The boundary value problems for non-linear elliptic equations and the maximum principle for Euler-Lagrange equations

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In the present paper we investigate the boundary value problems for non-linear elliptic equations connected with infinite convex hypersurfaces. The asymptotic cone of such hypersurfaces is very important in our considerations. We study the problem of existence of solutions for Monge-Ampere equations on the entire space with prescribed asymptotic cone and the maximum principle and estimates for Euler-Lagrange equations. These problems have natural mutual connections.
§ 1. Infinite convex hypersurfaces and their asymptotic cones.

1.1 The main definitions

Let \( E^{n+1} \) be a \((n+1)\)-dimensional Euclidean space with Cartesian coordinates \( x_1, x_2, \ldots, x_n, z \equiv x_{n+1} \) and let \( E^n \) be the hyperplane \( z = 0 \) in \( E^{n+1} \).

The graphs \( S_u \) of convex functions \( z = u(x), x \in k^n \) are called complete infinite convex hypersurfaces in \( E^{n+1} \). Thus from this definition it follows that we consider only convex hypersurfaces in \( E^{n+1} \) projected one-to-one on the entire hyperplane \( E^n \).

Let \( H_u \) be the convex body defined by equation

\[
z \leq u(x)
\]

for all \( x \in E^n \), where \( u(x) \) is a convex function. Clearly

\[
\partial H_u = S_u.
\]

Let \( A \) be any point of \( H_u \). The union of all closed rays starting from \( A \) and belonging to \( H_u \) is called the asymptotic cone of \( H_u \) at the point \( A \).

Let \( \phi_\lambda : E^{n+1} \rightarrow E^{n+1} \) be the homothety of \( E^{n+1} \) with the center \( A \) and coefficient \( \lambda > 0 \). Then

\[
K_u(A) = \lim_{\lambda \rightarrow 0} \phi_\lambda(H_u).
\]

For every two points \( A, A' \in H_u \) the convex cones \( K_u(A) \) and \( K_u(A') \) are congruent and can coincide by parallel translation in the direction of \( z \)-axis. The well known proof is based on the formula (1.3). Therefore we will denote the asymptotic cone \( K_u(A) \) by \( K_u \), if we do not need to use the special choice of the point \( A \).

The asymptotic cone \( K_u \) is called non-degenerate if \( 3K_u \) is projected one-to-one on the entire hyperplane \( E^n \). For non-degenerate asymptotic cones it is convenient to replace them by their boundary. In this case the notation \( K_u \) will be used for the boundary of a solid asymptotic cone and we will say that
$K_u$ is the asymptotic cone of the function $u(x)$.

1.2. Normal mapping and $R$-curvature of convex functions.

The normal mapping and $R$-curvature are the natural generalizations of the tangential mapping and the integral Gaussian curvature for arbitrary continuous convex functions (hyper surfaces). These concepts were introduced and studied by the author in [1] (more detailed exposition see in [2], [3]).

The normal mapping of every convex function $u(x)$ is constructed by supporting hyperplanes of its graph $S_u$.

Let $p^n$ be a $n$-dimensional Euclidean space with Cartesian coordinates $p_1, p_2, \ldots, p_n$ and let $p = (p_1, p_2, \ldots, p_n)$ be a point of $p^n$. If

$$\alpha : z = p_1^o x_1 + p_2^o x_2 + \ldots + p_n^o x_n$$

(1.4)

is a supporting hyperplane of $S_u$, then the point

$$x(\alpha) = (p_1^o, p_2^o, \ldots, p_n^o)$$

(1.5)

is called the normal image of the hyperplane $\alpha$. For any set $e \in E^n$ the normal image $x_u(e)$ is defined as

$$x_u(e) = \bigcup_{\alpha} x(\alpha)$$

(1.6)

where $\alpha$ is any supporting hyperplane of $S_u$, whose intersection with $S_u$ has points with projections in the set $e$.

