ISOLATED SINGULARITIES OF THE SOLUTIONS
OF THE SCHröDINGER EQUATION WITH A RADIAL POTENTIAL

By
J.L. Vazquez
AND
C. Yarur
ISOLATED SINGULARITIES OF
THE SOLUTIONS OF THE SCHRÖDINGER
EQUATION WITH A RADIAL POTENTIAL

by

J.L. Vazquez

Division de Matematicas
Universidad Autonoma de Madrid
28049 Madrid, SPAIN

and

C. Yarur

Facultad de Matematicas
Universidad Complutense
28040 Madrid, SPAIN

This paper has been written while the first author was a long-term visitor at
the Institute for Mathematics and its Applications, partly supported by
USA-Spain Cooperation Agreement under Joint Research Grant CCB-8402023.
We study the behaviour near an isolated singularity, say $x=0$, of the solutions of the equation (1) $\Delta u(x) = V(x)u(x)$, defined in a neighbourhood of $0$, in terms of the potential $V$ that we assume to be radially symmetric. For a class of potentials $P_1$ that we completely characterize there is only a unique singularity with constant sign and it behaves like the fundamental solution of the Laplace operator. For a larger class $P_2$ we yet obtain a unique type of nonnegative singularity that is radially symmetric. If $V(r) = cr^\theta$, with $c>0$, $\theta \in \mathbb{R}$, $P_1$ means $\theta > -2$ and $P_2$ means $\theta > -2$. The results fail for $\theta < -2$.

Using the Kelvin transformation our results give an equivalent description of the asymptotic behaviour of solutions of (1) defined for $|x| > R > 0$. 
INTRODUCTION

In this paper we study the behaviour near an isolated singularity of the solutions of the time-independent Schrödinger equation

\[(0.1) \quad \Delta u = V(x)u\]

defined in a domain of $\mathbb{R}^N$, $N > 2$. We shall assume that the potential $V$ is radially symmetric, $V = V(|x|)$, and not very negative. For convenience we shall also assume that $V$ is continuous except possibly at $x = 0$. We study in terms of $V$ the occurrence of a unique type of isolated singularity at $x = 0$ for solutions of (0.1) that have constant sign near the origin.

The behaviour of solutions of (0.1) around an isolated singularity is well-known in the case $V = 0$. In fact for every nonnegative harmonic function defined in a punctured ball $B^*_R = B_R(0) - \{0\}$ there exists $c > 0$ such that

\[(0.2) \quad u(x) = cE_N(x) + g(x)\]

where $E_N$ is the fundamental solution of the Laplace operator, see (1.1), and $g$ is a $C^\infty$ harmonic function in $B_R$.

In Section 1 we prove that this behaviour persists if $V$ is not very singular. The class $\mathcal{P}_1$ of potentials for which an isolated singularity with constant sign behaves like $E_N(x)$ is shown to coincide (for $N > 3$) with the Kato class $K_N$ (Theorem A). The proof relies heavily on the study of the corresponding ODE (Section 2). For potentials of the form $V(r) = cr^\nu$ with $c, \nu \in \mathbb{R}$ $V \in \mathcal{P}_1$ if and only if $\nu = 0$ or $\nu > -2$.

It follows in particular from this study that when $V$ is negative and $V \notin \mathcal{P}_1$ there are no radial singularities with constant sign. On the other side Willett [W] proves that the condition $V^- = \max(-V, 0) \in \mathcal{P}_1$ is sufficient to exclude the appearance of radial solutions that oscillate infinitely many times near $r = 0$. We shall also need $V^- \mathcal{P}_1$ to transform unilateral bounds into absolute bounds in Theorems A and C. Therefore we shall make in the sequel the assumption $V^- \in \mathcal{P}_1$ unless mention to the contrary.
We also study the conditions under which a singularity is removable both in the case $V \in P_1$ (Corollary 1.1) and in the case $V \notin P_1$ (Theorem B). All these results are proved in Section 3.

Then we consider a larger class of potentials $P_2$ for which there is only one type of singular behaviour for the nonnegative solutions of (0.1). Moreover this singularity is isotropic, i.e. the singular solution $u(x)$ becomes radially symmetric as $r \to 0$. In fact we prove (Theorem C) that for every not very negative singular solution of (0.1) with $V \in P_2$ we have

$$u(x) = \overline{u}(r) + o(1),$$

where $\overline{u}$ is the angular average of $u$, i.e.

$$\overline{u}(r) = |S^{N-1}|^{-1} \int_{S^{N-1}} u(r, \sigma) d\sigma.$$

Here $(r, \sigma) \in (0, \infty) \times S^{N-1}$ are spherical coordinates in $\mathbb{R}^N$, $d\sigma$ is the standard measure on $S^{N-1} = \{ x \in \mathbb{R}^N : |x| = 1 \}$ and $|S^{N-1}|$ is the area of $S^{N-1}$.

We also have for a singular $u$

$$\lim_{r \to 0} \overline{u}(r)/E_N(r) \neq 0$$

and $|u|$ grows at most like a power of $r$ as $r \to \infty$. The class $P_2$ contains, apart from $P_1$, the inverse square potentials $V(r) = cr^{-2}$, that play an important role in Quantum Mechanics, if $c > 0$. $P_2$ is also closed under addition. Complete details are given in Section 4 where we also prove Theorem C.

The sharpness of this theorem is shown in Section 5, where we construct infinitely many positive singular solutions of (0.1) exhibiting different singular behaviours for the potentials of a class $P_3$ which is nearly the complement of $P_2$ in the set of positive radial potentials. $P_3$ contains in particular the potentials of the form $V(r) = cr^\theta$ with $c > 0$ and $\theta < -2$. For these latter potentials there are radial solutions $u_0(r), u_1(r)$ of (0.1) such
that, as \( r \to 0 \),

\[(0.6.a) \quad u_1(r) = r^k \exp \left( \frac{\xi}{pr^p} \right), \text{ and} \]

\[(0.6.b) \quad u_0(r) = r^k \exp \left( -\frac{\xi}{pr^p} \right), \]

where \( p = \frac{1}{2} (\nu + 2) > 0 \) and \( k = \frac{1}{2} (p + 2 - N) \), see end of Section 2.

The study of the behaviour of the solutions of (0.1) near an isolated singularity can be transformed into the study of the asymptotic properties of the solutions of (0.1) as \( |x| \to \infty \) by means of the Kelvin transformation. In fact, let \( u = R^N - B_R(0) \) for some \( R > 0 \) and consider the solutions \( u(x) \) of (0.1) in \( u \), where now \( V \in C((R, \infty); \mathbb{R}) \). If we set

\[(0.7) \quad w(y) = |y|^{2-N} u(y |y|^{-2}) \]

then \( w \) is a solution of the equation

\[(0.8.a) \quad \Delta w(y) = W(y)w(y) \text{ for } 0 < |y| < 1/R \]

with

\[(0.8.b) \quad W(y) = |y|^{-4} V(|y|^{-1}). \]

In that way we obtain in Section 6 a description of the asymptotic behaviour of \( u \) when \( V \) belongs to the class \( P^\infty_{-1}, P^\infty_2 \) or \( P^\infty_3 \), which corresponds to \( W \in P^{-1}_1, P_2 \), or \( P_3 \). In particular we prove that a nonnegative solution of (0.1) behaves as \( r = |x| \to \infty \) either like a constant or like \( r^{2-N} \) if \( N > 3 \) (resp. like \( \log(r) \) or like a constant if \( N=2 \)) if and only if \( V \in P^\infty_{-1} \), cf. Theorem A'. For instance if \( |V(r)| = r^u \) this means \( u < -2 \) and in this case we have
\[(0.9.a)\]
\[u(x) = C_1(1 + O(r^{\theta+2})) + C_2 r^{2-N} + o(r^{2-N})\]

if \(N \geq 3\) and

\[(0.9.b)\]
\[u(x) = c_1(\log(r) + O(r^{\theta+2})) + C_2 + o(1)\]

if \(N=2\), for some constants \(c_1, c_2 \in \mathbb{R}\).

Finally we use these results to study in Section 7 the conditions under which the solutions of (0.1) in \(\Omega = \mathbb{R}^N - \{0\}\) are radially symmetric.

Two main restrictions are made in this paper. First, both the potentials and the solutions are assumed not to be very negative. This is made so as to eliminate all oscillatory solutions from the present discussion. Second, the potentials are radially symmetric, \(V(x) = V(|x|)\). Though many of our results have natural counterparts for nonradial potentials we only have a complete picture of the situation in the case of radial potentials. Moreover our treatment relies on the study of the radial solutions of (0.1), where ODE techniques are used. We shall consider the case of nonradial \(K_N\) - potentials in an upcoming paper using different techniques.

The study of isolated singularities and asymptotic behaviour of the solutions of elliptic equations, among them (0.1), is a classical subject that has been pursued by many authors, in particular Serrin, cf. [GS], [S1], [S2], [SW]. Recently Brezis and Lions [BL] proved that nonnegative solutions of \(\Delta u < cu\) with \(c > 0\) in \(B^*\) either have a removable or a weak singularity in the notation of Section 1. This is related to our Theorem 1. Caffarelli and Littman [CL] studied the positive solutions of (0.1) in \(\mathbb{R}^N\) with \(V(x) = 1\) (therefore \(V \in P_3^\infty\) in our notation). Murata [M] studies the behaviour of positive solutions as \(|x| \to \infty\) under conditions on \(V\) different from ours. The fact that the inverse-square potentials, \(V(r) = cr^{-2}\), are a sort of borderline case for
different phenomena is well-known, cf. for instance [RS], [BG]. Recently Garofalo and Lin [GL] have also encountered this threshold behaviour in the study of the property of unique continuation of solutions of (0.1). Finally dal Maso and Mosco [MM] study equation (0.1) using a variational approach, obtaining in particular moduli of continuity for nonsingular solutions of (0.1).

Isolated singularities of nonlinear equations of the form $\Delta u = \phi(u)$, with $\phi$ continuous and increasing, have been studied by Brezis, Véron and the first author, among others, cf. [BV], [VV], [Ve]. Even in the case $\phi(u) = u^p$ with $p>1$ and $u>0$ the results depart strongly from the linear case.

An announcement of results by the authors on this problem has appeared in [VY]. The present paper contains not only a full account but also a substantial improvement on that note. The authors are grateful to J. Serrin for his interest in this work.
1. BEHAVIOUR AT THE ORIGIN FOR SMALL POTENTIALS

Let $B_R = B_R(0)$ be the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$, $N \geq 2$, and let $B_R^* = B_R - \{0\}$. We consider solutions $u \in C^2(B_R^*)$ of equation (0.1) for $0 < |x| < R$ and describe their behaviour as $|x| \to 0$ depending on the singular character of $V$ at $0$. We shall use indistinctly the notations $V = V(x) = V(r)$, $r = |x|$, and so on, whenever no confusion arises.

Let us first briefly examine the possible behaviours of $u$ near $0$. In case $u$, and $V \cdot u \in L^1_{\text{loc}}(B_R)$ and (0.1) is satisfied in $D'(B_R)$ we say that $u$ is a solution of (0.1) in $B_R$ or that $u$ is nonsingular at $0$ or also that the singularity at $0$ is removable.

Actually the nonsingular solutions that appear in our main results are continuous functions in $B_R$. This is not the case for some very negative potentials (cf. Prop. 2.2).

