X$. From the representations above, it suffices to check that $(-\Delta)^{-1}$ is bounded from $L^2$ to $W^{1,\infty}(0,1)$, so that it is compact from $L^2$ to $L^\infty$. For this, just verify that the Green's function for $-\Delta$, which satisfies

$$(-\Delta)^{-1} f(x) = \int_0^1 G(x,y)f(y)dy$$

satisfies also

$$\int_0^1 G_x^2(x,y)dy < C \quad \text{independent of } x.$$

But explicitly, $G(x,y) = 1-y$ if $x < y$, $1-x$ if $x > y$, so this holds.

Now applying Henry ch. 5, A.1, we conclude that any point $\lambda$ in the spectrum of $A-B$ is either in the essential range of $\sigma'(\bar{\omega}(x))$ (a finite set), or is an eigenvalue on $X$.

We proceed with step two. Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $A-B$ not in the essential range of $\sigma'(\bar{\omega}(x))$, with eigenfunction $z(x) = (p,q)(x)$. Then the real and imaginary parts of $p$ are in $W^{2,2}_b = D(-\Delta)$, and we have

$$\lambda p + p_{xx} + \sigma'(\bar{\omega})(p+q) = 0$$

$$\lambda q - \sigma'(\bar{\omega})(p+q) = 0 \ .$$

Solve for $q$ in terms of $p$ and substitute, obtaining

$$p_{xx} + \beta(\lambda, \sigma'(\bar{\omega}))p = 0$$

where

$$\beta(\lambda, s) = \frac{\lambda^2}{\lambda - s} \ .$$

Multiply by $\bar{p}$ and integrate by parts, obtaining
\[ \int_0^1 \left[ \beta(\lambda, \sigma'(w(x))) \left| p(x) \right|^2 - \left| p_x(x) \right|^2 \right] dx = 0 . \]

Now apply the sharp Poincaré inequality valid for any \( p \in W^{1,2}, \ p(1) = 0, \)
\[ \int_0^1 \left| p_x \right|^2 dx / \int_0^1 \left| p \right|^2 dx \geq \beta_0 = \frac{\pi^2}{4} . \]

Then we have
\[ (4.1) \quad 0 \geq \int_0^1 \left[ \beta_0 - \beta(\lambda, \sigma'(w(x))) \right] \left| p(x) \right|^2 dx . \]

Note that in this inequality, \( \beta \) may take complex values. It is clear that if \( \lambda \) is real with \( \lambda < \sigma_0, \) then \( \beta(\lambda, \sigma'(w)) < 0 \) a.e., which forces \( p = 0, \) so \( \lambda \) is not an eigenvalue.

For the sequel, label the points of the essential range of \( \sigma'(\tilde{w}(x)) \) in increasing order: \( s_1 < s_2 < \ldots < s_N. \) (In Theorem 4.1, we may assume \( \sigma_0 = s_1. \))

Consider the imaginary part of (4.1),
\[ \int_0^1 \text{Im} \beta(\lambda, \sigma'(w(x))) \left| p(x) \right|^2 dx = 0 . \]

Clearly, if \( \lambda \) is an eigenvalue, \( \text{Im} \beta \) must be zero or change sign as \( x \) varies.

Compute
\[ \text{Im} \beta(\lambda, s) = \text{Im} \lambda \frac{2(\lambda - s) / |\lambda - s|^2}{\left| \lambda - s \right|^2} \frac{\left( \text{Im} \lambda \right)}{\left| \lambda - s \right|^2} = \frac{\left( \text{Im} \lambda \right)}{\left| \lambda - s \right|^2} (|\lambda|^2 - 2s \text{Re} \lambda) . \]

Since \( \sigma'(\tilde{w}(x)) > s_1 > 0 \) a.e. it is immediate that if \( \text{Re} \lambda < 0 \) with \( \text{Im} \lambda \neq 0, \) then \( \lambda \) is not an eigenvalue. But furthermore, we find that if \( \text{Im} \lambda \neq 0 \) and \( \lambda \) is an eigenvalue with \( \text{Re} \lambda > 0, \) then
\[ |\lambda - s_1| > s_1 \quad \text{and} \quad |\lambda - s_N| < s_N . \]
(Square these inequalities and expand, recalling that $s_1 \leq \sigma'\left(\hat{w}(\lambda)\right) \leq s_N$ a.e.) Therefore, nonreal eigenvalues are confined to a compact set of the positive half-plane which lies between two nested circles (see Fig. 4).

We now consider the real part of (4.1). Actually, it is important to consider a mixture of real and imaginary parts: Whenever $-\pi/2 \leq \theta \leq \pi/2$, we find

$$0 \geq \text{Re} \left[ e^{i\theta} \int_{0}^{1} \left( \beta_0 - \beta(\lambda, \sigma'(\hat{w})) \right) \rho^2 \, dx \right].$$

We shall show that for each $\lambda$ with $0 < \text{Re} \lambda < \min\{s_1, \beta_0/2\}$, there is a choice of $\theta$ in $[-\pi/2, \pi/2]$ such that

$$\text{Re} \left[ e^{i\theta}(\beta_0 - \beta(\lambda, s_j)) \right] > 0 \quad \text{for } j = 1, \ldots, N.$$

Geometrically, this simply means that:

For $0 < \text{Re} \lambda < \min\{s_1, \beta_0/2\}$, there is some line through the point $\beta_0 = \pi^2/4$ in the complex plane, such that the $N+1$ numbers $0, \beta(\lambda, s_1), \ldots, \beta(\lambda, s_N)$ lie all on one side of that line.

This, then, will force $p = 0$, so such $\lambda$ cannot be eigenvalues, and will finish the proof of the theorem. (We shall see that the line may not always be chosen vertical in (4.2), corresponding to $\theta = 0$ above.)

To establish (4.2), fix $\lambda = a + ib$ with $0 < a < \min\{s_1, \beta_0/2\}$ and for convenience take $b > 0$. We will consider the curves $\beta(a + it, s_j)$ for $t > 0$, $j = 1, \ldots, N$, describing their shape and the relative positions along these curves of the points $\beta(a + ib, s_j)$.

In particular, we claim:

(1) With $t^*(s) = \sqrt{s(a - 2s - a)}$ (for any $s > a$) one has

$$\text{Im} \beta(a + it, s) \leq 0 \quad \text{for } 0 \leq t \leq t^*(s),$$
\[ \text{Im} \beta(a+it, s) > 0 \quad \text{for} \ t > t^*(s) , \]
\[ \beta(a+it^*(s), s) = 2a , \quad \beta(a, s) = \frac{-a}{s-a} < 0 . \]

(2) The curve \( \beta(a+it, s) \), for \( s > a \) fixed, is convex to the left, that is,
\[
\arg \frac{d\beta}{dt} = \arg i \frac{\partial \beta}{\partial \lambda} (a+it, s)
\]
is a strictly increasing function of \( t \) for \( t \geq 0 \), with
\[
\arg \frac{d\beta}{dt} \bigg|_{t=0} = -\frac{\pi}{2} < \arg \frac{d\beta}{dt} < \frac{\pi}{2} \quad \text{for all} \ t > 0 .
\]
(Hence \( \text{Re} \beta(a+it, s) \) is increasing in \( t \), so \( \text{Re} \beta < 2a \) (resp. > 2a) if \( t < t^*(s) \) (resp. > \( t^*(s) \)).)

(3) If \( a < s_i < s_j \), the curves \( \beta(a+it, s), \ s = s_i, s_j \), are nested as follows:
If \( \text{Re} \beta(a+it_1, s_i) = \text{Re} \beta(a+it_2, s_j) < 2a \),
then \( \text{Im} \beta(a+it_1, s_i) < \text{Im} \beta(a+it_2, s_j) < 0 \).

(That is, the first curve lies below the second, for \( \text{Re} \beta < 2a \)).
If \( \text{Re} \beta(a+it_1, s_i) = \text{Re} \beta(a+it_2, s_j) > 2a \),
then \( \text{Im} \beta(a+it_1, s_i) > \text{Im} \beta(a+it_2, s_j) > 0 \)
(so the first curve lies above the second for \( \text{Re} \beta > 2a \)).

Assuming these claims hold, let us establish (4.2). Since \( t^*(s) \) is monotonically increasing in \( s \), for each \( t > 0 \) there exists \( j, 1 \leq j \leq N \), so that
\[
\text{Im} \beta(a+it, s_k) > 0 \quad \text{if} \ k < j
\]
\[
\text{Im} \beta(a+it, s_k) < 0 \quad \text{if} \ k > j .
\]
Now the convexity and nesting properties (2) and (3) imply that the tangent line to the curve \( \beta(a+it, s_j) \) at \( t = t^*(s_j) \) (where \( \beta = 2a \)) is a line with the property we desire: the points \( 0, \beta(a+it, s_k), \ k = 1, \ldots, N \) lie all above this line (which has positive slope), and the point \( \beta_0 > 2a \) lies below (see Fig. 5). (4.2) follows.
Remark. If in fact $\beta_0 = 2a$ but $\beta(\lambda, s_k) \neq \beta_0$ for all $k$, the argument above shows again $\lambda$ is not an eigenvalue. Then a neighborhood of such a $\lambda$ must be in the resolvent set of the operator $A - B$.