We shall use the following well known properties of the normal mapping.

Property 1.

$$x_u(E^n) = x_{K_u}(E^n)$$

(1.7)

for every convex function $u(x)$ defined on the entire hyperplane $E^n$. 
Property 2.

For every Borel subset \( e \) of \( \mathbb{E}^n \) the set \( x_u(e) \) is Lebesgue measurable.

Property 3.

Let \( R(p) > 0 \) be locally summable in \( \mathbb{E}^n \), then

\[
\omega(R,u,e) = \int_{x_u(e)} R(p) \, dp \tag{1.8}
\]

is a completely additive non-negative set function on the ring of Borel subsets of \( \mathbb{E}^n \).

If \( u(x) \in C^2(\mathbb{E}^n) \), then

\[
\omega(R,u,e) = \int_{x_u(e)} R(u)\det(u_{ij}) \, dx
\]

The set function \( \omega(R,u,e) \) is called the \textit{R-curvature} of a convex function \( u(x) \).

Thus the \( R \)-curvature is the generalization of the Monge-Ampere operator \( R(u)\det(u_{ij}) \) for the set of all convex functions.

If

\[
R(p) = 1,
\]

then

\[
\omega(1,u,e) = \text{mes} \, x_u(e), \tag{1.9}
\]

where the right part of (1.9) is the \( n \)-dimensional Lebesgue measure in the space \( \mathbb{E}^n \).

If \( R(p) = (1 + |p|^2)^{-\frac{n+1}{2}} \), then \( \omega(R,u,e) \) is the area (Lebesgue measure) of the image of \( e \) concerning the Gaussian mapping of \( S_u \) in the \( n \)-dimensional unit sphere \( S^n \).
1.3 Remarks.

In this Subsection we consider a few properties of the normal mapping of convex functions.

**Property 4.** Let \( u(x) \) be any convex function, then the function

\[
v(x) = u(x) + \sum_{i=1}^{n} a_i x_i + b,
\]

where \( a_1, a_2, ..., a_n, b \) are constants, is convex and the set \( x_v(e) \) can be obtained from \( x_u(e) \) by the parallel translation of \( p_n \) on the vector

\( a = (a_1, a_2, ..., a_n) \).

**Property 5.** the normal image of any convex cone \( K \) is a closed convex set in \( p^n \) whose dimension can take values \( 0, 1, 2, ..., n \).

**Property 6.** If \( K \) is a non-degenerate convex cone, then \( x_K(E^n) \) is a bounded closed \( n \)-dimensional convex domain with interior points.

The following Remarks follow from Properties 4, 5, 6.

It is sufficient to consider only convex functions \( u : E^n \rightarrow R \) whose normal images contain the origin of \( p^n \). If the asymptotic cone of such function is non-degenerate then we can additinally assume that the origin \( \theta' \) of \( p^n \) is the interior point of \( x_{K_u}(E^n) \). If we also assume that the vertex of this non-degenerate convex cone \( K_u \) is at the origin \( \theta \) of \( E^n \), then the equation of \( K_u \) is as follows

\[
z = k(x),
\]

where \( k(x) \) is non-negative convex function in \( E^n \) and \( k(x) = 0 \) only at the point \( \theta \).

Since any point of \( E^n \) can be taken as origin, then it is sufficient to consider asymptotic non-degenerate convex cones whose vertices are projected in
the point \( \theta \in E^n \). The equations of such cones are as follows

\[
z = k(x) + b,
\]

whose \( b \) is any constant.

\[\text{§2. The second boundary value problem for elliptic solutions of Monge-Ampere equations.}\]

\[\text{2.1 The statement of the second boundary value problem.}\]

We consider elliptic solutions \( u(x) \) of the Monge-Ampere equation

\[
\det (u_{ij}) = f(x,u,Du)
\]

in the entire space \( E^n \). We assume that \( f(x,u,p) \) is continuous in \( E^n \times \mathbb{R} \times \mathbb{R}^n \). If \( u(x) \) is a \( C^2 \) - elliptic solution of equation (2.1), then \( u(x) \) is necessarily either a convex or a concave function. It is sufficient to consider only convex solutions of equation (2.1). We should impose the assumption of positiveness of \( f(x,u,p) \) in \( E^n \times \mathbb{R} \times \mathbb{R}^n \) if we investigate convex solutions of equation (2.1).