If the above does not hold $u$ is singular at $0$. We are interested in describing singularities with constant sign. In the simplest case $V(x) = 0$, i.e. when $u$ is a harmonic function, it is well-known that every singular solution with constant sign behaves as $|x| \to 0$ like the fundamental solution $E_N(x)$ of the Laplace operator,

$$E_N(x) = k(N)|x|^{2-N} \text{ if } N > 3,$$
(1.1)

$$E_2(x) = (2\pi)^{-1/2}\log(1/|x|),$$

in the sense that there exists a constant $c \in \mathbb{R}$ such that

$$\lim_{|x| \to 0} \frac{u(x)}{E_N(x)} = c.$$  
(1.2)

Moreover $u$ satisfies the equation $-\Delta u = c\delta$ in $D'(B_R)$, where $\delta$ is Dirac's mass at the origin. If $c = 0$ the singularity is removable.
We say that a singular solution of (0.1) that satisfies (1.1) has a \textit{weak singularity} at 0. If (1.1) is true with $c = +\infty$ or $c = -\infty$ we say that $u$ has a \textit{strong singularity} at 0.

Strong singularities do not exist if $V \equiv 0$. In fact all singularities with constant sign are of weak type if $V \in L^q(B_R)$ with $q > N/2$ according to the results of Serrin [S2].

We want to describe the class of radial potentials for which every singular solution with constant sign exhibits a weak singularity. We shall prove that this is precisely the class of $P_-$ of potentials $V \in C((0, R); \mathbb{R})$ such that

(1.3) $V(|x|) E_N(x) \in L^1(B_R)$.

For radial functions $P_-$ coincides with the class $K_N$ introduced by Kato [K], see [AS, Proposition 4.10]. An important property of the class $K_N$, proved by Aizenman and Simon [AS], is the Harnack inequality, that we shall use in the proof of Theorems 1 and 3. Remark that $K_N \ni L^q(B_R)$ if and only if $q > N/2$. In particular if $V(r) = c r^\theta$ with $c \neq 0$, $\theta \in \mathbb{R}$, $V \in P_-$ if $\theta > -2$.

We obtain the following result

\textbf{THEOREM A.} Let $V \in P_-$ and let $u \in C^2(B_R)$ be a solution of (0.1) such that

(1.4) $u(x) > o(|x|^{1-N})$ or $u(x) < o(|x|^{1-N})$, $|x| \to 0$.

then there exists a constant $c$ such that

(1.5) $u(x) = cu_1(|x|) + u_2(x)$,

where $u_1(r)$ is a radial solution of (0.1) such that $u_1(r)/E_N(r) + 1$ as $r = |x| \to 0$ and $u_2 \in C(B_R)$ is a nonsingular solution of (0.1). Moreover $u$ and $Vu \in L^1_{\text{loc}}(B_R)$ and
(1.6) 
\[ -\Delta u + Vu = c_0 \quad \text{in} \quad D'(B_R). \]

The proof of Theorem 1 relies on the study of the radially symmetric solutions of (0.1) that is done in Section 2. In particular it follows from these results that \( p_1 \) can be considered as the class of "small" perturbations of the null potential. These results also allow to show that the two hypotheses made in the theorem are optimal. Consider for instance the unilateral condition (1.4). We can always construct oscillatory solutions of (0.1) that behave at 0 like \( O(|x|^{1-N}) \) by separation of variables. Thus if we set

(1.7) 
\[ u(x) = H_n(\sigma)U_n(r), \]

where \((r, \sigma)\) are spherical coordinates in \( \mathbb{R}^N \) and \( H_n(\sigma) \) is an eigenfunction of the Laplace-Beltrami operator in \( S^{N-1} = \{ x \in \mathbb{R}^N : |x| = 1 \} \) with eigenvalue \( \lambda_n = n(n+N-2) \), is a solution of (0.1) if \( U_n \) satisfies the equation

(1.8) 
\[ U''(r) + \frac{N-1}{r} U'(r) = (\frac{\lambda}{r^2} + V)U(r). \]

According to Proposition 2.3, the behaviour of \( U_n \) as \( r \to 0 \) is equivalent to the behaviour of the solutions of (0.1) with potential \( \lambda_n r^{-2} \) (again \( V \in p_1 \) is a small perturbation). But for potentials \( V = cr^{-2} \) we find the radially symmetric solutions

(1.9.a) 
\[ u_1(r) = r^{-p}, \quad p = \frac{N-2}{2} + \left( \frac{N-2}{2} \right)^2 + c \right) \frac{1}{2}, \]

(1.9.b) 
\[ u_0(r) = r^q, \quad q = -\frac{N-2}{2} + \left( \frac{N-2}{2} \right)^2 + c \right) \frac{1}{2}, \]

if \( c > c_* = -\left( \frac{N-2}{2} \right)^2 \). Observe that if \( c > 0 \) \( u_1(r) \) is a solution with a strong singularity and \( u_0(r) \) is a continuous nonsingular solution. In particular for the first eigenvalue \( \lambda_1 = N-1 \) we obtain a solution \( U_1(r) = r^{-p}, p = N-1, \) which is the oscillatory solution of the form (1.7) with slowest growth at 0.
Consider now that we have a weakly singular solution of (0.1). Since 
\( V u \in L^1_{1oc}(B_R) \) and \( u \sim E_N(x) \) near \( x = 0 \) we conclude that \( V E_N \in L^1_{1oc}(B_R) \), i.e. \( V \in _{1oc}^1 \).

Theorem 1 implies in particular the following removability result

COROLLARY 1.1. If \( u \) is a solution of (0.1) with \( V \in _{1oc}^1 \) and \( u(x) = o(E_N(x)) \) as \( x \to 0 \) then \( u \) is a continuous solution of (0.1) in \( B_R(0) \).

Removability results are somewhat different if \( V \notin _{1oc}^1 \). Assume for instance that \( V^+ = \max(V, 0) \notin _{1oc}^1 \) and \( V^- \in _{1oc}^1 \) and let \( u_0 \) and \( u_1 \) be two radially symmetric solutions of (0.1) such that \( u_0(r) \to 0 \) and \( u_1(r)/E_N(r) \to \infty \) as \( r \to 0 \). Such solutions are constructed in Section 2. Then we have

THEOREM 1. Let \( u \in C^2(B_R^*) \) be a solution of (0.1) with \( V^+ \notin _{1oc}^1 \), \( V^- \in _{1oc}^1 \).

The following conditions are equivalent

i) \( u(x) = o(u_1(r)), r = |x| \to 0 \),

ii) \( V u \in L^1_{1oc}(B_R) \),

iii) \( u(x) = o(u_0(r)), r \to 0 \),

iv) \( u \) is a continuous solution of (0.1) in \( B_R \).

In contrast to these conditions every weak singularity satisfies \( V u \in L^1_{1oc}(B_R) \) if \( V \in _{1oc}^1 \). Moreover by the Harnack inequality \( u(0) > 0 \) for an nonnegative solution of (0.1) while \( u(0) = 0 \) characterizes the nonsingular solutions in Theorem B. Theorems A and B will be proved in Section 3 after the basic properties of the radial solutions are investigated in Section 2.
2. SOME RESULTS FOR RADIAL SOLUTIONS

2.1 This section is devoted to study behaviour near $r = 0$ of the solutions $u(r)$ of the ODE

\[(2.1) \quad u''(r) + \frac{N-1}{r} u'(r) = V(r) u(r), \quad 0 < r < R.\]

We are especially interested in the consequences of condition (1.3), i.e.

\[(2.2.a) \quad \int_0^R r V(r) dr < \infty \text{ if } N > 3,\]

\[(2.2.b) \quad \int_0^R r |\log(r)| V(r) dr < \infty \text{ if } N = 2.\]

By means of the change of variables

\[(2.3) \quad u(r) = V(s)/s, \quad s = r^{N-2}\]

we transform the equation (2.1) if $N > 3$ into the (N=1)-version

\[(2.4) \quad \ddot{v}(s) = X(s) v(s), \quad 0 < s < S = R^{N-2},\]

where dot means $d/ds$ and the new potential $X(s)$ is defined by

\[(2.5) \quad X(s) = (N-2)^{-2} \frac{V(r)(r/s)^2}{s^{N-2}}.\]

Note that for $N=3$ we have $s = r, X = V$. For $N=2$ the appropriate change of variables is

\[(2.3') \quad u(r) = v(s)/s, \quad s = (\log(1/r))^{-1}\]

and then (2.4) holds with

\[(2.5') \quad X(s) = V(r)r^2 s^{-4}.\]

In every case condition (2.2) becomes
\[ (2.6) \quad \int_0^s sX(s) ds < \infty, \]

i.e. \( V \in \mathcal{P}_1 \) if and only if \( X \in \mathcal{P}_1 \).

It is known, cf. [N, Corollaries 2.3 and 3.3] that whenever \( X \in \mathcal{P}_1 \) there exist two solutions, \( v_0(s) \) and \( v_1(s) \), of (2.4) such that as \( s \to 0 \) we have

\[ (2.7.a) \quad v_0(s) = s + o(s), \quad v_0(s) = 1 + o(1), \]
\[ (2.7.b) \quad v_1(s) = 1 + o(1), \quad v_1(s) = o(1/s). \]

It follows that for every \( V \in \mathcal{P}_1 \) there exist two solutions \( u_0(r) \) and \( u_1(r) \), of (2.1) such that

\[ (2.8.a) \quad \lim_{r \to 0} u_0(r) = 1, \quad \lim_{r \to 0} r u_0'(r) = 0, \]
\[ (2.8.b) \quad \lim_{r \to 0} \frac{u_1(r)}{E_N(r)} = s, \quad \lim_{r \to 0} r^{N-1} u_1'(r) = -(n-2)k(N). \]

The fact that the condition \( V \in \mathcal{P}_1 \) is essential in this result is shown by the following converse result

**PROPOSITION 2.1** Let \( V \) be a nonnegative potential that does not belong to \( \mathcal{P}_1 \). Then there exist two positive solutions of (2.1), \( u_0 \) and \( u_1 \), such that

\[ (2.9.a) \quad \lim_{r \to 0} u_0(r) = \lim_{r \to 0} r u_0'(r) = 0 \]
\[ (2.9.b) \quad \lim_{r \to 0} \frac{u_1(r)}{E_N(r)} = \lim_{r \to 0} (-r^{N-1} u_0'(r)) = \infty. \]

Moreover \( u_0(r) u_1(r) < E_N(r) \) and

\[ (2.10) \quad V(r) u_0(r) E_N(r) \in L^1(B_R) \]

**REMARKS.** 1) \( u_0 \) is a regular solution of (0.1) at \( x = 0 \).

2) (2.10) is not true for \( u = u_1 \). On the contrary if \( V \in \mathcal{P}_1 \) it holds for all solutions of (2.1).
Proof of the Proposition. Using the transformation (2.3) we are reduced to construct positive solutions of (2.4) with \( X > 0, X \in \mathcal{P}_1 \), such that as \( s \to 0 \) \( v_0(s)/s \to 0 \) and \( v_1(s) \to \infty \).

Let us begin with \( v_1(s) \): we take the solution of (2.4) with initial condition

\[
v_1(s) > 0, \quad \dot{v}_1(s) = -1
\]

with \( S = \mathbb{R}^{N-2} \). By (1.4) \( v \) is convex, namely \( \dot{v} \) is an increasing function in \((0,S)\), hence

\[
\dot{v}_1(s) < -1 \quad \text{in } 0 < s < S.
\]

We conclude that the function \( v_1 \) is positive and decreasing in \((0,S)\) and that there exists the limit \( \lim_{s \to 0} v_1(s) = p, \quad 0 < p < \infty \). We have to prove that \( p = \infty \).

We take a number \( c \in (0,S) \). Since \( v_1 \) is strictly decreasing we have \( v(s) > v(c) > 0 \) for \( 0 < s < c \). Integrating twice (2.4) between \( s \) and \( c \) with \( 0 < s < c/2 \) we have

\[
(2.11) \quad \dot{v}_1(c) (c-s) - v_1(c) + v_1(s) > v_1(c) I(s),
\]

where

\[
I(s) = \int \frac{c}{y} \ dt X(t) = \int \frac{c}{s-t} \ X(t) \ dt > \int \frac{c}{2s} \ X(t) \ dt.
\]

Since \( X \notin \mathcal{P}_1 \) it follows that \( I(s) \to \infty \) if we let \( s \to 0 \). Going back to (2.11) we have \( v_1(s) \to \infty \) as \( s \to 0 \). Since

\[
v_1(c) - v_1(s) = \int_s^c X(t) v_1(t) \ dt > \frac{v_1(c)}{c} \int_s^c X(t) \ dt.
\]
we have \( \dot{v}_1(s) + \infty \) as \( s + \infty \).