It remains to establish the claims (1), (2), (3). Recall that when $\text{Im}\lambda > 0$,

$$\text{sgn Im} \beta(\lambda, s) = \text{sgn}(\lambda)^2 - 2s\text{Re}\lambda)$$

For $\lambda = a + it$, this equals $\text{sgn}(a(a - 2s) + t^2)$, which is positive or negative as $t$ is greater or less than $t^*(s)$. When $t = t^*(s)$,

$$\text{Re} \beta(\lambda, s) = \text{Re} \frac{\lambda^2(\lambda - s)}{|\lambda - s|^2} = \frac{\lambda^2\text{Re}\lambda - s((\text{Re}\lambda)^2 - (\text{Im}\lambda)^2)}{|\lambda - s|^2}$$

$$= \frac{\text{Re}\lambda(\lambda^2 - 2s\text{Re}\lambda) + s|\lambda|^2}{|\lambda - s|^2} = \frac{s(a + t^2)}{(a - s)^2 + t^2} = 2a$$

To check the convexity property (2), compute

$$\frac{\partial \beta}{\partial \lambda}(\lambda, s) = \frac{\lambda(\lambda - 2s)}{(\lambda - s)^2}.$$ 

Now

$$\arg \frac{d\beta}{dt}(a + it, s) = \arg \frac{\partial \beta}{\partial \lambda} = \frac{\pi}{2} + \arg\lambda - 2\arg(\lambda - s) + \arg(\lambda - 2s)$$

$$= - \frac{\pi}{2} + \tan^{-1}(t/a) + 2\tan^{-1}\left(\frac{t}{s-a}\right) - \tan^{-1}\left(\frac{t}{2s-a}\right).$$

Then

$$\frac{d}{dt} \left( \arg \frac{d\beta}{dt}(a + it, s) \right) = \frac{a}{a^2 + t^2} + \frac{2(s - a)}{(s - a)^2 + t^2} - \frac{(2s - a)}{(2s - a)^2 + t^2}$$

$$> \frac{a + 2(s - a) - (2s - a)}{(2s - a)^2 + t^2} = 0$$

if $s > a$. Since clearly $\lim_{t \to \infty} \arg \frac{d\beta}{dt}(a + it, s) = \pi/2$, the fact that $-\pi/2 < \arg \frac{d\beta}{dt}(a + it, s) < \pi/2$ follows.
To check the nesting property (3), it is enough to show that the direction of \( \frac{\partial \beta}{\partial s}(a+it, s) \) is to the left (resp. right) of the tangent \( \frac{d\beta}{dt} = i \frac{\partial \beta}{\partial \lambda}(a+it, s) \), whenever \( \text{Re} \beta < 2a \) (resp. \( > 2a \)). This follows from the fact

\[
\text{sgnRe} \frac{\partial \beta}{\partial \lambda} \frac{\partial \beta}{\partial s}(a+it, s) = \text{sgnIm}(a+it, s)
\]

which we now check:

\[
\frac{\partial \beta}{\partial \lambda} \frac{\partial \beta}{\partial s} = \frac{\lambda(\lambda - 2s)}{(\lambda - s)^2} \left( \frac{\lambda}{\lambda - s} \right)^2 = \frac{|\lambda|^2(\lambda - 2s\lambda)}{|\lambda - s|^4}
\]

So \( \text{sgnRe} \frac{\partial \beta}{\partial \lambda} \frac{\partial \beta}{\partial s} = \text{sgn}(|\lambda|^2 - 2s\text{Re} \lambda) = \text{sgnIm} \beta \). This finishes the proof of Theorem 4.1.

We conclude this section with a criterion for instability:

**Remark 4.2.** If in 4.1 the equilibrium solution \((0, \tilde{w}(x))\) satisfies, instead of (2.5),

\[
\sigma'(\tilde{w}(x)) = P \text{ a.e., } \sigma'(\tilde{w}(x)) < 0 \text{ for } x
\]

(4.3)

in a set of positive measure in \((0, 1)\)

then \( \tilde{z}(x) = (0, \tilde{w}(x)) \) is unstable in the sense that there exists \( \epsilon_0 > 0 \) and a sequence of initial data \( z_0 \to \tilde{z} \) in \( X^{1/2} \) so that, if \( z_0(t) \) denotes the solution of (3.1) with initial data \( z_0^n \), for all \( n \) we have

\[
\sup_{t \geq 0} \left\| z_n(t) - \tilde{z} \right\|_{X^{1/2}} > \epsilon_0 > 0 ,
\]

the supremum taken over the maximal interval of existence of \( z_n(t) \). This fact is an application of Henry 5.1.3, the point being that if (4.3) holds, then the essential spectrum of \( A - B \) intersects the unstable half plane \( \{ \text{Re} \lambda < 0 \} \), since the essential spectrum is the essential range of \( \sigma'(\tilde{w}(x)) \) as demonstrated in the proof of 4.1.

5. **Global existence and convergence to equilibrium**

The method we use to establish global existence of the solution of (2,10) is the same as that of Andrews and Ball; we include a discussion for completeness. In this method, the sign of the stress for extreme ranges of strain is restricted
in a reasonable way, and also the energy of the initial data is restricted (see 5.2 below). However, if \( \sigma(w) \) is defined for all \( w \in \mathbb{R} \), and satisfies, for some \( M > 0 \),
\[
(5.1) \quad w(\sigma(w) - P) > 0 \quad \text{if} \ |w| > M,
\]
then the energy of the initial data is unrestricted.

As we begin, let us first establish the energy identity:

**Proposition 5.1.** Assume \( \sigma(w) \) is locally Lipschitz. Suppose a solution \((p,q)\)
to problem (2.10) exists for \( 0 \leq t \leq T \) with \( p_0 \in W^{1,2}, \ p_0(1) = 0, \) and \( q_0 \in L^\infty \).
Then the energy identity (2.14) holds and each term is continuous in \( t \) for \( t > 0 \).

**Proof:** Multiply equation (2.10) for \( p \) by \( p_{xx} = (p+q)_t \) and integrate over \( x \).
Using the regularity of the solution as established in Theorem 3.1 and the boundary conditions in (2.10), we find that for \( t > 0 \),
\[
\int_0^1 p_{xx}^t(x,t)\,dx = - \int_0^1 p_x^t \frac{1}{2} p^2(x,t)\,dx = - \left( \int_0^1 \frac{1}{2} p_x^2(x,t)\,dx \right)_t
\]
and
\[
\int_0^1 (\sigma(p+q) - P)(p+q)_t(x,t)\,dx = \left( \int_0^1 (W(p+q) - P(p+q)) \,dx \right)_t.
\]

Integrating from \( t = \varepsilon > 0 \) to \( T \) and letting \( \varepsilon \to 0 \), (2.14) follows.

In the global existence theorem below, we fix a compact interval \([w_-, w_+] \subset J\) (the domain of \( \sigma \)), and fix \( M, \ 0 < M < (w_+-w_-)/2, \) so that also \([w_--M, w_++M] \subset J\).
We require that \( \sigma(w) \) satisfy
\[
(5.2) \quad \sigma(w) - P < 0 \quad \text{if} \ |w-w_-| < M
\]
\[
\sigma(w) - P > 0 \quad \text{if} \ |w-w_+| < M
\]
Adjusting the stored energy function \( W(w) \) by a constant if necessary, we may assume
\[
\min\{W(w) - Pw \mid w \in [w_--M, w_++M]\} = 0.
\]
Theorem 5.2 (Global existence). Assume \( \sigma(w) \) is locally Lipschitz. Suppose the initial data for (2.10) satisfy \( p_0 \in W^{1,2} \), \( p_0(1) = 0 \), \( q_0 \in L^\infty \), and

\[
q_0(x) \in \left[ w_-, w_+ \right] \text{ a.e.}
\]

(5.3)

\[
\int_0^1 \left( \frac{1}{2} p_0^2 + W(p_0 + q_0) \right)(x)dx < M^2/2 .
\]

Then the solution \((p, q)\) to the problem (2.10) exists globally for \( t \geq 0 \) and satisfies, for all \( t \geq 0 \),

\[
|p(x, t)| < M \quad \text{for all } x \in (0, 1)
\]

(5.4)

\[
q(x, t) \in \left[ w_-, w_+ \right] \text{ for a.e. } x \in (0, 1) .
\]

Corollary 5.3 (Classical solutions). Suppose that \( \sigma(w) \) is \( C^1 \) and \( \sigma'(w) \) is locally Lipschitz. In addition, to the hypotheses of Theorem 5.2, assume \( p_0 \in W^{2,2}(0, 1) \), \( p_0' (0) = 0 \), and \( q_0 \in C^1[0, 1] \). Then \( q(x, t) \) is \( C^1 \) in \((x, t)\) for all \( t \geq 0 \), so that the solution \( u(x, t) \) of (2.4) obtained from \((p, q)\) remains classical for all \( t > 0 \).

The proof of the corollary will be postponed until after the proof of Theorem 5.4 below.

Proof of 5.2: Assume the solution exists for \( 0 \leq t < T \) and \( T \) is maximal. We claim that the bounds (5.4) along with the inequality \( W(p+q) - P(q+q) \geq 0 \) a.e. hold for \( 0 \leq t < T \). In particular we will show that the set of \( t_0 \) such that (5.4) and (5.5) hold for \( 0 \leq t \leq t_0 \) is open and closed in \([0, T)\). Suppose (5.4) holds for \( 0 \leq t \leq t_0 \).

Since \( p(x, t) \) is continuous, there is \( \epsilon > 0 \) so \( |p(x, t)| < M - \epsilon \), \( 0 < x < 1 \), \( 0 \leq t \leq t_0 \). Then since \( q \) is continuous into \( L^\infty \), there is \( \delta > 0 \) so \( (p+q)(x, t) \in [w-M, w+M] \) for \( 0 \leq t \leq t_0 + \delta \), and for a.e. \( x \). This implies that \( W(p+q) - P(p+q) \geq 0 \) for a.e. \( x, 0 \leq t \leq t_0 + \delta \), and the energy identity (2.14) now yields

\[
|p(x, t)| \leq \left\| p_x(t) \right\|_{L^2} < M \quad \text{for } 0 \leq x < 1, \ 0 \leq t \leq t_0 + \delta .
\]

(5.5)

Using Corollary 3.5, for each \( x \) in a set of full measure we may apply Lemma 1.1 to the function \( q(t) = q(x, t) \), \( x \) fixed, because of the hypothesis (5.2), and conclude
that the interval \([w_-, w_+]\) is invariant for the ODE \((2.10_2)\) for \(0 \leq t \leq t_0 + \delta\).