The solution \( v_0 \) is obtained by the method of variation of constants:

\[
(2.12) \quad v_0(s) = v_1(s) \int_0^s \frac{dt}{v_1^2(t)}.
\]

Since \( v_1 \) is decreasing in \((0,S)\) we have

\[
(2.13) \quad v_0(s)v_1(s) < s,
\]

i.e. \( u_0(r)u_1(r) < E_N(r) \). (2.10) is equivalent to prove that the integral \( \int x(t)v_0(t)dt \) is convergent at 0 and this follows from

\[
(2.14) \quad \dot{v}_0(s) - \dot{v}_0(s) = \int_s^S x(t)v_0(t)dt,
\]

letting \( s + 0 \). In fact since \( v_0 \) is convex and \( v_0(s)/s + 0 \) we have \( \dot{v}_0(s) + 0 \) as \( s + 0 \).

The conclusions for \( u_0' \) and \( u_1' \) in (2.9) follow now from the formulas

\[
(2.15a) \quad ru'(r) = (N-2)[\dot{v}(s) - \frac{v(s)}{s}] \text{ if } N > 3, \\
(2.15.b) \quad ru'(r) = \dot{v}(s)s - v(s) \text{ if } N=2.
\]

The situation is different for negative potentials:

**Proposition 2.2.** Let \( V \) a nonpositive potential that does not belong to \( P_1 \). Then either all the solutions to (2.1) oscillate infinitely many times near \( r = 0 \) or they satisfy

\[
(2.16.a) \quad \lim_{r \to 0} u(r) = +\infty, \\
(2.16.b) \quad \lim_{r \to 0} \frac{u(r)}{r^N} = 0, \\
\text{and} \\
(2.16.c) \quad \lim_{r \to 0} r^{N-1} u'(r) = 0.
\]
Proof. Using the change of variables (1.3), (1.5) (or (1.3'), (1.5') if N=2) we can consider solutions \( v(s) \) of (2.4) in \( 0 < s < S \) with \( X < 0 \), \( X \notin P_1 \). It is standard that either all the solutions oscillate an infinite number of times around the 0-level as \( s \to 0 \) or every solution is either positive or negative in a small neighbourhood of 0. Let us assume to be specific that \( v \) is a positive solution of \( \ddot{v} = X(s)v \) for \( 0 < s < S \). Since \( X < 0 \), \( v \) is a concave function and there exists the limit

\[
p = \lim_{s \to 0} v(s).
\]

We have \( 0 < p < 0 \). We want to prove that \( p = 0 \), i.e. (2.16.a). In fact if \( p > 0 \) we integrate the equation as in Proposition 2.1 to obtain for \( 0 < s < c < S \)

\[
v(s) = v(c) + \dot{v}(c)(s-c) + \int_s^c \dot{v}(y) X(t)v(t)dt.
\]

For \( c \) small enough we have \( v(t) > p/2 \) if \( 0 < s < c \). Hence

\[
v(s) > v(c) + \dot{v}(c)(s-c) + \frac{p}{2} \int_s^c \dot{v}(y) X(t)v(t)dt
\]

As in Proposition 2.1 the last integral is infinite, which leads to contradiction. Therefore \( p = 0 \) and \( v \) is increasing in a certain neighbourhood of \( s=0 \).

By concavity there also exists

\[
\dot{v}(0) = \lim_{s \to 0} \dot{v}(s)
\]

and \( 0 < \dot{v}(0) < \infty \). It is then clear that \( v(s) > a.s \) in a neighbourhood of \( 0 \) for a certain \( a>0 \). Since for \( 0 < s < c < S \) we have

\[
\dot{v}(s) = \dot{v}(c) + \int_s^c (-X)(t)v(t)dt
\]
we easily conclude from $X \notin P_1$ that $v(0) = \infty$ so that
\[
\lim_{s \to 0} \frac{v(s)}{s} = \infty,
\]
i.e. (2.16.a). Finally from (2.15) and the fact that $0 < v(s) - \dot{v}(s)s < v(s)$ we get (2.16.c).

Proposition 2.2 implies that for negative potentials that do not belong to $P_1$ the radial solutions of (0.1) belong to one of those two types: i) all of them oscillate infinitely many times as $r \to 0$, ii) all of them are nonsingular (by (2.15.c)) and discontinuous (by 2.15.a). For potentials of the form
\[
V(r) = cr^{-2}, \; 0 < c
\]
the first situation occurs if $c < c_* = -((N-2)/2)^2$ while the second one happens if $0 > c > c_*$, see formulas (1.9).

Our next result shows how a perturbation of the potential affects the nature of the singularity at $0$ of a radially symmetric solution of (0.1).

PROPOSITION 2.3. Let $V = \hat{V} + \nabla$ be a potential in $(0,R)$ with $\hat{V} > 0$ and let $\hat{u}_0, \hat{u}_1$ be the solutions of $\hat{u}''(r) + ((N-1)/r)\hat{u}'(r) = \hat{V}(r)\hat{u}(r)$ constructed in Proposition 2.1. Then i) if $V$ satisfies
\[
\int_0^R |\nabla(V(r))\hat{u}_0(r)\hat{u}_1(r)r^{N-1} dr < \infty
\]
for $i = 0,1$ there exist solutions $u_0, u_1$ of (2.1) such that
\[
\lim_{r \to 0} \frac{u_i(r)}{\hat{u}_i(r)} = 1, \; i = 0,1.
\]
ii) Assume now that

\[ (2.19.a) \quad \int V^+(r) \hat{u}_0(r) \hat{u}_1(r) r^{N-1} dr = - \infty, \]

\[ (2.19.b) \quad \int V^{-}(r) \hat{u}_0(r) \hat{u}_1(r) r^{N-1} dr < - \infty. \]

Then there exist solutions \( u_0, u_1 \) of (2.1) such that

\[ (2.20) \quad \lim_{r \to 0} \frac{u_0(r)}{\hat{u}_0(r)} = 0, \lim_{r \to 0} \frac{u_1(r)}{\hat{u}_1(r)} = - \infty. \]

**Remark 2.1.** Condition (2.17) is in particular implied by \( V \in P_1 \) by virtue of Proposition 2.1. In fact (2.17) is equivalent to \( V \in P_1 \) if \( \hat{V} \in P_2 \) (see (2.8)) or if \( \hat{V}(r) = cr^{-2} \) with \( c > 0 \) (see (1.9)). We shall see later that the equivalence holds if \( \hat{V} \in P_2 \). On the contrary if \( \hat{V}(r) = r^\theta, \theta < -2 \), there are solutions \( \hat{u}_0, \hat{u}_1 \) as in Proposition 2.1 such that

\[ \hat{u}_0(r) \hat{u}_1(r) = r^{P+2-N} \text{ if } r = 0, \]

where \( p = -\frac{1}{2}(\theta+2) > 0 \), see (0.6). Therefore (2.17) means

\[ (2.21) \quad \int_0^R |V(r)| r^{-\theta/2} dr < - \infty, \]

a class larger than \( P_1 \).

**Proof of the Proposition.** We assume that \( N > 3 \), the case \( N = 2 \) being similar (cf. Prop. 2.1).

**Part i)** We perform the change of variables \( u(r) = v(s)/s, \hat{u}(r) = y(s)/s, s = r^{N-2} \). As in Proposition 2.1 we obtain the equations

\[ \ddot{v}(s) = X(s) v(s), \ddot{y}(s) = X(s) y(s) \text{ for } 0 < s < R^{N-2}, \]

with \( X, \hat{X} \) defined as in (2.5). Let \( y_0(s), y_1(s) \), be the two solutions of
the last equation obtained in Proposition 2.1. We have to prove that there exist solutions $v_0, v_1$ of $v(s) = X(s)v(S)$ such that $v_i(s)/y_i(s) + 1$ as $s \to 0^+$, $i = 0, 1$.

This is done as follows. Consider the function

$$h(t) = \frac{v(s)}{y_1(s)} \text{ with } t = \frac{y_0(s)}{y_1(s)}$$

for $0 < s < c < R$, $c$ being sufficiently small so that $y(s)$ is positive in $(0, c)$. $h(t)$ satisfies the equation

$$\frac{d^2 h(t)}{dt^2} = \overline{X}(s(t)) y_1^4(s(t)) h(t), \ 0 < t < t_0 = t(c).$$

If the potential $W(t) = \overline{X}(s(t)) y_1^4(s(t))$ belongs to the class $\mathcal{P}_1$, there exist two solutions $h_0(t), h_1(t)$ of (2.23) such that as in (2.7) we have

$$h_0(t) = t + o(t), \ h_0'(t) = 1 + o(1)$$

$$h_1(t) = 1 + o(t), \ h_1'(t) = o(1/t)$$

as $t \to 0$. Setting for $i = 0, 1$, $v_i(s) = h_i(t)/y_1(s)$ and $u_i(s) = v_i(s)/s$ the result (2.18) for $u_0, u_1$ would follow. Therefore we have to check that $W \in \mathcal{P}_1$.

$$\int_0^t t |W(t)| dt = \int_0^c t |\overline{X}(s(t)) y_1^4(s)| dt =$$

$$= \int_0^c |X(s)| y_0(s)y_1(s) ds = \int_0^R |\overline{V}(r)| \hat{u}_0(r) \hat{u}_1(r) r^{N-1} dr < \infty.$$  

**Part ii).** Assume first that $V = 0$. Then, with the notations of part i) we have

$$\int_0^t tw(t) dt = \int_0^R \overline{V}(r) \hat{u}_0(r) \hat{u}_1(r) r^{N-1} dr = + \infty.$$
Therefore by Proposition 2.1 there exist two solutions, $h_0$ and $h_1$ of (2.23) such that as $t \to 0$ \( h_0(t)/t \to 0 \) and $h_1(t) \to \infty$. Reversing the changes of variables $y_1(s)h_i(t) = v_i(s)$ and $v_i(s) = su_i(r)$, $i = 0, 1$, we obtain (2.19). Remark also that $h_0(t)h_1(t) < 1$, i.e. $v_1(s)v_0(s) < y_0(s)y_1(s)$ or even $u_0(r)u_1(r) < \hat{u}_0(r)\hat{u}_1(r)$.

In case $\nabla \neq 0$ we proceed in two steps. First we consider the potential $V_1 = \hat{V} + \nabla^+$ and argue as above. From (2.19.b) and the remark of the last paragraph it follows that $\nabla$ satisfies condition (2.17) with respect to a pair of solutions of (2.1) with potential $V_1$. Therefore by part i) of this proof we conclude that no change of behaviour occurs upon passing from $V_1 = \hat{V} + \nabla^+$ to $V = V_1 - \nabla^-$.

We give next a simple comparison result for singular solutions corresponding to different potentials.

**Proposition 2.4** Let $V$, $\nabla$ be two potentials such that $V(r) > \nabla(r)$ for $0 < r < R$. If there is a positive singular solution $u_1$ of (2.1) with potential $V$ there is also a singular solution $\bar{u}_1$ of (2.1) with potential $\nabla$ such that $\bar{u}_1 > u_1$ near $r = 0$.