So \((5.4)\) holds for \(0 \leq t \leq t_0 + \delta\).

It is easy to check that if \((5.4)\) holds for \(0 \leq t < t_0 < T\) then it holds for \(0 \leq t \leq t_0\). So we have shown that \((5.4)\) holds for \(0 \leq t < T\). We may now apply a continuation theorem, Henry 3.3.4, to conclude that \(T = +\infty\), because \((5.4)\) implies that \(\{z(t) \mid z(t) = (p, q)(t), \ 0 < t < T\}\) is contained in a closed, bounded subset of the domain of definition of \(f(z) = (\sigma(p+q)-P, -\sigma(p+q)+P)\) in \(X^{1/2} = (W_{b}^{1,2}, L_{b}^{\infty})\), on which \(f\) is bounded in \(X = (L_{b}^{2}, L_{b}^{\infty})\). This completes the proof of 5.2. We note that we have shown that the solution satisfies, for some \(M_1 > 0\),

\[
(5.6): \quad \left\| \sigma(p+q)(t) - P \right\|_{L_{b}^{\infty}} \leq M_1
\]

Our main result in this section is:

**Theorem 5.4 (Approach to equilibrium).** Under the hypotheses of Theorem 5.2, the solution \((p, q)\) of \((2.10)\) has the following asymptotic behavior as \(t \to \infty\):

\[
\left\| p(t) \right\|_{W_{b}^{2,2}} \to 0
\]

\(q(x, t) \to w^{\infty}(x)\) boundedly a.e., where \(w^{\infty} \in L_{b}^{\infty}\) and \(\sigma(w^{\infty}(x)) = P\) a.e.

\[
\sigma(p+q)(x, t) \to P \text{ boundedly a.e.}
\]

\[
\left\| S(t) \right\|_{C^{1, \nu}} \to 0 \text{ for any } \nu < 1/2
\]

If \(\sigma(w)\) is \(C^1\) with \(\sigma'(w)\) locally Lipschitz, then also

\[
\left\| S(t) \right\|_{W_{b}^{2,2}} \to 0
\]

We shall see in §6 that in general, one cannot expect \(\sigma\), the elastic part of the stress, to approach \(P\) uniformly in \(x\). We point out that in terms of the solution \(u(x, t)\) of \((2.4)\) obtained from \((p, q)\), the results of the theorem above yield

\[
\left\| u_t(t) \right\|_{W_{b}^{1,2}} \to 0
\]

\(u_x(x, t) \to w^{\infty}(x)\) boundedly a.e.
Proof: The first step is to establish that

\[(5.7) \quad \|p(t)\|_{W^{1,2}} \to 0 \quad \text{as} \quad t \to \infty .\]

From the Poincaré inequality, using the boundary condition \(p_x(0, t) = 0\), we have

\[
\int_0^\infty \int_0^1 p_x^2(x, t) \, dx \, dt \leq \int_0^\infty \int_0^1 p_{xx}^2(x, t) \, dx \, dt \leq M^2/2 ,
\]

this last following from the energy identity. We claim that \(\|p_x(t)\|_{L^2}^2\) has bounded continuous time derivative on \([\delta, \infty)\) for any \(\delta > 0\), so that (5.7) follows. We derive this claim from Lemma A.3 in the Appendix, an abstract smoothing result which sharpens Henry 3.5.2 to achieve some explicit bounds: For our abstract equation (3.1), it follows from A.3 and the proof of 3.1 that \(z_t \in C((0, \infty), X^{1/2})\), with

\[
\|z_t(t)\|_{X^{1/2}} \leq C_x(T)(t-t_0)^{-1} \left(\|z(t_0)\|_{X^{1/2}} + \sup_{t_0 \leq s \leq t} \|f(z(s))\|_{X} \right)
\]

for any \(t, t_0\) with \(0 < t - t_0 < T\) (T fixed). Taking \(t-t_0 = 1\) and using the established bounds (5.4), (5.5), and (5.6) to get bounds for \(z(t)\) and \(f(z(t))\), we conclude that for some \(M_2 > 0\),

\[(5.8) \quad \|p_t(t)\|_{W^{1,2}} \leq M_2 \quad \text{for all} \quad t \geq 1 .\]

This yields the desired result (5.7), but also has some independent interest since \(p_t = S\), the total stress.

Step two in our proof (perhaps the most significant) is to reconsider the ODE (2.10) \(q_t = -\sigma(p+q)+P\). Since from (5.7), \(\|p(t)\|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty ,\) this ODE is "asymptotically autonomous." A simple lemma describes the asymptotic behavior of its solutions:

Lemma 5.5. Suppose \(q(t)\) is \(C^1\) for \(t \in (0, \infty)\), with \(|q(t)| \leq C\) for all \(t\), \(p(t)\) is continuous with \(p(t) \to 0 \quad \text{as} \quad t \to \infty ,\) and \(q_t = -\sigma(p+q)+P\). Then there is a constant \(q^\infty\) so that \(q(t) \to q^\infty \quad \text{as} \quad t \to \infty ,\) and \(\sigma(q^\infty) = P\).
The author thanks J. Ball for the proof below, which is simpler and more
general than the original proof:

Let \( q = \lim_{t \to \infty} q(t), \overline{q} = \lim_{t \to \infty} q(t) \). Supposing \( q < \overline{q} \), choose any \( q_0 \in (q, \overline{q}) \).

Then there exist two sequences \( t_i^+ \to \infty \) such that \( q(t_i^+) = q_0 \) and \( q(t_i^+) > 0 \to q(t_i^-) \)
for all \( i \). Since \( q(t_i^+) = -\sigma(q_0 + p(t_i^+)) + P \), letting \( i \to \infty \) we find \( \sigma(q_0) = P \). Thus \( \sigma \equiv P \) on \([q, \overline{q}]\), since \( q_0 \) was arbitrary. But now if \( T \) is so large that
\[ |p(t)| < \epsilon(\overline{q} - q) \text{ for } t > T, \]
we have \( q(t) = 0 \) whenever \( q \in (q+\epsilon, \overline{q}-\epsilon) \). This is not possible if \( q < \overline{q} \). Hence \( q = \overline{q} = \lim_{t \to \infty} q(t) \).

Recall that for \( x \) in a set of full measure in \((0, 1)\) the ODE (2.10) is
satisfied classically for \( t > 0 \). With \( x \) fixed, the lemma above applies to \( q(t) = q(x, t) \),
and we conclude that as \( t \to \infty, q(x, t) \to q^\infty(x) \text{ boundedly almost everywhere pointwise in } x \).
In particular, \( q^\infty \in L^\infty(0, 1) \), and \( \sigma(q^\infty(x)) = P \) a.e. We also conclude
that \( \sigma(p+q)(x, t) \to P \) as \( t \to \infty \text{ boundedly a.e.} \), so \( \sigma \to P \) in \( L^2 \) and \( q_t \to 0 \) in \( L^2 \) as \( t \to \infty \).

Our next step is to show that as \( t \to \infty \)

\[ \|p_t\|_{C^{1, \nu}} \to 0 \text{ for any } \nu < 1/2 \]

From equation (2.10) it then follows that \( p_{xx} = p_t - \sigma + P \to 0 \text{ in } L^2 \) (but not in \( L^\infty \), see §6).

To establish (5.6) we shall regard (2.10) as an abstract equation

\[ p_t = \Delta p + g(p, t) \]

in the space \( Y = L^2 \), with \( \Delta \) and \( D(\Delta) \) as in the proof of 3.1, and with
\( g(p, t) = \sigma(p+q(t)) - P \). As in §3, \( Y^\gamma = D((-\Delta)^\gamma) \subset C^1, \nu \) if \( 2\gamma > \nu + \frac{3}{2} \) and
\( Y^{1/2} = W^{1, 2}_b \). Observing that

\[ \|q(t) - q(s)\|_{L^2} \leq K(s)(t-s) \]

for \( t > s > 0 \) where \( K(s) = \sup_{s < t < \infty} \|q_t(t)\|_{L^2} \to 0 \text{ as } s \to \infty \), we have

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\[ \left\| g(p(t), t) - g(p(s), s) \right\|_2 \leq L \left( \left\| p(t) - p(s) \right\|_2 + K(t_0)(t - s) \right) \]

whenever \( t_0 < s < t \) (where \( L \) depends on the bounds in (5.4)). Applying Lemma A.3 in this context, we find, for \( 0 < t_0 < t < t_0 + T \) (\( T \) fixed), and any \( \gamma < 1 \),

\[ \left\| p(t) \right\|_{Y \gamma} \leq C_{\gamma}(T)(t - t_0)^{-\gamma/2} \left( \left\| p(t_0) \right\|_{Y \gamma}^{1/2} \right) + \sup_{t_0 \leq t \leq t_0 + T} \left\| g(p(t), t) \right\|_Y + K(t_0). \]

Fixing \( t - t_0 = 1 \) and letting \( t_0 \to \infty \), (5.9) follows.

Our last goal is to show that \( p_{\text{max}} = S_{xx} \to 0 \) in \( L^2 \) as \( t \to \infty \), provided \( \sigma(w) \) is \( C^1 \) and \( \sigma'(w) \) is locally Lipschitz. Since \( S \) is a strong solution of

\[ S_t = S_{xx} + \sigma'(p+q)(S - \sigma(p+q) + p) \]

for \( t > 0 \) (cf. §3), it suffices to establish that

\[ (5.10) \quad \left\| S_t(t) \right\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty. \]

But the proof of this goes the same as the proof of (5.9). This completes the proof of Theorem 5.4.