*Proof.* Take a point $c \in (0,R)$ and let $\bar{u}_1$ be the solution of (2.1) with potential $\nabla$ and initial data

$$\bar{u}_1(c) = u_1(c), \quad \bar{u}'_1(c) = u'_1(c).$$

For every $r \in (0,c)$ we have

$$d \left( r^{N-1}(\bar{u}_1(r)u_1(r) - \bar{u}_1(r)u_1'(r)) \right) = r^{N-1}(V(r) - \nabla(r))\bar{u}_1(r)u_1(r).$$

Therefore on every interval $I = (a,c) \subset (0,c)$ where $\bar{u}_1$ is positive the function $r^{N-1}(\bar{u}_1(r)u_1(r) - \bar{u}_1(r)u_1'(r))$ is nondecreasing. Since it vanishes at $r = c$ we conclude that
(2.26) \[
\frac{\overline{u}_1'(r)}{\overline{u}_1(r)} < \frac{u_1'(r)}{u_1(r)}
\]

in \((a,c)\). We conclude that \(a = 0\) and \(\overline{u}_1(r)/u_1(r)\) is nonincreasing in \((0,c)\).

Since \(\overline{u}_1(c) = u_1(c)\) we get \(\overline{u}_1(r) > u_1(r)\) for \(0 < r < c\).

In the study of the asymptotic behaviour in Chapter 6 it will be interesting to know how much the solutions corresponding to a potential \(V = P_1\) depart from the ones for \(V = 0\). The next result settles this question as \(r \to 0\).

**Proposition 2.5.** Let \(V = P_1\) and let \(u_0, u_1\) be as above. Then as \(r \to 0\)

(2.27) \[
u_0(r) = 1 + p_0(r)(1 + o(1))
\]

(2.28) \[
u_1(r) = E_N(r) + p_1(r)(1 + o(1)) + C,
\]

where

(2.29) \[
p_0(r) = \int_0^r s^{1-N} \int_0^s t^{N-1} V(t) dt \, ds = o(1)
\]

and

\[
p_1(r) = \begin{cases} k(N) \int_0^r s^{1-N} \int_0^s t V(t) dt \, ds & \text{if } N > 3, \\ \frac{1}{2\pi} \int_0^r s^{-1} \int_0^s t |\log t| V(t) dt \, ds & \text{if } N=2, \end{cases}
\]

hence \(p_1(r) = o(E_N(r))\).

**Proof.** We know that \(u_0(r) = 1 + o(1)\). Hence integrating (2.1) we get

\[
u_0'(r) = (r^{1-N} \int_0^r t^{N-1} V(t) dt) (1+o(1)),
\]

so that

\[
u_0(r) = (\int_0^r s^{1-N} \int_0^s t^{N-1} V(t) dt) (1+o(1)),
\]
i.e. (2.27), \( p_0(r) \) being well-defined by (2.29) since \( V \in \mathcal{P}_1 \).

Let us now prove (2.28) when \( N > 3 \). As before, if \( E_N(r) = k(N)r^{2-N} \), we get for a constant \( C \):

\[
   u_1'(r) = Cr^{1-N} + (kr^{1-N} \int_0^r tV(t)dt)(1+o(1)).
\]

Hence for another constant \( C \)

\[
   u_1(r) = k(N)r^{2-N} - (k(N) \int_0^r tV(t)dt) s^{1-N}(1+o(1)) + C,
\]

since we know that \( u_1(r) = E_N(r) \). Finally \( p_1(r) = o(E_N(r)) \) follows easily from the fact that \( V \in \mathcal{P}_1 \).

The case \( N=2 \) is similar.

\( \square \)

**COROLLARY 2.6.** If \( V(r) = O(r^\theta) \) for some \( \theta > -2 \) we have as \( r \to 0 \)

\[
   (2.31) \quad u_0(r) = 1 + O(r^{\theta+2}),
\]

\[
   (2.32) \quad u_1(r) = E_N(r)(1 + O(r^{\theta+2})) + c,
\]

for some \( c \in \mathbb{R} \).

2.2. Using the Kelvin transformation

\[
   w(s) = s^{2-N}u_1(1/s), \quad W(s) = s^{-4}V(1/s),
\]

We can easily apply the perturbation result, Proposition 2.3, to the behaviour at infinity of the solutions of

\[
   w'' + \frac{N-1}{s} w' = W(s)w \quad \text{in} \quad S < s < \infty.
\]

In particular condition (2.17) transforms into

\[
   (2.17') \quad \int_S^\infty |W(s)|w_0(s)w_1(s)s^{N-1} ds < \infty.
\]
with obvious notation. Many perturbation results can be found for the case \( N=1, W=1 \) in [B].

As an application we obtain the behaviour of the solutions of (2.1) as \( r \to 0 \) when \( V(r) = r^\theta, \theta < -2 \). The change of variables

\[
(2.33) \quad u(r) = r^k w(s), \quad s = \frac{1}{p} r^{-p}
\]

with \( p = -\frac{\theta+2}{2} \) and \( k = \frac{D+2-N}{2} \) transforms (2.1) into

\[
(2.34) \quad \ddot{w}(s) = (1+\bar{w}(s))w(s), \quad S < s < \infty
\]

with \( \bar{w}(S) = Cs^{-2} \), \( C = C(N, \theta) > 0 \). Since two solutions of \( \ddot{w}(s) = w(s) \) are \( e^S \) and \( e^{-S} \) and (2.17') is satisfied, Prop. 2.3 at infinity implies that there exist two solutions of (2.34), \( w_0 \) and \( w_1 \), such that as \( s \to \infty \)

\[
(2.35) \quad w_0(s) = e^{-S}, \quad w_1(s) = e^{S},
\]

(cf. [B], pg. 125). Therefore we obtain two solutions of (2.1), \( u_0, u_1 \), which behave as \( r \to 0 \) like (0.6).

The estimates (0.6) are yet obtained if \( V = r^\theta + V \) and \( \bar{V} \) satisfies (2.21). This is the case for instance if

\[
(2.36) \quad V = r^\theta + O(r^\mu), \quad \mu > -1 + \theta/2.
\]

REMARK. An explicit formula for the solutions \( u_0, u_1 \) when \( V(r) = r^\theta, \theta < -2 \), can be found for instance in [GL].
3. PROOF OF THEOREMS A AND B

In the sequel $u_0$ and $u_1$ are the radial solutions of (2.1) constructed in the previous section and $U_1(r)$ is a positive singular solution of (1.8) with $\lambda = N-1$, so that $U_1(r)$ grows like $r^{1-N}$ as $r \to 0$ if $V \geq 0$. Given a solution $u(x)$ of (0.1) defined in $B^*_R$ we denote by $\overline{u}(r)$ the angular mean of $u$ as defined in (0.4).

In order to prove Theorem A we need to control the nonradial part of $u$, $u(x) - \overline{u}(r)$. This is done in the next lemma that has an independent interest and exploits ideas introduced by Véron [Ve] and then used by Véron and one of the authors [VVJ] in dealing with semilinear elliptic equations.

**LEMMA 3.1** Let $u \in C^2(B^*)$ be a solution of (0.1) such that $u(x) = o(U_1(r))$ as $r = |x| \to 0$. Then

\[ u(x) - \overline{u}(r) = o(u_0(r)). \]

**Proof:** 1) The first step consists in obtaining a bound for $u(r,\sigma) - \overline{u}(r)$ in $L^2(S_{N-1})$. Let

\[ g(r) = (\int_{S_{N-1}} (u(r,\sigma) - \overline{u}(r))^2 d\sigma)^{1/2}. \]

Let $A = \{r \in (0, R_0): g(r) > 0\}$, where $R_0 \in (0, R)$ is so small as to have $U_1(r) > 0$ in $(0, R_0]$. On every connected component of $A$, say $(\alpha, \beta)$, we have (see [Ve] or [VVJ])

\[ g''(r) + \frac{N-1}{r} g'(r) > \left(\frac{N-1}{r^2} + V(r)\right)g(r). \]

Consider now the function

\[ h(s) = g(r)/U_1(r), \]
where \( s = U_0(r)/U_1(r), U_0(r) = U_1(r) \int_0^r \frac{1}{t^{1-N} u_1^2(t)} dt. \) \( s = s(r) \) in an increasing bijection from \((0,R_0)\) onto \((0,s(R_0))\). (3.4) implies that

\[
\dot{h}(s) > 0 \quad \text{for} \quad s(\alpha) < s < s(\beta) \quad (\cdot = \frac{d}{ds}),
\]

i.e. \( h \) is convex, therefore either \( s(\alpha) = 0 \) and \( \alpha = 0 \) or \( s(\beta) = s(R_0) \), i.e. \( \beta = R_0 \). We also now that \( h(s) = o(1) \) as \( s \to 0 \) by the assumption on \( u \).

Therefore if \( A \ni (0,\mu) \) then necessarily \( \mu = R_0 \). Summing up, either \( A = (0,R_0) \) or \( A = (\alpha,R_0) \) with \( 0 < \alpha < R_0 \). In this last case we have \( g(r) = 0 \) and \( u = \bar{u} \) for \( 0 < r < \alpha \). In the other case \( A = (0,R_0) \), \( h(0) = 0 \) and \( h \) is convex in \((0,s(R_0))\), therefore \( h(s) < Cs \) for a certain \( C > 0 \). This means that

\[
g(r) < Cu_0(r), \quad 0 < r < R_0
\]

2) We next obtain a bound in \( L^\infty(S_{n-1}) \): Let \( C > 0 \) be a constant such that

\[
|u(x) - u(x)| < Cu_0(R_0) \quad \text{for} \quad |x| = R_0.
\]

Let now

\[
\dot{w}(x) = ((u(x) - \bar{u}(r)) - Cu_0(r))^+.
\]

Clearly \( w \in H^1(B^*_R) \). By Kato's inequality [K] \( w \) satisfies

\[
\Delta w > Vw \quad \text{in} \quad D'_R(B^*),
\]

hence its angular mean \( \overline{w} \) satisfies

\[
\overline{w}'' + \frac{N-1}{r} \overline{w}' > V\overline{w} \quad \text{in} \quad D'(0,R_0).
\]

As above if we set

\[
h_1(s) = \overline{w}(r)/u_1(r), \quad s(r) = u_0(r)/u_1(r).
\]
then \( h_1 \) turns out to be convex and there exists the limit

\[
\lim_{s \to 0} h_1(s) = \lim_{r \to +} \frac{w(r)}{u_1(r)} = a \quad [0, \infty].
\]

Since \( w(r) = 0 \) \((u_0(r) + U_0(r)) \) (by (3.7), (3.9)) and \( U_0(r) < u_0(r) \) for small we conclude that \( a = 0 \), i.e. \( h_1(0) = 0 \). Also \( h_1(R_0) = 0 \), therefore \( h_1 \equiv 0 \) and \( w \equiv 0 \) in \((0, R_0)\). \( w \) being nonnegative, it follows that \( w \equiv 0 \), i.e.

\[
u(x) < w(r) + C u_0(r)
\]

Since the same argument can be applied to \(-u\) (3.2) holds. \(\square\)

**Proof of Theorem A.** By Propositions 2.1 and 2.3 we may assume that \(-u \equiv 0\) by studying \(u - \bar{u}\) instead of \(u\).