**Proof of 5.3:** As in the proof of 5.2, we need to show that if \( T \) is finite, the set \( \{ z(t) \mid z(t) = (p, q)(t), \ 0 < t < T \} \) is contained in a closed, bounded subset of the domain of definition of \( f(z) \) in \( X^\alpha = \left( D((-\Delta)^{\alpha/2}), C^1 \right) \) (\( 3/4 < \alpha < 1 \), as in the proof of 3.6) on which \( f \) is bounded in \( X = (L^2, C^1) \). Beyond the argument in the proof of 5.2, it suffices to show that the quantities \( \left\| p_{xx} \right\|_{L^2} \) and \( \left\| q_x \right\|_{L^2} \) cannot blow up in finite time.

Now \( p_{xx} = p_t - \sigma(p+q) + P \), so in fact \( \left\| p_{xx} \right\|_{L^2} \leq M_1 + M_2 \) for \( t \geq 1 \) by (5.6) and (5.8). On the other hand, \( q_x \) satisfies a linear ODE,

\[ q_{xt} = -\sigma'(p+q)(q_x + p_x), \]

with bounded coefficients and forcing, so \( q_x \) grows at an exponential rate at worst.

Q.E.D.
6. Discontinuous asymptotic states for smooth solutions

According to the results of §5, each solution of (2.4) approaches some unique equilibrium asymptotic state as $t \to \infty$. The stability result of §4 identifies the asymptotic state when the initial data are close to an equilibrium which satisfies (2.5). But if the equilibrium has discontinuous strain $\tilde{w}(x)$, then Theorem 4.1 applies only if the initial strain is close to equilibrium in $L^\infty$, hence is discontinuous itself. On the other hand, smooth initial data yield classical solutions (see Dafermos (1969) and 3.6) with continuous strain for all $t$. What happens if the initial data are smooth, but approximate (in $L^1$, say) some equilibrium satisfying (2.5) and having discontinuous strain? The result of this section shows that the asymptotic state for such a classical solution can be identified at many points $x$ (though not all), and can guarantee that the asymptotic state has discontinuous strain. Essentially, we show that equilibria satisfying (2.5) are stable to perturbations with small energy, in a sense.

Fix an equilibrium state $(0, \tilde{w}(x))$ for (2.10), so $\sigma(\tilde{w}(x)) = \mathbb{P} \ a.e.$ Suppose at first that $\sigma(w)$ is $C^1$, and that $\sigma'(\tilde{w}(x)) \geq \sigma_0 > 0$ (so (2.5) holds). Then the essential range of $\tilde{w}(x)$ must be a finite set $\{w_1, \ldots, w_N\}$ (we assume $w_1 < w_2 < \ldots < w_N$) and we may choose $E_j > 0$ for $j = 1, \ldots, N$ so that, with $E = \min_j \{E_j\}$:

If $|w - w_j| < E_j + E$, then

$$\begin{align*}
\sigma(w) > \mathbb{P} & \text{ if } w > w_j \\
\sigma(w) < \mathbb{P} & \text{ if } w < w_j
\end{align*}$$

(6.1)

In the theorem below, we assume only that $\sigma(w)$ is locally Lipschitz, but require that the essential range of $\tilde{w}(x)$ be a finite set as above and that (6.1) holds. We note that by Lemma 2.1, provided that $|p(x, t)| < E$ for all $(x, t)$, the balls $B(w_j; E_j) = (w_j - E_j, w_j + E_j)$ are all positively invariant for the ODE (2.10a).

Below, $\mu(S)$ denotes the Lebesgue measure of the set $S$.

**Theorem 6.1 (Stability of equilibria).** Assume that the initial data $u_0 \in L^\infty$, $u_1 \in L^2$ for the problem (2.4) satisfy, with $q_0(x) = u_0(x) - \int_0^x u_1(y) dy$, ...
(6.2) \[ q_0(x) \in [w_1 - E_1, w_N + E_N] \quad \text{a.e.}, \]

(6.3) \[ \int_0^1 \frac{1}{2} u_1^2(x) \, dx + \int_0^1 (W(u_0(x)) - Pu_0(x)) \, dx < \frac{E^2}{2} + \int_0^1 W_m(q_0(x)) \, dx, \]

and

either \[ \sup_{0 \leq x \leq 1} \left| \int_1^x u_1(y) \, dy \right| < E \]

(6.4) \[ \text{or} \]

\[ \int_0^1 (W(u_0(x)) - Pu_0(x)) \, dx \geq \int_0^1 W_m(q_0(x)) \, dx, \]

where

\[ W'(w) = \sigma(w), \quad 0 = \inf \left\{ W(w) - Pw \middle| w \in (w_1 - 2E_1, w_N + 2E_N) \right\}, \]

and

\[ W_m(q) = \begin{cases} W(w_j) - Pw_j & \text{if } |q - w_j| < E_j \\ 0 & \text{if } q \notin \bigcup_{j=1}^N B(w_j; E_j) \end{cases}. \]

Then the solution \( u(x, t) \) of (2.4) exists for all \( t > 0 \), and \( u(x, t) \) converges boundedly a.e. as \( t \to \infty \) to an equilibrium state \( w^\infty(x) \) with \( \sigma(w^\infty(x)) = P \) a.e., and in particular,

(6.5) \[ w^\infty(x) = w_j \quad \text{if } |q_0(x) - w_j| < E_j, \quad \text{for a.e. } x. \]

Thus, if

(6.6) \[ \mu \left( \left\{ x \mid |q_0(x) - \hat{w}(x)| > E \right\} \right) < \epsilon \]

then

(6.7) \[ \mu \left( \left\{ x \mid w^\infty(x) \neq \hat{w}(x) \right\} \right) < \epsilon. \]

This theorem identifies a certain "basin of stability" for an equilibrium state satisfying (6.1) and achieves pointwise control over the asymptotic state of solutions.
in (6.5). We emphasize that this basin contains smooth data \((u_0, u_1)\) which satisfies (6.3), (6.4) and (6.6), for any positive \(E\) and \(\epsilon\). In particular, given a sequence \(\{(u_0^n, u_1^n)\}\) with \(u_1^n \to 0\) in \(L_x^2\), \(u_0^n(x) \to \tilde{w}(x)\) boundedly a.e., then (6.3), (6.4) and (6.6) must hold for \(n\) sufficiently large. If the equilibrium strain \(\tilde{w}(x)\) is not constant, and the initial strain \(u_0(x)\) is smooth and satisfies (6.6) for \(\epsilon\) sufficiently small, then \(q_0(x)\) is close to \(\tilde{w}(x)\) except in some "transition layers" of small measure. Then (6.5) identifies the asymptotic strain \(w^\infty(x)\) pointwise from the initial data except in these transition layers, and in particular \(w^\infty(x)\) is forced to be nonconstant and discontinuous, since \(\sigma(w^\infty(x)) = P\) a.e. Since smooth initial data yield classical solutions with \(C^1\) strain, convergence of the strain as \(t \to \infty\) does not occur uniformly. We note that the behavior of the strain in the "transition layers" which must persist is governed by the ODE (2.102).

**Proof of 6.1:** Of course, we work with the corresponding solution \((p, q)\) of the system (2.10). Recall \(u_t = p_x, u_x = p + q\), and the energy identity (2.14). We claim that the solution \((p, q)\) exists globally in \(t\), satisfying for \(t > 0\)

\[
(6.8) \quad |p(x, t)| \leq \left\|p_x(t)\right\|_{L_x^2} < E \quad \text{for } 0 \leq x \leq 1,
\]

\[
(6.9) \quad q(x, t) \in \left[1 - E^{-1}, w_N - E_N\right] \quad \text{for a.e. } x,
\]

\[
(6.10) \quad \left(W(p + q) - P(p + q)\right)(x, t) \geq W_{\min}(q_0(x)) \quad \text{for a.e. } x.
\]

As in the proof of Theorem 5.2, it suffices to establish these inequalities for \(t \in [0, T]\), where \([0, T]\) is the maximal interval of existence. As before, our strategy is to show that the set of \(t_0 < T\) such that (6.8) - (6.10) hold for \(0 \leq t \leq t_0\) is open and closed in \([0, T]\). It is easy to show this set is closed, using the energy identity to get (6.8). Now suppose (6.8) - (6.10) hold for \(0 \leq t \leq t_0 < T\). Then by continuity, (6.8) holds for \(0 \leq t \leq t_0 + \delta\) for some \(\delta > 0\). Using Lemma 2.1, this implies that the intervals \(B(w_j; E_j), j = 1, \ldots, N\), and the interval \(\left[1 - E_1, w_N + E_N\right]\) are all positively invariant for the ODE (2.102) for \(0 \leq t \leq t_0 + \delta\), for \(x\) in a set of full measure in \([0, 1]\). Thus (6.9) holds. Now certainly \(W(p + q) - P(p + q) \geq 0\) for a.e. \(x\), for \(0 \leq t \leq t_0 + \delta\). But also, for a.e. \(x\) such that \(q_0(x) \in B(w_j; E_j)\) we

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must then have \( q(x, t) \in B(w_j; E_j) \) for \( 0 \leq t \leq t_0 + \delta \). Using (6.8) and (6.1) we find that for such \( x \),
\[
(W(p+q) - P(p+q))(x, t) \geq W(w_j) - Pw_j = \inf\{W(w) - Pw \mid |w - w_j| < 2E_j\}
\]
\[
= W_m(q_0(x)) .
\]
So the lower bound (6.10) on the stored energy holds for \( 0 \leq t \leq t_0 + \delta \) as well.

Global existence follows, and indeed the conclusions of Theorem 5.4 hold for this solution as well. We conclude that \( \|p(t)\|_{W^{2,2}} \to 0 \) as \( t \to \infty \), and that for \( x \) in a set of full measure, \( q(x, t) \to w^\infty(x) \) as \( t \to \infty \), where \( \sigma(w^\infty(x)) = P \). Then
\[
u_x \to (p+q)(x, t) \to w^\infty(x) \quad \text{as} \quad t \to \infty .
\]
Since the intervals \( B(w_j; E_j) \) are invariant for the ODE (2.10), it follows that if \( |q_0(x) - w_j| < E_j \) for some \( j \), then \( w^\infty(x) \in B(w_j; E_j) \). But by (6.1), \( w_j \) is the only solution of \( \sigma(w) = P \) in \( B(w_j; 2E_j) \), so we must have \( w^\infty(x) = w_j \). This finishes the proof of Theorem 6.1.