We first transform the unilateral bound (1.4) into an absolute bound. Let \(x_0 \in B_{r/2}\) and let \(\rho_0 = |x_0|/2\). We apply Kato's inequality to \(u\) in \(B_{\rho_0}(x_0) \subset B_{r}\):

\[
\Delta |u| > u \cdot \text{sign}(u) = V|u| > -V^{-}(r)|u(x)|.
\]

Therefore \(|u|\) is a positive subsolution of the equation \(-\Delta z - V^{-}(r)z = 0\). Aizenman and Simon have proved, [AS], that the Harnack inequality

\[
u(x_0) < C \int_{B_{\rho_0}(x_0)} |u(x)| dx < \delta
\]

holds provided that the potential \(V^{-}\) belongs to the Kato class \(K_N\), hence in our case \(V^{-} \in P_1\). \(C\) and \(\delta\) are positive constants depending on the norm of \(V^{-}\) in \(K_N\), cf. [AS, Theorems 3.8 and 6.1]. A careful scrutiny of their proof shows in our case we can take

\[
C = k \delta^{-N}, \quad 0 < \delta < \delta_0
\]

with \(k, \delta_0\) positive constants not depending on \(x_0\).
Assuming now that \( u^+(x) = o(r^{1-N}) \) and using (3.11), (3.12), we have for all \( |x_0| \) small \( (\rho_0 = |x_0|/2) \)

\[
|u(x_0)| \leq k\rho_0^{-N} \int_{\rho_0(x_0)} |u(x)| dx \leq |S_{N-1}| k\rho_0^{-N} \int \rho_0^{-N-1} u(r)^{N-1} dr
\]

\[
= 2|S_{N-1}| k\rho_0^{-N} \int \rho_0^{-N} u^+(r)r^{N-1} dr = o(|x_0|^{1-N}).
\]

Since \( U_1(r) = r^{1-N} \), Lemma 3.1 can be applied. Thus we get

\[
u(x) = u(x) - \overline{u}(x) = o(u_0(r)),
\]

therefore \( u \) is bounded. It follows that \( Vu \in L^1(B_R) \) and that (0.1) is satisfied in \( D'(B_R) \). By Theorem 1.5 of [AS] we conclude that \( u \) is continuous.

Finally proving (1.6) is equivalent to prove that \( \Delta u_1 + u_1 = 0 \) in \( D'(B_R) \) and this follows by standard arguments from the properties of \( u_1 \).

We end this section with the proof of Theorem B:

i) \( \Rightarrow \) iii) This is a consequence of Lemma 3.1 and the observation that

\[
u_1(r) = O(U_1(r)) \]

that follows easily from the equations satisfied by both of them.

iii) \( \Rightarrow \) i) and iii) \( \Rightarrow \) ii) See Proposition 2.1, 2.3.

iv) \( \Rightarrow \) ii) By definition.

iii) \( \Rightarrow \) iv) This follows from a calculus lemma: "if \( u \in L^{N/N-2}(B_R) \) (resp. \( u \in L^\infty(B_R) \) and \( u \to 0 \) as \( |x| \to 0 \) if \( N=2 \), \( f \in L^1(B_R) \) and \( \Delta u = f \) in \( D'(B_R) \), then \( \Delta u = f \) in \( D'(B_R) \)" (see for instance [BV]).

ii) \( \Rightarrow \) iii) This follows from
LEMMA 3.2. Let \( u \in C^2(B_R^c) \) be a solution of (0.1) with \( V^+ \notin P_1 \) and suppose that \( Vu \in L^1_{\text{loc}}(B_R) \). Then

\[
(3.17) \quad u(x) = O(u_0(r)) \quad \text{as} \quad r \to 0.
\]

The proof of Lemma 3.2. is parallel to Step 2 of Lemma 3.1, putting \( w(x) = (u(x) - cu_0(r))^+ \) and using \( V^+ \notin (P_1) \), \( Vu \in L^1_{\text{loc}}(B_R) \) and \( u_1(r)/E_N(r) \to \infty \) instead of (3.7). \( \square \)
4. ISOTROPIC SOLUTIONS

In this section we consider a class of potentials $p_2$ for which it can be proved that a singular solution with constant sign becomes radially symmetric as $x \rightarrow 0$. This class is defined as follows: a potential $V \in C((0,R) : \mathbb{R})$ belongs to $p_2$ if $V^- \in p_1$ and there exists a singular radially symmetric solution of (0.1) that satisfies the following power-growth condition: there is a constant $K > 0$ such that

$$(4.1) \quad r|u_1'(r)| < Ku_1(r)$$

if $r$ is small enough. It follows from (2.8) that every potential in $p_1$ belongs to $p_2$. Moreover $u_1$ satisfies in that case

$$(4.2) \quad \lim_{r \to 0} \frac{ru_1'(r)}{u_1(r)} = 2-N.$$ 

A second important example of potentials in $p_2$ consists in those $V$'s of the form $V(r) = cr^{-2}$ with $c > 0$. According to (1.9) we have

$$(4.3) \quad \lim_{r \to 0} \frac{ru_1'(r)}{u_1(r)} = -p \in (2-N, -\infty).$$

The condition (4.1) implies that $u_1(r)$ grows at most like $r^K$ as $r \to 0$. Therefore the potentials $V(r) = cr^\theta$ with $c > 0$, $\theta < -2$, do not belong to $p_2$, since then the singular solution $u_1(r)$ grows exponentially, see (0.6).

Using the results of Section 2, if $V^- \in p_1$ we can obtain a regular radial solution of (0.1) that is positive for small $r$ by means of the formula

$$(4.4) \quad u_0(r) = u_1(r) \int_0^r \frac{dt}{t^{n-2}u_1^2(t)}$$

and $u_0(r) \to 0$ as $r \to 0$ if and only if $V \not\in p_1$. Let $U_1$ be a positive singu-
lar solution of (1.8) with $\lambda_n = N - 1$. By Proposition 2.4 we may assume that $U_1 > u_1$ near 0 and $U_1(r)$ grows at least like $r^{1-N}$ as $r \to 0$. We shall see later that $U_1(r)/u_1(r) \to \infty$ as $r \to \infty$.

We can state now our isotropy result.

**THEOREM C.** Let $V \in P_2$ and let $u \in C^2(B_R)$ be a solution of (0.1) such that

$$u(x) > o(U_1(r)) \text{ or } u(x) < o(U_1(r))$$

as $|x| = r \to 0$. Then we have

$$u(x) = \overline{u}(r) + g(x),$$

where $g$ is a nonsingular solution of (0.1) in $B_R$. Moreover $g \in C(B_R)$ and $g(x) = o(u_0(r))$ as $r \to 0$.

Since $u(r) = c_1 u_1(r) + c_0 u_0(r)$ for some constants $c_0, c_1 \in H_R$, the nonsingular part of $u$, $c_0 u_0(r) + g(x)$ is $O(u_0)$. $u$ is singular if $c_1 \neq 0$ and in this case $u = c_1 u_1$ has a singularity at 0 of weak type if $V \in P_1$, of strong type otherwise. Remark that if $V \notin P_1$ any nonsingular solution vanishes at $x = 0$.

Before giving the proof of Theorem C we pursue the study of the class $P_2$. We first derive a comparison result.

**PROPOSITION 4.1.** Let $V$ and $\hat{V}$ be potentials such that $V^- \in P_1$, $\hat{V} \in P_2$ and $V < \hat{V}$. Then $V \in P_2$.

**Proof.** Let $V = -V^-$ and let $\overline{u_1}(r)$ be a singular positive solution of (2.1) with potential $\overline{V}$. Choose a point $R_0 > 0$ such that $\overline{u_1}(R_0) < 0$ and $\overline{u_1}(r) > 0$ for $0 < r < R_0$. Consider the solution $u_1$ of (2.1) with potential $V$ and initial values
\[ u_1(R_0) = \overline{u}_1(R_0), \quad u_1'(R_0) = \overline{u}_1'(R_0). \]

We use the comparison technique of Proposition 2.4 to compare \( u_1 \) and \( \overline{u}_1 \). In fact it follows from it that if

\[ G(r) = r^{N-1} \{ u_1'(r) \overline{u}_1(r) - u_1(r)\overline{u}_1'(r) \} \]

we have \( G(R_0) = 0, G'(r) > 0 \) for \( 0 < r < R \), \( u_1(r) > \overline{u}_1(r) > 0 \) in \((0,R)\) and

\[ (4.7) \quad - \frac{ru_1'(r)}{u_1(r)} > - \frac{ru_1'(r)}{\overline{u}_1(r)}. \]

By the same argument, applied now to \( V \) and \( \hat{V} \) instead of \( \overline{V} \) of \( V \), and have

\[ (4.8) \quad - \frac{ru_1'(r)}{u(r)} < - \frac{ru_1'(r)}{\hat{u}_1(r)}, \]

where \( \hat{u}_1 \) is a singular positive solution of (0.1) with potential \( \hat{V} \) and \( u = c_1u_1 + c_0u_0 \) is a singular solution of (0.1) with potential \( V \) that satisfies suitable initial conditions. Since \( c_1 > 0 \) and \( u_0(r), ru_0'(r) = o(u_1(r)) \) as \( r \to 0 \), we have

\[ (4.9) \quad \frac{ru_1'(r)}{u(r)} = \frac{ru_1'(r)}{u_1(r)} (1 + o(1)). \]

It follows from (4.7) - (4.9) that \( u_1 \) satisfies (4.1).

In fact it follows from the above comparisons that for every potential \( V \) such that \( V^- \in P_1 \) we have

\[ (4.10) \quad \lim_{r \to 0} \inf \frac{ru_1'(r)}{u_1(r)} < 2-N. \]
Also the limit

\[ (4.11) \quad K_1 = \limsup_{r \to 0} \left( - \frac{ru_1'(r)}{u_1(r)} \right) \]

is independent of the singular solution \( u_1(r) \) and depends only on \( V \), \( K_1 = K_1(V) \). We prove next that the class \( P_2 \) is closed under addition

**Proposition 4.2.** \( P_2 + P_2 \subseteq P_2 \).

**Proof.** Let \( V = \hat{V} + \overline{V} \) with \( \hat{V}, \overline{V} \in P_2 \). Choose singular solutions \( \hat{u}_1, \overline{u}_1 \) of \( (2.1) \) with potential \( \hat{V}, \overline{V} \) resp. as in the definition of \( P_2 \) and set 
\( u(r) = \hat{u}_1(r)\overline{u}_1(r) \). We have

\[ (4.12) \quad u''(r) + \frac{N-1}{r} u'(r) = (\hat{V}(r) + \overline{V}(r))u(r) + 2\hat{u}'(r)\overline{u}'(r). \]

Since for \( N \geq 3 \), \( \hat{u}'_1(r) < 0 \) and \( \overline{u}'_1(r) < 0 \) for \( r \) small enough we see that \( u \) is a solution of \( (2.1) \) with potential

\[ (4.13) \quad V_1(r) = \hat{V}(r) + \overline{V}(r) + \frac{2\hat{u}'_1(r)\overline{u}'_1(r)}{\hat{u}_1(r)\overline{u}_1(r)} > \hat{V}(r) + \overline{V}(r) \]

Moreover since

\[ \frac{ru'(r)}{u(r)} = \frac{ru'(r)}{\hat{u}_1(r)} + \frac{ru'(r)}{\overline{u}_1(r)}, \]

\( V_1 \in P_2 \) and \( K_1(V_1) < K_1(V_1) + K_1(V_2) \). By Proposition 4.1 we have \( V \in P_2 \) and \( K_1(V) < K_1(V_1) + K_2(V_2) \).

In case \( N=2 \) we do not necessarily have \( \hat{u}'_1(r), \overline{u}'_1(r) < 0 \) but since

\[ \liminf_{r \to 0} \frac{ru_1'(r)}{\hat{u}_1(r)}, \liminf_{r \to 0} \frac{ru_1'(r)}{\overline{u}_1(r)} > 0 \]

we can make a similar argument with \( u(r) = u_1(r)u_2(r)r^{-p} \) for any \( p > 0 \). \( \square \)
The above comparison methods allow us in particular to estimate the ratio $U_1(r)/u_1(r)$ for the functions $U_1, u_1$ appearing in Theorem C. We obtain first a general result.

**PROPOSITION 4.3.** Let $V \in P_2, c > 0$ and $\hat{V} = V + cr^{-2}$. For every $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$p_1 - \varepsilon < \frac{ru_1'(r)}{u_1(r)} - \frac{ru_1'(r)}{u_1(r)} < p_2 + \varepsilon$$

where $p_2 = -\frac{(N-2)}{2} + \left(\frac{N-2}{2}\right)^2 + c \right)^{1/2}$ and $p_1$ is the positive solution of

$$p^2 + p(2K_1(V) - N + 2) = c$$

**Proof.** For $p > 0$ the function $Z(r) = u_1(r)r^{-p}$ is a solution of the Schrödinger equation with potential

$$V + \frac{p}{r^2} \left[ p - (N-2) + 2\left(\frac{ru_1'(r)}{u_1(r)}\right) \right]$$

If $p > p_2$, since $-ru_1'(r) > (N-2)u_1(r)$ for $r$ small, this potential is larger than $\hat{V}$ and the conclusion (4.14) - right follows by comparison (see Proposition 4.1). Since $-ru_1'(r)/u_1(r) < K_1(V) + \varepsilon$ for $r$ small the potential is smaller than $\hat{V}$ if $0 < p < p_1$.

**COROLLARY.** With the notations of Theorem C we have for every small $\varepsilon > 0$

$$0 < p_1 - \varepsilon < \log(U_1(r)/u_1(r))\log(r) < 1 + \varepsilon$$

if $0 < r < r_\varepsilon$, $p_1$ is determined from (4.15) with $c = N-1$.\]
We now derive some properties of the regular solution $u_0(r)$.

**PROPOSITION 4.4.** Let $V \in P_2$ and $u_1$ satisfy (4.1). Then there exists a nonsingular solution $u_0(r)$ given by

$$u_0(r) = u_1(r) \int_0^r \frac{dt}{t^{N-1} E_N^2(t)} ,$$

(4.17)

and the following properties hold: $u_0$ is bounded and there exists a constant $C > 0$ such that for all small $r > 0$

$$(4.18.a) \quad \frac{1}{C} r^{2-N} < u_0(r)u_1(r) < Cr^{2-N} \quad \text{if } N > 3 ,$$

$$(4.18.b) \quad \frac{1}{C} < u_0(r)u_1(r) < C|\log r| \quad \text{if } N = 2 ,$$

$$(4.19) \quad r|u_0'(r)| < Cu_0(r).$$

**Proof.** By (2.8), (2.9) and Proposition 2.3 there exists the limit

$$L = \lim_{r \to 0} \frac{u_1(r)}{E_N(r)}$$

(4.20)

and $0 < L < \infty$. Therefore if $r$ is small we have

$$\frac{u_1(t)}{E_N(t)} > \frac{u_1(r)}{2E_N(r)},$$

if $0 < t < r$. Hence

$$u_1(r)u_0(r) < 4E_N^2(r) \int_0^r \frac{dt}{t^{N-1} E_N^2(t)} = CE_N(r),$$

where $C = C(N) > 0$. This proves the right-hand side of (4.8). To prove the left-hand side we observe that since $|ru_1'(r)| < Cu_1(r)$ we have for $r/2 < t < r$
\[-33-\]

\[|\log\left(\frac{u_1(t)}{u_1(r)}\right)| = \int_t^r \frac{|u_1'(r)|}{u_1(r)} \, dr < C_1 \log(2).\]

Therefore \(u_1(t) \leq 2C_1 u_1(r)\) and

\[u_0(r) u_1(r) \geq 4^{-C_1} \int_r^{r/2} \frac{dt}{t^{N-1}} = \frac{1}{C} r^{2-N} .\]

Finally (4.19) follows from the equation

\[(4.21) \quad \frac{ru_0'(r)}{u_0(r)} - \frac{ru_1'(r)}{u_1(r)} = \frac{1}{KN-2} \frac{u_0(r)u_1(r)}{u_0(r)u_1(r)} ,\]

using (4.1) and (4.18).

\[\square\]

We end this study by proving that \(P_1\) is a class of small perturbations in \(P_2\) in a strong sense.

**PROPOSITION 4.5.** Let \(u_1, \hat{u}_1\) be radial singular solutions of (2.1) with potentials \(V, \hat{V}\) resp. and assume that \(V, \hat{V} \in P_2\) and \(V = V - \hat{V} \in P_1\). Then

\[(4.22) \quad \lim_{r \to 0} \left( \frac{ru_1'(r)}{u_1(r)} - \frac{ru_1'(r)}{\hat{u}_1(r)} \right) = 0 .\]

**Proof.** Using the notations of Proposition 2.3, if \(u_1(r) = h(t)\hat{u}_1(r)\) and \(t = u_0(r)/\hat{u}_1(r)\) we have

\[(4.23) \quad \frac{ru_1'(r)}{u_1(r)} - \frac{ru_1'(r)}{\hat{u}_1(r)} = \frac{th'(t)}{h(t)} \phi(r) ,\]

where \(\phi(r) = C(N)(r^{N-2} u_0(r)u_1(r))^{-1}\), \(C(N) = N-2\) if \(N > 3\), \(C(2) = 1\). By (4.18) \(\phi\) is bounded. On the other hand \(h(t)\) satisfies the ODE

\[h''(t) = W(t)h(t), \quad 0 < t < t_0\]

where (see (2.23)) \(V \in P_1\) implies \(W \in P_1\). Therefore, since \(h(t) = 1\) for \(t = 0\) (see Proposition 2.3), it follows from (2.7) that \(h'(t) = o(1/t)\) and the second member of (4.23) vanishes as \(r \to 0\).

\[\square\]
The class $P_{2}$ has other nice properties. As an example, it is easy to prove, arguing much as above, that for every potential $V \in P_{2}$ and constants $a, b > 0$, the potential $\tilde{V}(r) = aV(br)$ belongs again to $P_{2}$.

We return now to the Proof of Theorem C. As in Theorem A we may assume that $u = 0$ by considering $g = u - \bar{u}$ instead of $u$. Most of the rest of the proof is also similar. Thus we can prove that (4.5) implies

$$|u_{0}(x)| = o(|x_{0}|^{-N}) \int_{\mathbb{R}^{N}}^{3\rho_{0}} r^{N-1} U_{1}(r) dr$$

for $x_{0} \in B_{\mathbb{R}/2}$ with $|x_{0}|$ small enough and $\rho_{0} = \frac{1}{2} |x_{0}|$. (Here the condition $V^{+} \in P_{1}$ is used). Since by Proposition 4.2 $U_{1}$ also satisfies an inequality like (4.1) we have for some $c > 0$

$$U_{1}(\rho_{0}) > 3^{c} U_{1}(\rho) \text{ if } \rho_{0} < \rho < 3\rho_{0},$$

and

$$U_{1}(\rho_{0}) < 2^{c} U_{1}(2\rho_{0}).$$

Hence it follows from (4.24) that $u(x_{0}) = o(U_{1}(|x_{0}|))$.

Our next step consists in checking that Lemma 3.1 is yet true under the present circumstances. In this way we get $u(x) = o(u_{0}(x))$.

To prove that indeed $u(x) = o(u_{0}(r))$ as $r \to 0$ we observe that the function $h(x) = u(x)/u_{0}(r)$ satisfies in $B_{\mathbb{R}/0}^{*}$, $R_{0}$ small, the equation

$$\Delta h + 2 \frac{u_{0}(|x|)}{u_{0}(|x|)} \sum_{i=1}^{\infty} \frac{x_{i}}{|x|} \frac{\partial h}{\partial x_{i}} = 0.$$ (4.25)

The limit of $h(x)$ as $|x| \to 0$ exists by lemma 4.6 below. Since $\bar{R}(r) = \bar{u}(r)/u_{0}(r) = 0$ we conclude that this limit is $0$ and the proof is done. □
**Lemma 4.6.** Let \( h \in L_{\text{loc}}^\infty(B_R) \cap C^2(B_R) \) be a solution of the equation

\[
(4.26) \quad \Delta h + \sum_{i=1}^N c_i(x) \frac{\partial h}{\partial x_i} = 0 \quad \text{for} \quad 0 < |x| < R
\]

with \( |c_i(x)| < k|x|^{-1} \). Then there exists the limit of \( h(x) \) as \( |x| \to 0 \).

**Outline of the proof.** We may assume that \( h > 0 \). As in [Ve, Lemma 1.5], it can be proved that

\[
\max_{|x|=r} h(x) \leq C \min_{|x|=r} h(x), \quad 0 < r < R/2
\]

with \( C > 0 \) independent of \( r \), as a consequence of the Harnack inequality of [GT, pg. 189]. Then we can use the method of [GS, Theorem 2], to conclude that the limit exists. \( \square \)
5. ANISOTROPIC SINGULAR SOLUTIONS

5.1. In this section we use the method of separation of variables mentioned in
Section 1 to construct infinitely many linearly independent solutions, both
singular and regular, that are nonisotropic and yet of constant sign near
\( r = 0 \), for the class of potentials \( P_3 \) defined as follows. Let \( V \) be a non-
negative potential and let \( u(r), u_1(r) \) be a regular and a singular nonnegative
solution of

\[
(5.1) \quad u''(r) + \frac{N-1}{r} u'(r) = V(r)u(r), \quad 0 < r < R,
\]

as constructed in Proposition 2.1. Then \( V \in P_3 \) if and only if

\[
(5.2) \quad \int_0^r r^{N-3} u_0(r)u_1(r)dr < +\infty.
\]

(We could replace \( V \gg 0 \) by \( V^- \in P_1 \) and nothing would change in what follows).

Let us consider, as in (1.7), solutions of (0.1) of the form

\[
u_n(x) = H_n(\sigma)u_n(r), \text{ where } u_n \text{ satisfies}
\]

\[
(5.3) \quad u'' + \frac{N-1}{r} u' = (V(r) + \frac{\lambda_n}{r^2})u, \quad 0 < r < R.
\]

It follows from Proposition 2.3 that there exist solutions \( u_0^n(r), u_1^n(r) \) of
(4.3) that behave as \( r \to 0 \) like \( u_0(r), u_1(r) \). resp. if and only if (5.2)
holds. In that case positive, non-isotropic, singular solutions of (0.1) can be
constructed in the form

\[
(5.4) \quad u(x) = u_1(r) + \sum_{i=1}^k c_i H_i(\sigma) U_i^1(r)
\]

provided that \( \sum |c_i|H_i \|_{\infty} < 1 \). Replacing the index 1 by 0 in (5.4) we
obtain positive, non-isotropic, regular solutions of (0.1).

5.2 The condition (5.2) is not directly verifiable on \( V \). For that reason the
introduce a set of conditions on \( V \) under which we shall prove that \( V \in P_3 \).
These conditions are
\{(C_3)\}

\[\begin{align*}
\text{i) } V \text{ is a } C^1 \text{ positive function in } (0,R), \\
\text{ii) } \lim_{r \to 0} \frac{d}{dr} (V^{-1/2}(r)) = 0 \quad \text{and} \\
\text{iii) } r^{-2} V^{-1/2} \in L^1(0,R) \text{ if } N > 3, \quad \text{and } (r | \log r |)^{-2} V^{-1/2} \in L^1(0,R) \text{ if } N = 2.
\end{align*}\]

It is clear that a potential of form \( V(r) = c r^\theta \) satisfies \((C_3)\) if and only if \( c > 0 \) and \( \theta < -2. \)

We prove first an analogon of Proposition 4.4.