Note. Weinberger (1982) has described the phenomenon of an uncountable infinity of discontinuous, stable, asymptotic states for classical solutions in a population model with nearly the same structure as the system (2.10). Here, though, the special structure of (2.10) makes it possible to obtain pointwise information about the asymptotic state with the simple techniques used above.

7. Hysteresis and creep in a load-deformation experiment

Our goal in this section is to exhibit hysteresis and creep phenomena in appropriately idealized dynamic processes in a viscoelastic bar.

Let us take \( \sigma(w) \) of the form indicated in Fig. 1 and settle some notation: \( \sigma \) should be locally Lipschitz, defined on \( \mathbb{R} \) for convenience, strictly increasing for \( w < \alpha \) and \( \beta < w \) with \( \alpha < \beta \) and strictly decreasing for \( \alpha < w < \beta \). Set \( P_\alpha = \sigma(\alpha), \quad P_\beta = \sigma(\beta) \). For \( P < P_\alpha \) denote by \( w_\alpha(P) \) the unique \( w \leq \alpha \) satisfying

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\( \sigma(w) = P \), and for \( P \geq P_\beta \) similarly define \( w_\beta(P) \) as that \( w \geq \beta \) so \( \sigma(w) = P \).
Also for \( P_\beta < P < P_\alpha \) set

\[
E_\alpha(P) = \frac{1}{2} (w_0 - w_\alpha(P)), \quad E_\beta(P) = \frac{1}{2} (w_\beta(P) - w_0),
\]

where \( w_0 \) is the middle root of \( \sigma(w) = P \). For \( P < P_\beta \), set \( E_\alpha(P) = \infty \), and for \( P > P_\alpha \), set \( E_\beta(P) = \infty \).

We will model the load-deformation experiment very simply, as a chain of initial-boundary value problems, allowing discretely imposed changes in the load \( P \) and small "fluctuations" imposed on the solution itself (as one model accounting for unknown physical influences on the bar).

Select a sequence of loads \( P_0, P_1, \ldots \) (nondecreasing for now), a sequence of "fluctuations" \( (u_j^0, u_j^1), j = 0, 1, \ldots \) in \( (L^\infty, L^2) \), and a sequence of times \( 0 = t_0 < t_1 < \ldots \). Under suitable restrictions, we will construct global in time solutions \( u_j^j(x, t) \) to the initial-boundary value problems below, for \( j = 0, 1, 2, \ldots \):

\[
\begin{align*}
(7.1) \quad u_j^{jj} &= (\sigma(u_j^j) + u_j^{j-1})_x, \quad 0 < x < 1, \ t > t_j \\
(7.2) \quad u_j^0(0, t) &= 0, \quad (\sigma(u_j^j) + u_j^{j-1})(1, t) = P_j \quad \text{for } t > t_j \\
(7.3) \quad u_j^1(x, t_j) &= u_j^{j-1}(x, t_j) + u_j^0(x) \quad \text{for } 0 \leq x \leq 1. \\
& \quad u_j^1(x, t_j) &= u_j^{j-1}(x, t_j) + u_j^0(x)
\end{align*}
\]

(For \( j = 0 \) in (7.3), take \( u_x^{-1} = 0 = u_t^{-1} \).) The displacement of the bar under the influence of the varying loads and imposed fluctuations is represented by the function

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\[ u(x, t) = u^j(x, t) \quad \text{if} \quad t_j \leq t < t_{j+1}. \]

As a "measurement" of the bar, we will consider the sequence of asymptotic states determined by \( u^j(x, t) \): Theorem 6.1 will guarantee that there exist \( w^j_\infty \in L^\infty(0, 1) \) so that

\[ u^j_x(x, t) \to w^j_\infty(x) \quad \text{as} \quad t \to \infty \quad \text{for a.e.} \quad x \in (0, 1). \]

The "asymptotic length" of the bar while \( t_j < t < t_{j+1} \) may then be defined by

\[
L^j = \int_0^1 w^j_\infty(x) \, dx
\]

and may be considered to correspond to the quantity typically measured in experiment.

**Hysteresis.** Let us illustrate how hysteresis can occur in this model upon slowly raising, then lowering, the load. Suppose the initial load is small, \( P_0 < P^* \). There is a unique equilibrium state at this load, for which the strain \( u_x \) is constant, so \( w^0_\infty(x) = L^0 = w^0_\alpha(P_0) \). Roughly speaking, we shall show that if the load is raised slowly, and the fluctuations are small, then the measured deformations \( L^j \) increase along the curve \( w^0_\alpha(P) \) until \( P \) is close to \( P^\alpha \), then jump to the curve \( w^0_\beta(P) \) when the load \( P \) exceeds \( P^\beta \). If then the load is decreased slowly, the observed deformations decrease along the curve \( w^0_\beta(P) \) until \( P \) is close to \( P^\beta \), then jump back to the curve \( w^0_\alpha(P) \) when \( P \) drops below \( P^\beta \).

We proceed with more precision: Given any \( P_f < P^\alpha \), we will show that if the stress increments \( P_{j+1} - P_j \) are small (see (7.10)), the fluctuations \( (u^j_0, u^j_1) \) are small (see (7.11), (7.12)) and the time increments \( t_{j+1} - t_j \) large (depending on the solutions \( u^j(x, t) \) themselves), then the sequence of asymptotic states will be a sequence of constant states \( w^j_\infty(x) \equiv L^j = w^j_\alpha(P_j) \) which in finitely many steps \( N \) can achieve \( P_N = P_f \) with \( w^N_\infty(x) \equiv L^N = w^N_\alpha(P_f) \). Thus, "proceeding slowly and carefully enough," the asymptotic state can be put as close as desired to \( w^\alpha_\alpha(P^\alpha) = \alpha \).

Precise conditions on the load increments and fluctuations are as follows:

To guarantee global existence of the \( u^j(x, t) \) and ensure that
(7.4) \[ w^j_\infty(x) \equiv w^j_\alpha(P_j), \]

we require (cf. Theorem 6.1)

(7.5) \[
\int_0^1 \left( \frac{1}{2} u^j_t + W(u^j_x) - P_j u^j_j \right) (y, t) dy < \frac{1}{2} E^\alpha(P_j)^2 + W(w^j_\alpha(P_j)) - P_j w^j_\alpha(P_j)
\]

(7.6) \[
\left| \int_1^x u^j_t(y, t) dy \right| < E^\alpha(P_j) \quad 0 < x < 1
\]

and

(7.7) \[
q^j(x, t_j) = u^j_x(x, t_j) + \int_1^x u^j_t(y, t_j) dy \in B(w^j_\alpha(P_j); E^\alpha(P_j))
\]

for a.e. \( x \) in \( (0, 1) \).

If \( P_j \) is to be chosen less than \( P_\beta \), so that \( E^\alpha(P_j) = \infty \), then no restriction need be imposed on the fluctuation \( (u^j_0, u^j_1) \) or the time \( t_j > t_{j-1} \) to ensure (7.4).

If \( P_j > P_\beta \), though, \( E^\alpha(P_j) \) will be finite. We may assume (as an induction hypothesis) that \( w^j_{\infty}(x) \equiv w^j_\alpha(P_{j-1}) \) and

(7.8) \[
\int_0^1 \left( \frac{1}{2} u^j_{j-1} + W(u^j_{j-1}) - P_{j-1} u^j_{j-1} \right) (y, t) dy = R_{j-1}(t) + W(w^j_{\alpha}(P_{j-1})) - P_{j-1} w^j_{\alpha}(P_{j-1})
\]

where \( R_{j-1}(t) \to 0 \) as \( t \to \infty \).

From the initial conditions (7.3) for \( u^j_j \), the left side of (7.5) may be expressed as

\[
\left\{ R_{j-1}(t_j) - (P_j - P_{j-1}) \left( \int_0^1 \left( u^j_{j-1}(y, t_j) - w^j_{\alpha}(P_{j-1}) \right) dy \right) \right\}
\]

(7.9) \[
+ \left\{ \int_0^1 \left[ u^j_t(y, t_j) u^j_j + \frac{1}{2} u^j^2 + W(u^j_x) + u^j_0 \right] - W(u^j_{x}(t_j)) - P_j u^j_0 \right\} (y) dy
\]

\[
+ \left\{ W(w^j_{\alpha}(P_{j-1})) - P_j w^j_{\alpha}(P_{j-1}) \right\}
\]
Require that $P_j$ satisfy

\begin{equation}
(7.10) \quad \left| W(w_\alpha(P_{j-1})) - W(w_\alpha(P_j)) - P_j(w_\alpha(P_{j-1}) - w_\alpha(P_j)) \right| < \frac{1}{6} E_\alpha(P_j)^2
\end{equation}

Indeed, the loads $P_j$ may be chosen inductively to satisfy (7.10), achieving $P_N = P_f < P_\alpha$ in finitely many steps $N$, since for $P < P_f$, $E_\alpha(P) > E_\alpha(P_f) > 0$.

We now need to restrict $t_j$ and $(u_{0_j}^j, u_1^j)$ so that the first and second brackets in (7.9) are each less than $\frac{1}{6} E_\alpha(P_j)^2$, and so that (7.6) and (7.7) hold.