**Proposition 5.1.** Let \( V \) be a potential that satisfies \((C_3)\) and let \( u_0(r), \) \( u_1(r) \) be solutions of (2.1) as in Section 2. Then \( V(r) r^2 \to \infty \) as \( r \to 0 \) and for \( i = 0,1 \) we have

\[
\lim_{r \to 0} \frac{u_i'(r)}{V^{1/2}(r) u_i(r)} = (-1)^i
\]

and there exists a constant \( k > 0 \) such that

\[
\lim_{r \to 0} V^{1/2}(r) r^{N-1} u_0(r) u_1(r) = k \quad \text{if } N > 3.
\]

**Proof.** Let us use the notation of Section 2 and consider only the case \( N > 3. \) (5.2) can be written in terms of \( v_0 \) and \( v_1 \) as

\[
\int_0^s s^{-2} v_0(s) v_1(s) ds < \infty.
\]

First we remark that if \( V \in (C_3) \) then \( V^{-1/2}(r) = c + o(r). \) Since \( r^{-2} V^{-1/2}(r) \) is integrable we get \( c = 0 \) and

\[
V(r) r^2 \to \infty \quad \text{as} \quad r \to 0.
\]

From this it follows easily that \( X(s) = c V(r) (r/s)^2, \quad s = r^{N-2}, \quad c = c(N), \) belongs to \((C_3).\)

We now prove that

\[
\lim_{s \to 0} X^{-1/2}(s) \frac{v_1(s)}{v_1(s)} = -1
\]
that is equivalent to (5.5) for $i = 1$ thanks to (5.8). In fact we prove (5.9) for every solution of (2.4) such that $v(s) \to \infty$ as $s \to 0$. Since $X \in C_3$ for every $\varepsilon > 0$ there exists $s_0 = s_0(\varepsilon) > 0$ such that

$$-\varepsilon < -X^{-3/2}(s)\dot{x}(s) < \varepsilon \quad \text{if} \quad 0 < s < s_0. \tag{5.10}$$

Now let $v_\varepsilon(s)$ be the solution of (2.4) with $v_\varepsilon(s_0) = 1$, $\dot{v}_\varepsilon(s_0) = -kX^{1/2}(s_0)$, $k = 1 - \varepsilon > 0$. Then, arguing as above, we get $v_\varepsilon(s) > 0$, $\dot{v}_\varepsilon(s) < 0$ in $(0, s_0)$ and $v_\varepsilon(s) \to \infty$ as $s \to 0$.

We consider the functions $p(s) = \frac{\dot{v}_\varepsilon(s)}{v_\varepsilon(s)}$ and $H(s) = -kX^{1/2}(s)$. The former satisfies in $(0, s_0)$ the equation

$$\ddot{p}(s) + p^2(s) = X(s). \tag{5.11}$$

On the other hand $\ddot{H}(s) + H^2(s) = -\frac{k}{2}X^{-1/2}\dot{X}(s) + kX^2(s)$. Using (5.10) and choosing $\varepsilon < 1$ we obtain

$$\ddot{H}(s) + H^2(s) < X(s). \tag{5.12}$$

Since $H(s_0) = p(s_0)$ it follows that $p(s) < H(s)$ for $0 < s < s_0$, therefore

$$\lim_{s \to 0} \sup \frac{X^{-1/2}(s)}{v_\varepsilon(s)} \frac{\dot{v}_\varepsilon(s)}{V_\varepsilon(s)} < -1 + \varepsilon. \tag{5.13}$$

Now every solution $v_1(s)$ of (2.4) such that $v_1(s) \to \infty$ is of the form $v_1(s) = av_\varepsilon(s) + bv_0(s)$, $a > 0$. Since $v_0(s) = o(s)$ and $\dot{v}_0(s) = o(1)$, cf. Prop. 2.1, (5.13) is true for $v_1(s)$ instead of $v_\varepsilon(s)$. Letting $\varepsilon \to 0$ we obtain the upper bound part of (5.9). A similar argument proves the other inequality.

Finally since the wronskian of $v_0$ and $v_1$, $v_0 \dot{v}_1 - v_1 \dot{v}_0$, is constant, the function $Q(s) = v_0(s)/v_1(s)$ satisfies
for a certain $k > 0$. Then, using L'Hôpital's rule, we get

$$\lim_{s \to 0} x^{1/2}(s)v_0(s)v_1(s) = \lim_{s \to 0} \frac{Q(s)}{v_1^{-2}(s)x^{-1/2}(s)} = \lim_{s \to 0} \frac{2k}{-2v_1^{-1}(s)v_1(s)x^{-1/2}(s) - \frac{1}{2} x^{-3/2}(s)x'(s)} = k.$$ 

This proves (5.6). Finally (5.5) for $i = 0$ follows from

$$x^{-1/2} \left( \frac{\dot{v}_0}{v_0} - \frac{\dot{v}_1}{v_1} \right) = \frac{2kx^{-1/2}}{v_0v_1},$$

using (5.9) and (5.15).

## COROLLARY 5.1

Every $V$ that satisfies $(C_3)$ belongs to $P_3$.

**Proof.** It follows from (5.6) and the assumption

$$r^{-2}v^{-1/2}(r) \in L^1(0,R).$$

### 5.3. In particular cases one can find solutions that have a stronger anisotropy property, namely positive solutions $u(x)$ such that

$$\lim_{|x| \to 0} \sup \frac{u(x)}{u_1(x)} = \infty, \quad \lim_{|x| \to 0} \inf \frac{u(x)}{u_1(x)} = 0.$$ 

In all the examples we take $V(r) = r^\theta$, $\theta < -2$, $p = -\frac{\theta + 2}{2}$ and $u(x)$ has the form

$$u(x) = r^{2-N} \exp(f(x)), \quad r = |x|.$$ 

**Example 1.** $N > 2$, $\theta = -4$. Take $f(x) = \frac{x_1}{|x|^2}$. 
Example 2. \( N = 2n \) even, \( \theta = -6 \). Take

\[
f(x) = i(x^2 + \ldots + x_n^2) - (x_{n+1}^2 + \ldots + x_{2n}^2) \frac{1}{pr^{p-2}}
\]

Example 3. \( N = 2 \), \( p \) positive integer. Take

\[
f(x) = (\cos p\alpha) \frac{1}{pr^p}, \ (r, \alpha) \text{ polar coordinates in } \mathbb{R}^2.
\]
6. BEHAVIOUR AT INFINITY

By means of the Kelvin transformation (0.7) we can rephrase the results obtained in previous sections as a study of the behaviour as \(|x| \to \infty\) of the solutions of

\[(6.1) \quad -\Delta u + V(|x|)u = 0, \quad |x| > R\]

where \(V \in C(R,\infty)\) is a radial potential. In particular we define the classes \(P_1^\infty, P_2^\infty, P_3^\infty\) through the condition that the transformed potential \(W(r) = r^{-4}V(r^{-1})\) belongs to \(P_1, P_2, P_3\) resp. In terms of \(V\) this means

\[P_1^\infty: \quad \int_{|x| > R} |V(|x|)| E_N(x)|dx < \infty.\]

\[P_2^\infty: \quad V^- \in P_1^\infty \text{ and there exists a positive radial solution of (0.1) in an interval } (R, \infty), \text{ such that } E_N(r)/u(r) \text{ is bounded and } |ru'(r)| < Cu(r) \text{ for a certain constant } C > 0.\]

\[P_3^\infty: \quad \int_{R}^{\infty} r^{N-3} u_0(r)u_1(r)dr < \infty.\]

Here \(u_0, u_1\) are two linearly independent, positive solutions of (6.1) for \(r = |x| > R\) such that

\[(6.2) \lim_{r \to \infty} r^{N-2}u_0(r) = 1 \text{ if } V \in P_1^\infty, = 0 \text{ if } V \notin P_1^\infty\]

\[(6.3) \quad \begin{align*}
\text{N>3: } & \lim_{r \to \infty} u_1(r) = 1 \text{ if } V \in P_1^\infty, = \infty \text{ if } V \notin P_1^\infty. \\
\text{N=2: } & \lim_{r \to \infty} \frac{u_1(r)}{\log(r)} = 1 \text{ if } V \in P_1^\infty, = \infty \text{ if } V \notin P_1^\infty.
\end{align*}\]

All this follows from (0.7) and Sections 2, 4.
In case $V \in P_1^\infty$ we can derive from Prop. 2.6 more precise estimates on the behaviour of $u_0$ and $u_1$:

**PROPOSITION 6.1.** As $r \to \infty$ we have

\begin{equation}
(6.4) \quad u_0(r) = r^{2-N} + \overline{p}_0(r)(1+o(1)),
\end{equation}

\begin{equation}
(6.5.a) \quad u_1(r) = 1 + \overline{p}_1(r)(1+o(1)) + Cr^{2-N} \quad \text{if } N > 3,
\end{equation}

\begin{equation}
(6.5.b) \quad u_1(r) = \log(r) + \overline{p}_1(r)(1+o(1)) + C \quad \text{if } N = 2,
\end{equation}

where

\begin{equation}
(6.6) \quad \overline{p}_0(r) = r^{2-N} \int_r^\infty \int_s^{s-N-3} j t^{3-N} V(t) dt ds
\end{equation}

and

\begin{equation}
(6.7) \quad \overline{p}_1(r) = \begin{cases} 
  r^{2-N} \int_r^\infty J_s^{N-3} (t V(t) dt) ds & \text{if } N > 3, \\
  \int_r^\infty s^{-1} (t |\log t| V(t) dt) ds & \text{if } N = 2.
\end{cases}
\end{equation}

Similarly $(C_3^\infty)$ is obtained from $(C_3)$ by replacing $r \to 0$ by $r \to \infty$. In particular if $V(r) = ar^\theta$, $a > 0$, $\theta \in \mathbb{R}$ we have

\begin{equation}
V \in P_1^\infty \quad \text{if } \theta < -2,
\end{equation}

\begin{equation}
V \in P_2^\infty \quad \text{if } \theta < -2 \quad \text{and}
\end{equation}

\begin{equation}
V \in (C_3^\infty) \quad \text{if } \theta > -2.
\end{equation}

The meaning at infinity of the theorems of Section 1 is the following. Let $\Omega = \mathbb{R}^N - B_R(0)$ and let $u \in C^2(\Omega)$ be a solution of (6.1):
THEOREM A'. If \( V \in P_1^\infty \) and \( u \) satisfies as \( |x| \to \infty \)

\[
(6.8) \quad u(x) \geq o(|x|) \text{ or } u(x) \leq o(|x|),
\]

then

\[
(6.9) \quad u(x) = u(r) + o(r^{2-N})
\]

Moreover there exists a constant \( C_1 \) such that if \( C_1 \neq 0 \)

\[
(6.10.a) \quad \overline{u}(r) = C_1 + O(p_1(r) + r^{2-N}) \text{ if } N > 3
\]

\[
(6.10.b) \quad u(r) = C_1 \log(r) + O(p_1(r) + 1) \text{ if } N = 2
\]

If \( C_1 = 0 \) there exists a constant \( C_0 \) such that

\[
(6.11) \quad \overline{u}(r) = C_0 r^{2-N} + p_0(r)(1 + o(1)).
\]

Finally \( V(x)u(x)r^{2-N} \in L^1(\bar{\Omega}) \).

As in Theorem A it can be noted that the conditions \( V \in P_1^\infty \) and (6.8) are optimal.

THEOREM B'. Assume that \( V \notin P_1^\infty \) but \( V^- \in P_1^\infty \). The following conditions are equivalent

i) \( u(x) = o(u_1(r)), \ r = |x| \to \infty \),

ii) \( r^{2-N}V(r)u(x) \in L^1(\mu) \),

iii) \( u(x) = O(u_0(r)), \ r \to \infty \).

If \( \tilde{U}_1 \) is the solution of (1.7) in \( R < r < \infty \) constructed as the \( U_1 \) in Section 1 we have

THEOREM C'. Let \( V \in P_2^\infty \) and assume that as \( r \to \infty \)

\[
(6.12) \quad u(x) \geq o(\tilde{U}_1(r)) \text{ or } u(x) \leq o(\tilde{U}_1(r)).
\]
Then

(6.13) \quad u(x) = \bar{u}(r) + g(x)

where \( g(x) = o(u_0(r)) \).

In the same way the results of Section 5 can be translated to study the existence of anisotropic solutions.
7. SOLUTIONS IN $\mathbb{R}^N\setminus\{0\}$.