For the second bracket in (7.9), it suffices to require

\begin{equation}
(7.11) \quad \int_0^1 u_1^j + \int_0^1 u_0^j < \frac{1}{16} E_\alpha^2(P_j)
\end{equation}

for some suitably large $M$ depending only on an interval containing all values of strain to be encountered. We require that the fluctuations also satisfy

\begin{equation}
(7.12) \quad |u_{0_j}^j(x)| < \frac{1}{2} E_\alpha(P_j) \quad 0 \leq x \leq 1.
\end{equation}

Now the first bracket and all the conditions (7.5) - (7.7) hold provided that $t_j$ is sufficiently large, depending on the solution $u_{j-1}(x, t)$ at $t = t_j$. (Note that the restrictions on the loads $P_j$ and the fluctuations $(u_{0_j}^j, u_1^j)$ do not depend on the solution, however.)

To summarize, we have shown that if the loads $P_j$ satisfy (7.10), the fluctuations $(u_{0_j}^j, u_1^j)$ satisfy (7.11), (7.12) and the increments $t_j - t_{j-1}$ are sufficiently large, then in finitely many steps $N$ we can achieve $w_\infty^N(x) \equiv w_\alpha(P_j)$ for any $P_f < P_\alpha$ (so $w_\alpha(P_f)$ may be as close as desired to $\alpha$).

To conclude our discussion of hysteresis, note that if the stress is raised more, so that for some $M > N$ we have $P_M > P_\alpha$, then we must have $w_\infty^M(x) \equiv w_\alpha(P_M)$, since this is the unique equilibrium state at stress $P_M$. The stress may now be lowered slowly to any level $\tilde{P}_f > P_\beta$, achieving $w_\infty^L(x) \equiv w_\beta(\tilde{P}_f)$ for some finite $L > M$, provided the fluctuations are restricted, in a manner similar to that above. If now the stress is lowered further to a level less than $P_\beta'$,
of course the asymptotic strain will return to the branch \( w_\alpha(P) \) (in the \( \alpha \) phase), completing a hysteresis loop.

**Creep.** As \( P \) approaches \( P_\alpha \) from below, the "metastable" state \( w_\alpha(P) \) becomes more sensitive to perturbations. That is, \( E_\alpha(P) \to 0 \) and the "basin of stability" of the homogeneous state \( w_\alpha(P) \) which can be guaranteed by the restrictions (7.11), (7.12) becomes smaller. If fluctuations are larger, nonhomogeneous states may appear, resulting in a measured deformation \( L \) which is some average of the available homogeneous states \( w_\alpha(P) \) and \( w_\beta(P) \). We will illustrate how a "creep" phenomenon may occur due to small-energy fluctuations of sensitive metastable states producing nonhomogeneous asymptotic states. (Such small-energy fluctuations might be considered a primitive model for the effect of Brownian motion in the bar.)

Consider the experiment (7.1) - (7.3) carried out at a constant load \( P < P_\alpha \), so \( P_j = P \) for all \( j \). Suppose that \( P \) is close to \( P_\alpha \), so that

\[
E = E_\alpha(P) \ll E_\beta(P).
\]

So that we may apply Theorem 6.1 (by verifying (6.3), (6.4)) to deduce the invariance of the interval \( B(w_\alpha(P); E) \) for the modified strain \( q \), we need two hypotheses: First, we require that the fluctuations have small energy in the sense that (7.11) is satisfied for some suitably large \( M \), which now may depend on how small \( E \) is. Second, we **assume** that each asymptotic state \( w^j_\infty(x) \) takes the "unstable" value \( w_u \) (where \( w_u \) is the middle solution of \( \sigma(w) = P, \ w_\alpha(P) < w_u < w_\beta \) on a set of negligible measure. (This assumption is physically reasonable in the sense that an equilibrium state \( \tilde{w}(x) \) having \( \tilde{w}(x) = w_u \) on a set of positive measure is unstable, by 4.2.)

To check that these hypotheses guarantee that (6.3) holds in the form

\[
(7.13) \quad \int_0^1 \left( \frac{1}{2} u^2_t + W(u^j_x) - Pu^j_x \right)(y, t) dy < \frac{1}{2} E_\alpha(P)^2 + \int_0^1 W_m(q^j(y, t)) dy,
\]

set

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\[ W_p(w) = W(w) - Pw, \]

observe that
\[ \int_0^1 W_p(u_{x}^{j-1}(t_j) + u_{0}^{j}) - W_m(q^{j}(t_j)) = \]

\[ \int_0^1 (W_p(u_{x}^{j-1}(t_j) + u_{0}^{j}) - W_p(u_{x}^{j-1}(t_j))) + \int_0^1 (W_p(w_{\infty}^{j-1}) - W_m(w_{\infty}^{j-1})) \]

\[ + \int_0^1 (W_m(w_{\infty}^{j-1}) - W_m(q^{j}(t_j)) + \tilde{R}_{j-1}(t_j) = T_1 + T_2 + T_3 + \tilde{R}_{j-1}(t_j). \]

Now estimate
\[ |T_1| < M_1 \int_0^1 |u_{0}^{j}| \]
\[ |T_2| < M_2 \mu \{ x \mid w_{\infty}^{j-1}(x) = w_u \} \]
\[ |T_3| < M_3 (\mu(S_1) + \mu(S_2 \cap S_3)) \]

where
\[ S_1 = \{ x \mid w_{\infty}^{j-1}(x) = w_\alpha^{(P)} \text{ and } |u_{x}^{j-1}(x, t_j) - w_{\infty}^{j-1}(x)| > E/4 \} \]
\[ S_2 = \{ x \mid w_{\infty}^{j-1}(x) = w_\alpha^{(P)} \text{ and } |u_{x}^{j-1}(x) - w_{\infty}^{j-1}(x)| < E/4 \} \]
\[ |S_3| = \{ x \mid |u_{0}^{j}(x)| > E/4 \}. \]

(Provided that \( \int u_{t}^{j-1}(t_j)^2 < E^2/16, \mid q^{j}(x, t_j) - w_\alpha^{(P)} \mid > E \) and \( x \in S_2 \) imply \( x \in S_3 \),)
Now $\mu(S_1) \to 0$ as $t_j \to \infty$, $\mu(S_2)$ is negligible by assumption, and
$$\mu(S_3) \leq \frac{4}{E} \sum_{j=0}^{1} \left| u_0^j \right|.$$ So if $t_j$ is large, (7.13) holds.

The third hypothesis we impose on the experiment (7.1) - (7.3) in the present discussion of creep is to restrict the fluctuations in strain to satisfy

$$|u_0^j(x)| \leq M_0, \quad \text{where } 5E < M_0 < 2E\beta(p) - 4E.$$  

Thus the restriction (7.12) is considerably relaxed, since we may take $M_0 \gg E$. Roughly speaking, the restriction (7.14) will imply that fluctuations cannot "kick" the strain from the $\beta$ phase to the $\alpha$ phase, but that they can "kick" the strain at some points into the $\beta$ phase from the $\alpha$ phase (but only on a small set compatible with (7.11)). Given suitable perturbations (see (7.16)), the result is that the set of points $x$ such that $w_{\infty}^j(x) = w_{\beta}^j(P)$ grows, and the set of $x$ with $w_{\infty}^j(x) = w_{\alpha}^j(P)$ gets smaller, so that the asymptotic length $L^j$ slowly grows. This is the "creep" phenomenon desired.

To be more precise, set
$$S_{\beta}^{j-1} = \left\{ x \mid \left| q_{\beta}^{j-1}(x,t_j) - w_{\beta}(P) \right| < 2E \right\}$$

and observe
$$\left\{ x \mid w_{\infty}^j(x) = w_{\alpha}^j(P) \right\} \subset (S_{\beta}^{j-1})^c = [0,1] \setminus S_{\beta}^{j-1}.$$  

We will show that if $t_{j+1}$ is large enough, then

$$S_{\beta}^{j} \supseteq S_{\beta}^{j-1} \cup \left\{ x \mid u_0^j > 5E \right\}.$$  

It follows that if the fluctuations satisfy, for some $\epsilon > 0$,

$$\mu((S_{\beta}^{j-1})^c \cap \left\{ x \mid u_0^j > 5E \right\}) \geq \epsilon \mu((S_{\beta}^{j-1})^c),$$

then $\mu((S_{\beta}^{j})^c) \leq \mu((S_{\beta}^{j-1})^c(1-\epsilon)$, so that

$$\mu((S_{\beta}^{j})^c) \leq (1-\epsilon)^j$$

which implies "creep."
It remains to establish (7.15). We assume \( t_j \) is so large that
\[
q^j - 1(x, t_j) > w_\alpha(P) - E \text{ for all } x \in [0, 1].
\]
Then, because of the condition on \( M_0 \) in (7.14), and (7.11), we find that if \( u_0(x) > 5E \), then
\[
q^j(x, t_j) > w_\beta(P) - 2E \beta \beta(P) + 2E.
\]
Since \( \sigma(w) - P \) is strictly negative on \((w_\beta(P) - 2E \beta \beta(P) + E, w_\beta(P) - E)\) and strictly positive on \((w_\beta(P) + E, \infty)\), we can ensure that for \( t_{j+1} - t_j \) larger than some constant, we have
\[
|q^j(x, t_{j+1}) - w_\beta(P)| < 2E,
\]
that is, \( x \in S_\beta \). This establishes (7.15), and concludes our discussion of creep.

II. Admissibility and moving phase boundaries in an elastic bar

8. A viscosity criterion for admissibility of waves

If one deals with the elastic system (2.2), one must deal with discontinuous weak solutions. Here the ill-posedness of (2.2) is evident in the simplest initial-value problem involving discontinuous data, the Riemann problem: One seeks a centered wave solution \((w, v)(x/t)\) of (2.2) with initial data
\[
(w, v)(x, 0) = \begin{cases} 
(w^-, v^-) & \text{for } x < 0, \\
(w^+, v^+) & \text{for } x > 0.
\end{cases}
\]
Then with \( \sigma(w) \) of the form in Fig. 1, James (1980b) has shown that the Riemann problem can have a two-parameter family of centered wave weak solutions, so uniqueness fails in a rather bad way.

Resolution of this uniqueness issue seems to devolve on the question of what discontinuous waves for (2.2) of the form
\[
(8.1) \quad (w, v)(x, t) = \begin{cases} 
(w^-, v^-) & \text{for } x < st, \\
(w^+, v^+) & \text{for } x > st
\end{cases}
\]
are to be regarded as physically relevant, or admissible. The approach we will follow is to identify those waves which arise in the limit of vanishing viscosity.

For the wave (8.1) to be a weak solution of (2.2), the Rankine-Hugoniot jump conditions must hold:

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\[-s(w_+ - w_-) = (v_+ - v_-)\]
\[-s(v_+ - v_-) = \sigma(w_+ - \sigma(w_-)\]

The second equation may be replaced by

\[(8.2) \quad s^2 (w_+ - w_-) = \sigma(w_+ - \sigma(w_-)\]

We will identify such a wave as \textit{admissible according to the viscosity criterion} provided that

\[(w, v)(x, t) = \lim_{\mu \to 0^+} (w^\mu, v^\mu)(x, t)\]

where \((w^\mu, v^\mu)(x, t)\) are \textit{traveling wave solutions} of the associated viscoelastic system derived from (2.3):

\[(8.3)\]
\[w_t = v_x\]
\[v_t = (\sigma(w) + \mu w_t)_x\]

The traveling wave solutions may be taken in the form \((\phi, \psi)((x - st)/\mu)\) where \(\phi, \psi\) satisfy

\[(8.4)\]
\[-s(\phi - w_-) = \psi - v_-\]
\[-s(\psi - v_-) = \sigma(\phi) - \sigma(w_-) - s\phi_x\]

The last equation may be replaced by

\[(8.5) \quad s\phi_x(\xi) = \sigma(\phi) - \sigma(w_-) - s^2(\phi - w_-)\]

There are now two cases. If \(s \neq 0\) and \((w_+, w_-, s)\) satisfies (8.2), then (8.5) is an ODE, and it is well known and easily checked that a solution exists satisfying

\[\lim_{\xi \to \pm \infty} \phi(\xi) = w_\pm\] if and only if the following \textit{chord condition} is satisfied:

The chord connecting the points \((w_+, \sigma(w_+))\) should lie above (resp. below) the graph of \(\sigma(w)\) if \((\text{sgn } s)\text{sgn}(w_+ - w_-)\) is negative (resp. positive).
This condition ensures that \( \text{sgn} \phi' = \text{sgn}(w_+ - w_-) \) when \( \phi \) is between \( w_+ \) and \( w_- \), and would be equivalent to Liu's (strict) entropy condition for shocks (Liu, 1976) if the system were strictly hyperbolic (which requires \( \sigma' > 0 \) everywhere).

In the second case, \( s = 0 \), so (8.5) reduces to an algebraic equation

\[
(8.7) \quad \sigma(\phi) = \sigma(w_-).
\]

A solution with the desired property is

\[
(8.8) \quad \phi(\xi) = w(\xi) = \begin{cases} 
w_- & \text{for } \xi < 0 \\
w_+ & \text{for } \xi > 0 
\end{cases}.
\]

As we have already noted, the viscoelastic equation (2.3) has the same equilibria as the elastic equation (2.1). So all elastic equilibria are trivial viscous limits.

To summarize, the viscosity criterion for admissibility of weak solutions of (2.2) of the form (8.1) may be stated as follows:

If \( s \neq 0 \), the wave is admissible if the chord condition (8.6) holds.

\[
(8.9) \quad \text{If } s = 0, \text{ the wave is admissible unconditionally.}
\]

Concerning the solution of the Riemann problem under the admissibility criterion (8.9), it is remarkable that a study already exists. For \( \sigma(w) \) as in Fig. 1, changing convexity at just one point, Shearer (1982) established existence and uniqueness for the solution, provided that discontinuities are required to satisfy exactly the admissibility condition (8.9). Shearer appears to have admitted the waves with \( s = 0 \) on an ad hoc basis. Here we have shown how these waves arise from the viscosity criterion.

Slemrod (1983) has previously dismissed a viscosity criterion of the sort above, his analysis forbidding the waves having \( s = 0 \). The waves remaining form a class insufficient to solve the Riemann problem. However, Slemrod rules out discontinuous viscous waves of the sort in (8.8) a priori. In the hindsight afforded by the results of §6 above, his requirement of continuity seems clearly too restrictive. Discontinuous stationary waves of this sort can be stable asymptotic limits of smooth solutions in the viscoelastic problem (2.4).
Shearer (1983) and also Slemrod (1983) have examined admissibility criteria based on other forms of dissipation mechanisms in the system \((8.3)\). Slemrod's viscosity-capillarity criterion admits just a single stationary wave, at the Maxwell line, where the stored-energy density at \(w_+\) and \(w_-\) are equal. (Also see Aifantis and Serrin (1983).) Typically, for moving waves, Slemrod's criterion implies that the metastable state is overtaken by the stable state. Shearer (to appear) has now shown, however, that the viscosity-capillarity criterion fails to imply uniqueness for solutions of some Riemann problems with states near the Maxwell line.

We finish with two remarks concerning the admissible waves selected by the viscosity criterion developed in this section. Fix \(\sigma\) of the form in Fig. 1, requiring that the convexity of \(\sigma\) changes at just one point.

We first remark that the closure of the set of triples \((w_+, w_-, s)\) which determine an admissible wave \((2,2)\) form a connected continuum in \(\mathbb{R}^3\). That is, any two triples \((w_+, w_-, s)\) and \((\tilde{w}_+, \tilde{w}_-, \tilde{s})\) which determine admissible waves can be connected by a path in this set. We leave the verification to the reader. Note that this includes waves with speed of any sign, so the zero speed "shocks" in particular are not exceptional from this point of view.

We shall have occasion in §9 to consider "interphase shocks," for which \(w_+\) and \(w_-\) lie in different components of \(\{w \mid \sigma'(w) > 0\}\). Waves of this sort exist with arbitrarily small speed \(s\). Consider the "shock structure" for these waves obtained from \((8.5)\). Our second remark is that if viscosity \(\mu > 0\) is fixed, the smooth "transition layer" obtained from \((8.5)\) for interphase shocks approaches zero thickness as \(s \to 0\). For \(s > 0\), \(w_+ < \alpha\), and \(w_- > \beta\) in Fig. , the condition \(s \to 0\) implies \(w_+ \to \alpha\) and \(w_- \to \gamma\). Our second remark means that, fixing the phase of the wave appropriately, as \(s \to 0\) we have

\[
\phi(\xi) \to \phi_0(\xi) = \begin{cases} 
\gamma & \text{for } \xi < 0 \\
\alpha & \text{for } \xi > 0 
\end{cases}
\]
9. Appearance of propagating phase boundaries

In the hysteresis process discussed in §7, a phase transition occurs approximately when the load level \( P \) crosses the level \( P_\alpha \). The bar, formerly in the "\( \alpha \) phase" with asymptotic strain \( w_\alpha(P) \), is forced into the "\( \beta \) phase" with asymptotic strain \( w_\beta(P) \). We describe here an idealized problem for the elastic bar indicating that this transition can be associated with a slowly propagating "phase boundary," corresponding to a wave of discontinuous strain for the system (2.2) which is admissible according to the chord condition (8.6). The mechanism we suggest has been examined by Pence (to appear) and suggested in the context of shear flow in polymeric fluids by Hunter and Slemrod (1983). However, we are able to assert the uniqueness of the solution to the problem (9.1) below within the class of centered wave solutions satisfying the admissibility criterion (8.9) derived from the limit of vanishing viscosity.

For ease in applying the results of Shearer (1982), we will in fact consider the situation in which the load \( P \) falls below the transition level \( P_\beta \) (notation as in §7), with the bar previously equilibrated in the \( \beta \) phase.

Suppose then that at \( t = 0 \) the elastic bar lies at equilibrium at a stress level \( P > P_\beta \), \( P \) near \( P_\beta \), so \( w(x,0) \equiv w_\beta(P) > \beta \), \( v(x,0) \equiv 0 \) in the system (2.2), for \( 0 \leq x \leq 1 \). Imagine that the load at the boundary \( x = 1 \) is suddenly lowered to a level \( P_+ < P_\beta \), and held constant. In the elastic model, this determines the strain at the boundary to be \( w(1,t) = w_\alpha(P_+) < \alpha \), since we must have \( \sigma(w(1,t)) = P_+ \).

We now have the following initial-boundary value problem (a Riemann problem at the boundary):

Find a weak solution of the elastic system (2.2) having
initial conditions

\[ \begin{align*}
(9.1) \quad w(x,0) &= w_\beta(P), \quad v(x,0) = 0 \quad 0 \leq x \leq 1
\end{align*} \]

and boundary condition

\[ w(1,t) = w_\alpha(P_+) \quad 0 < t \]
(For short time, we ignore the fixed-end condition \( v(0, t) = 0 \).) We can construct a solution to this problem for short time by finding a centered 1-wave solution to the ordinary Riemann problem for (2.2), centered at \((x, t) = (1, 0)\), connecting \((w_-, v_-) = (w_\beta(P), 0)\) on the left to \((w_+, v_+) = (w_\alpha(P), v_+\) on the right, for some \(v_+\) to be determined. That is, we seek \(v_+\) so that the solution of the Riemann problem for \((w_-, v_-), (w_+, v_+)\) contains only 1-waves, which have negative speed, so that the boundary condition in (9.1) will be satisfied. The resulting solution is valid until the leading wave impinges on the boundary \(x = 0\).