In this section we study the conditions under which a solution $u \in C^2(\mathbb{R}^N\setminus\{0\})$ of

$$(-\Delta u + V(x)u = 0, \ 0 < |x| < \infty)$$

is radially symmetric. For simplicity we deal only with the case $N > 3$. The differences for $N = 2$ are similar to those of previous sections. As above we assume that $V(x)$ is continuous, radially symmetric: $V = V(r), \ r = |x| > 0$, and we restrict the growth of $V^-$ as $r \to 0$ or $r \to \infty$ as follows:

$$\int_0^\infty V^-(r)r \ dr < \infty.$$

We also need another condition on the potential, namely that the ordinary differential equation

$$u''(r) + \frac{N-1}{r}u'(r) = V(r)u(r), \ 0 < r < \infty$$

be disconjugate in $[0,\infty)$, namely that every solution has at most one zero in that interval, cf. [W]. This is equivalent to the existence of a positive function $y \in C^2(0,\infty)$ satisfying

$$y''(r) + \frac{N-1}{r}y'(r) + V^-(r)y(r) < 0,$$

(cf. [W], Theorem 2.1). It is therefore clear that (7.3) is disconjugate if $V > 0$. More precisely, using the change of variables (2.5) and Corollary 2.1* of [W], it is easy to see that (7.3) is disconjugate in $[0,\infty)$ if

$$\int_0^\infty V^-(r)r \ dr < \frac{N-2}{2}, \ N > 3.$$

We begin with the study of the radial solutions of (7.3)
LEMMA 7.1 Under the above assumptions on $V^-$ there exist two positive solutions of (7.3), $u_1(r)$ and $u_2(r)$, such that as $r \to \infty$.

(7.4) \[ r^{N-2} u_1(r) + C_1, u_2(r) + C_2 \]
and as $r \to 0$

(7.5) \[ r^{N-2} u_1(r) + C_3, u_2(r) + C_4, \]

where $C_1 = 1$ if $V \in P_1^\infty$, $C_1 = 0$ if $V \notin P_1^\infty$, $C_2 = \text{positive constant depending on } V$ if $V \in P_1^\infty$, $C_2 = \infty$ if $V \notin P_1^\infty$, $C_3 = \text{positive constant depending on } V$ if $V \in P_1$, $C_3 = \infty$ if $V \notin P_1$ and $C_4 = 1$ if $V \in P_1$, $C_4 = 0$ if $V \notin P_1$.

Proof. i) Let $u_1$ be a solution of (4.3) such that $r^{N-2} u_1(r) + C_1$ as $r \to \infty$ with $C_1$ as given above. Now we perform the change of variables

\[ v_1(s) = u_1(r), s = r^{2-N}. \]

Then $v_1(s) \to 0$ as $s \to 0$. Since (7.3) is disconjugate the equation satisfied by $v_1$ also is and it follows that $v_1(s) \neq 0$ for $s > 0$, hence $u_1(r) \neq 0$ for every $r > 0$. We may assume that $u_1(r) > 0$.

ii) Now we let $u_2$ be a solution of (7.3) satisfying $u_2(r) + C_4$ as $r \to 0$ with $C_4$ as above. With the change of variables

\[ v_2(s) = r^{N-2} u_2(r), s = 1/r, \]

then $v_2(s)$ satisfies (7.3) and $s^{N-2} v_2(s) + C_4$ as $s \to \infty$. Using step i) we conclude that $v_2 \neq 0$ on $(0, \infty)$ and we may assume that $u_2(r) > 0$ for $0 < r < \infty$. 
Let \( u \) be a solution of (7.3) such that \( u(r)/E_N(r) + k_1 \) as \( r \to 0 \), with \( k_1 = 1 \) if \( V \in P_1 \), \( k_1 = \infty \) if \( V \notin P_1 \). Then we have

\[
u(r) = c_1 u_1(r) + c_2 u_2(r)
\]

with \( u_1, u_2 \) as above. Since \( u_2 \) is bounded near 0 it follows that the limit \( \lim_{r \to 0} r^{N-2} u_1(r) = C_3 \) is either positive or \( +\infty \).

Finally let \( u \) be a solution of (7.3) such that, as \( r \to \infty \), \( u(r) + k_2 \), with \( k_2 = 1 \) if \( V \in P_1 \), \( k_2 = \infty \) if \( V \notin P_1 \). Arguing as in iii) we obtain the existence of the limit \( \lim u_2(r) = C_2 \in (0, +\infty] \).

We are also interested in the solutions obtained by separation of variables as in Section 1. Therefore we want to study the solutions of the equation

\[
U''(r) + (N-1/r)U'(r) = ((N-1/r^2) + V(r))U(r), \quad 0 < r < \infty,
\]

Since the potential \( \frac{N-1}{r^2} + V(r) > V(r) \) and (7.3) is disconjugate, (7.6) also is. (See (7.4)). Besides, the change of variable \( w(r) = u(r)/r \) transforms solutions of (7.6) into solutions of

\[
w''(r) + (N+1/2)w'(r) = V(r)u(r).
\]

Therefore applying the results of Lemma 7.1 in dimension \( N+2 \) instead of \( N \) we obtain:

**Lemma 7.2.** There exist two positive solutions \( U_1, U_2 \) of (7.6) such that as \( r \to \infty \)

\[
r^{N-1} U_1(r) + c_1, \quad r^{-1} U_2(r) + c_2
\]

and as \( r \to 0 \)

\[
r^{N-1} U_2(r) + c_3, \quad r^{-1} U_2(r) + c_4
\]

with \( c_1, c_2, c_3, c_4 \) as in Lemma 7.1.
We can now state our main result

**THEOREM D.** Let \( u \in C^2(B^N_0) \) be a solution of (7.1) such that

\[
(7.10) \quad u(x) = o(U_1(r)) \quad \text{as} \quad r \to 0
\]

\[
(7.11) \quad u(x) = o(U_2(r)) \quad \text{as} \quad r \to \infty.
\]

Then \( u \) is radial: \( u(x) = \overline{u}(r) \) for every \( 0 < |x| = r < \infty \).

**Proof.** Since \( V_2 \) is nonsingular at \( 0 \) and \( U_1 \) is positive we may take \( U_2 \) to be

\[
U_2(r) = U_1(r) \int_0^r t^{1-N} u_1^{-2}(t) dt.
\]

Now we consider the function

\[
h(x) = \frac{\|u(r, \sigma) - u(r)\|_{L^2(S_{N-1})}}{U_1(r)} \quad s(r) = \frac{U_2(r)}{U_1(r)}.
\]

Repeating the argument of Lemma 3.1 we prove that \( h(r) \) is a convex function in \((0, \infty)\). Since \( h(s) = o(1) \) as \( s \to 0 \) and \( h(s) = o(1) \) as \( s \to \infty \) it follows that \( h \equiv 0 \), therefore \( u(r, \sigma) = \overline{u}(r) \).

For the potentials of the class \( P_2 \) we have proved that unilateral bounds imply absolute bounds.

Therefore we have

**COROLLARY 7.1.** Let \( V \in P_2 \cap P_2^\infty \) and let \( u \in C^2(B^N_0) \) be a solution of (7.1) such that

\[
(7.12) \quad u(x) > o(U_1(r)) \quad \text{or} \quad u(x) < o(U_1(r)) \quad \text{as} \quad r \to 0
\]

\[
(7.13) \quad u(x) > o(U_2(r)) \quad \text{or} \quad u(x) < o(U_2(r)) \quad \text{as} \quad r \to 0
\]

then \( u \) is radial.
In fact under those conditions \(|u(x)| = o(U_1(r))\) as \(r \to 0\) cf. Theorems A and C, and \(u(x) = o(U_2(r))\) as \(r \to \infty\), cf. Theorems A' and C'.

Observe that in the case \(V \in \overline{P_3} \cap \overline{P_3}^\infty\), i.e. if

\[
\int_0^\infty r^{N-3} u_1(r) u_2(r) dr < \infty,
\]

since \(u_1 = U_1\) as \(r \to 0\) and \(u_2 = U_2\) as \(r \to \infty\), Theorem D says that if a solution \(u\) is \(o(u_1)\) at 0 and \(o(u_2)\) at infinity it vanishes identically in \(\mathbb{R}^N\), a result that follows simply from Theorems B and B'.
BIBLIOGRAPHY


Recent IMA Preprints (continued)

120 D.R.J. Chillingworth, Three Introductory Lectures on Differential Topology and Its Applications

121 Giorgio Vergara Caffarelli, Green's Formulas for Linearized Problems with Live Loads

122 F. Clarezen and W. Garofalo, Unique Continuation for Nonnegative Solutions of Schrodinger Operators

123 J.L. Ericksen, Constitutive Theory for some Constrained Elastic Crystals

124 Minoru Murata, Positive solutions of Schrodinger Equations

125 John Maddock and Gareth P. Parry, A Model for Twinning

126 M. Kaneko and W. Wooders, The Core of a Game with a Continuum of Players and Finite Coalitions: Nonemptiness with Bounded Sizes of Coalitions

127 William Zamo, Equilibrium in Production Economies with an Infinite Dimensional Commodity Space

128 Myrna Holtz Wooders, A Tiebout Theorem

129 Abstracts for the Workshop on Theory and Applications of Liquid Crystals

130 Yoshikazu Giga, A Remark on A Priori Bounds for Global Solutions of Semilinear Heat Equations

131 N. Chipot and G. Vergara-Caffarelli, The N-Membranes Problem

132 P.L. Lions and P.E. Souganidis, Differential Games and Directional Derivatives of Viscosity Solutions of Isaacs' Equations II

133 G. Capriz and P. Giovine, On Virtual Effects During Diffusion of a Dispersed Medium in a Suspension

134 Fall Quarter Seminar Abstracts

135 Umberto Mosco, Wiener Criterion and Potential Estimates for the Obstacle Problem

136 Chi-Sing Man, Dynamic Admissible States, Negative Absolute Temperature, and the Entropy Maximum Principle

137 Abstracts for the Workshop on Oscillation Theory, Computation, and Methods of Compensated Compactness

138 Arie Leizarowitz, Tracking Nonperiodic Trajectories with the Overtaking Criterion

139 Arie Leizarowitz, Convex Sets in R^n as Affine Images of some Fixed Set in R^m

140 Arie Leizarowitz, Stochastic Tracking with the Overtaking Criterion

141 Abstracts from the Workshop on Amorphous Polymers and Non-Newtonian Fluids

142 Winter Quarter Seminar Abstracts

143 D.G. Aronson and J.L. Vazquez, The Porous Medium Equation as a Finite-speed Approximation to a Hamilton-Jacobi Equation

144 E. Sanchez-Palencia and H. Weinberger, On the Edge Singularities of a Composite Conducting Medium


146 Chi-Sing Man and H. Cohen, A Coordinate-Free Approach to the Kinematics of Membranes

147 J.L. Lions, Asymptotic Problems in Distributed Systems

148 R. Lauterbach, An Example of Symmetry Breaking with Submaximal Isotropy Subgroup

149 Abstracts from the Workshop on Natastability and Incompletely Posed Problems

150 B. Boquad and Jerry Bona, Wave-dominated Shelves: A Model of Sand-Ridge Formation by Progressive, Infragravity Waves

151 Abstracts from the Workshop on Dynamical Problems in Continuum Physics

152 Y.I. Golikov, The problem of Embedding in R^m with Prescribed Gauss Curvature and Its Solution by Variational Methods

153 R. Batra, The force on a Lattice Defect in an Elastic Body

154 J. Fleckinger and Michael Lapidus, Eigenvalues of Elliptic Boundary Value Problems with and Indefinite Weight Function

155 R. Kohn and M. Vogelius, Thin Plates with Rapidly Varying Thickness, and Their relation to Structural Optimization

156 M. Gurtin, Some Results and Conjectures in the Gradient Theory of Phase Transitions