The problem in (9.1) is now easily resolved utilizing Shearer's solution in (Shearer, 1982) of the Riemann problem for (2, 2) under the admissibility criterion justified in §8. We must assume \(\sigma''(w) < 0\) for \(w < \alpha\), \(\sigma''(w) > 0\) for \(w > \beta\). We find that a unique \(v_+ = v_b < 0\) with the property described above does always exist. In Shearer's notation, given the point \(U_0 = (w_-, v_-)\) in his Fig. 4, we require that the solution involve only 1-waves, so that the state \(U_1 = (w_+, v_+)\) must lie either on the curve \(S_1^{*}(U_0)\) or on the curve \(E\). The structure of the solution is as follows: If \(w_+ < w_*(w_-)\) where the tangent to the graph of \(\sigma(w)\) at \(w = w_-\) intersects the graph again at \(w_*(w_-) < \alpha\), then the solution is a single 1-shock connecting \((w_-, v_-)\) to \((w_+, v_+)\). If \(w_*(w_-) < w_+ < w_\alpha(P_\beta)\), then the solution consists of a rarefaction wave connecting \((w_-, v_-)\) to an intermediate state \((w_0, v_0)\), with \(\beta < v_0 < w_\alpha\) which is then immediately connected to \((w_+, v_+)\) by a 1-shock. Here \(w_0\) is determined by the requirement \(w_*(w_0) = w_+\), that is, the chord from \((w_+, \sigma(w_+))\) to \((w_0, \sigma(w_0))\) lies below the graph of \(\sigma(w)\) and is tangent to it at \(w_0\). As \(\alpha \rightarrow \beta\), or as \(w_+\) approaches \(w_\alpha(P_\beta)\), clearly the slope of this chord, which is equal to the squared speed of the shock connecting the two phases, approaches zero. So the interphase shock which appears in the solution can have an arbitrarily small speed, depending on how close \(\alpha\) is to \(\beta\). We also note that as \(P_+ \rightarrow P_\beta\), the velocity \(v_b \rightarrow 0\).

We conclude by illustrating a "necking" phenomenon arising out of a Riemann problem for (2, 2) obtained by applying a symmetry principle to the solutions just described. (Actually, true "necking" would arise in the companion
situation for loads near $P_\alpha$.) Denote the solution of the whole-line Riemann problem just used to solve (9.1), centered at $(x, t) = (0, 0)$, by $(\tilde{w}, \tilde{v})(x, t)$, so that

$$(\tilde{w}, \tilde{v})(x, 0) = \begin{cases} (w_\beta(P), 0) & \text{for } x < 0 \\ (w_\alpha(P_\alpha^+), v_b) & \text{for } x > 0 \end{cases}.$$ 

Then for $t \geq 0$ and all $x \geq 0$, we know $(\tilde{w}, \tilde{v})(x, t) = (w_\alpha(P_\alpha^+), v_b)$ as well. Due to the symmetry of the equations (2.2) and the chord condition (8.6), the function

$$(w, v)(x, t) = \begin{cases} (\tilde{w}, \tilde{v})(x, t) & \text{for } x < 0 \\ (\tilde{w}, 2v_b - \tilde{v})(-x, t) & \text{for } x > 0 \end{cases}$$ 

is the (unique admissible) solution of the Riemann problem for (2.2) with initial states

$$(w_-, v_-) = (w_\beta(P), 0), \quad (w_+, v_+) = (w_\alpha(P), 2v_b).$$

Physically, this means starting with homogeneous strain $w = w_\beta(P)$ at $t = 0$, but giving the right "half" of the bar a small velocity $2v_b < 0$. The solution then develops a slowly expanding region near $x = 0$ in the $\alpha$ phase with $w = w_\alpha(P_\alpha^+)$. 

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Appendix: Some estimates for the abstract parabolic equation.

Essentially, in this section we reorganize the proof of Henry (1981), 3.5.2 to obtain estimates needed in §3 and §5. Throughout, we assume $A$ is sectorial on a Banach space $X$, $f: U \to X$ is locally Lipschitz on an open set $U \subset \mathbb{R} \times X^\alpha$ for some $0 \leq \alpha < 1$, and $z(t)$ is a solution on $(0, T]$ of

$$ z_t + A z = f(t, z), \quad z(0) = z_0 $$

with $(0, z_0) \in U$. We assume

$$ \| f(t, z(t)) - f(s, z(s)) \| \leq K(t - s) + L \| z(t) - z(s) \|, $$

(Here $\|$ is the norm in $X$, $\| \|_\alpha$ the norm in $X^\alpha$.) We let

$$ M_z = \| z_0 \|_\alpha, \quad M_f = \sup_{0 < t < T} \| f(t, z(t)) \| . $$

Below, we rely heavily on standard estimates for fractional powers of sectorial operators (Henry 1.4.3).

**Lemma A.1.** If $\alpha < \beta < 1$, there exists $C = C(\alpha, \beta, T)$ so that

$$ \| z(t) \|_\beta \leq C(t^{\alpha - \beta} M_z + M_f) \quad \text{for} \quad 0 < T. $$

**Proof.** The solution $z(t)$ satisfies

$$ z(t) = e^{-At} z_0 + \int_0^t e^{-A(t-s)} f(s, z(s)) \, ds. $$

Using standard estimates (Henry 1.4.3), we find

$$ \| z(t) \|_\beta \leq C \| z_0 \|_\alpha + C \int_0^t e^{-s^\beta} \, ds. $$

The lemma follows.

**Lemma A.2.** There exists $C_\ast = C_\ast(\alpha, T, L)$ so that for $0 < t < t + h < T$ we have

$$ \| z(t+h) - z(t) \|_\alpha \leq h^{\frac{1}{\alpha}} C_\ast (M_z + M_f + K). $$

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Proof. Fix $0 < \tau < t$ and write, for $t > \tau$,

\[
    z(t+h) = e^{-A(t+h-\tau)} z(\tau) + \int_{\tau}^{t+h} e^{-A(t+h-s)} f(s, z(s)) \, ds
\]

\[
    z(t+h) - z(t) = (e^{-Ah} - 1_e^{-A(t-\tau)} z(\tau) + \int_{\tau}^{t+h} e^{-A(t+h-s)} f(s, z(s)) \, ds
\]

\[
    + \int_{\tau}^{t} e^{-A(t-s)} (f(s+h, z(s+h)) - f(s, z(s))) \, ds
\]

\[
    = T_1 + T_2 + T_3
\]

Now

\[
    \| T_3 \|_{\alpha} \leq C \int_{\tau}^{t} (t-s)^{-\alpha} (K_h + \| z(s+h) - z(s) \|_{\alpha} ) \, ds
\]

\[
    \leq CK_h + C \int_{\tau}^{t} (t-s)^{-\alpha} \| z(s+h) - z(s) \|_{\alpha} \, ds
\]

\[
    \| T_2 \|_{\alpha} \leq CM_f \int_{\tau}^{t+h} (t+h-s)^{-\alpha} \, ds \leq hCM_f (t-\tau)^{-\alpha}
\]

Choosing $\beta$ with $\alpha < \beta < 1$, and again using Henry 1.4.3,

\[
    \| T_1 \|_{\alpha} \leq hC(t-\tau)^{\beta-\alpha-1} \| z(\tau) \|_{\beta}
\]

Now we have

\[
    \| z(t+h) - z(t) \|_{\alpha} h^{-1} \leq (t-\tau)^{\beta-\alpha-1} C (\| z(\tau) \|_{\beta} + M_f + K)
\]

\[
    + CL \int_{\tau}^{t} (t-s)^{-\alpha} \| z(s+h) - z(s) \|_{\alpha} h^{-1} \, ds
\]

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Because $\beta - \alpha - 1 > -1$ and $0 \leq \alpha < 1$ we may now apply the generalized Gronwall inequality (Henry (1981), Ch. 6, ex. 4) to conclude that

$$\| z(t+h) - z(t) \|_{\alpha} \leq h C_* (t-\tau)^{\beta - \alpha - 1} (\| z(\tau) \|_\beta + M_f + K)$$

where $C_* = C_*(\alpha, \beta, T, L, C)$ is independent of $\tau$. Now choose $\tau = t/2$ and apply Lemma A.1 to obtain the result.

**Lemma A.3.** If $0 < \gamma < 1$, there exists $C_* = C_*(\alpha, \gamma, T, L)$ so that for $0 < t < T$,

$$\| z_t (t) \|_\gamma \leq C_* (t^{\alpha - \gamma - 1} M_z + t^{-\gamma} (M_f + K)) .$$

**Proof.** From Henry 3.2.1 we may write

$$z_t (t) = -A e^{-At} z_0 + e^{-At} \left( \int_0^{t/2} + \int_{t/2}^t \right) A e^{-A(t-s)} (f(t, z(t)) - f(s, z(s))) ds$$

$$= T_1 + T_2 + T_3 + T_4 .$$

We then estimate

$$\| T_1 \|_\gamma \leq C t^{\alpha - \gamma - 1} M_z$$

$$\| T_2 \|_\gamma \leq C t^{-\gamma} M_f$$

$$\| T_3 \|_\gamma \leq C \int_0^{t/2} (t-s)^{-\gamma - 1} ds \cdot 2M_f \leq C t^{-\gamma} M_f$$

$$\| T_4 \|_\gamma \leq C \int_{t/2}^t (t-s)^{-\gamma - 1} (K(t-s) + L \| z(t) - z(s) \|_\alpha) ds$$

$$\leq C_* \int_{t/2}^1 (t-s)^{-\gamma - 1} ds (M_z + M_f + K)$$

$$= t^{-\gamma} C_* \int_{1/2}^1 (1-s)^{-\gamma - 1} ds (M_z + M_f + K) .$$

The lemma follows.
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Fig. 1. Stress-strain relation.

Fig. 2. Stored-energy density.
Fig. 3. Invariant intervals for modified strain.
Fig. 4. $\lambda$-plane.

Fig. 5. $\beta(\lambda, s_j)$ plane.
Fig. 6. Pairs \((w_+, w_-)\) determining admissible waves for \(s \geq 0\).