The convex function \( u(x) \) is called a generalised solution of equation (2.1) if \( u(x) \) satisfies (2.1) almost everywhere and the set function \( \omega(1,u,e) \) is absolutely continuous. Note that

\[
\omega(1,u,e) = \text{mes } \chi_u(e)
\]

for any Borel subset \( e \) of \( E^n \).

The statement of the second boundary value problem is as follows. Let \( K \) be a non-degenerate convex cone. Find wide sufficient conditions of existence of at least one generalised solution of equation (2.1) for which \( K \) is
the asymptotic cone.

This problem was formulated by Alexandrov and Pogorelov (Conference of Global Differential Geometry, Leningrad, 1958) and was solved by Bakelman in 1984 (see [4]). Up to 1984 only two special cases of equation (2.1) were investigated. The first one was studied by Alexandrov [5] in 1942 and it is related to existence and uniqueness of a general complete infinite hypersurface with prescribed area of Gaussian mapping. The second Theorem was established by Bakelman [7] in 1956 and is related to generalized solutions of the Monge-Ampere equation

$$\det (u_{ij}) = \frac{y(x)}{R(Du)} \quad (2.2)$$

-$\frac{(n+1)}{2}$ (the particular case $R(Du) = [1+(Du)^2]^\frac{1}{2}$ corresponds to Alexandrov's Theorem.)

The necessary and sufficient condition of solvability of the second boundary value problem for equation (2.2) is as follows

$$\int_{\mathbb{R}^n} g(x)dx = \int_{\chi_K(\mathbb{R}^n)} R(p)dp \quad (2.3)$$

where $g(x) > 0$, $g(x) \in L(\mathbb{R}^n)$, $R(p) > 0$, $R(p) \in L_{loc}(\mathbb{R}^n)$ and $\chi_K(\mathbb{R}^n)$ is the normal image of prescribed non-degenerate convex asymptotic cone $K$. Note that the desired solution is defined to within an additive constant.

The solution of the second boundary value problem for the general Monge-Ampere equation

$$\det (u_{ij}) = f(x,u, Du)$$

is essentially more difficult, because the simple necessary and sufficient condition (2.3) of the solvability of the same problem for equation (2.2) must be
replaced by the complicated implicit necessary condition

\[ \text{mes } x_k (p^n) = \int_{\mathbb{E}^n} f(x, u(x), Du(x)) \, dx. \]  \hfill (2.4)

Moreover, the application of fixed points Theorems contains difficulties because \( \mathbb{E}^n \) is a non-compact set. The proof of Bakelman's recent existence Theorem for equation (2.1.1), presented in 2.3, is based on the construction of a new Monge-Ampère equation in some special introduced Banach space. This construction is significantly based on the properties of the asymptotic behavior of the function \( f(x, u, p) \) and its first derivatives by \( |x| \to + \infty \) and \( |u| \to + \infty \).

### 2.2 The Second boundary problem for Monge-Ampère equations.

Let \( y(x) > 0 \) be a summable function in \( \mathbb{E}^n \) and let \( R(p) > 0 \) be a locally summable function in \( p^n \).

**Theorem 1.** Let \( K \) be a non-degenerate convex cone in \( \mathbb{E}^{n+1} = \mathbb{E}^n \times \mathbb{R} \) and let

\[ z = k(x), \quad x \in \mathbb{R}^n \]

be the equation of \( K \). Let

\[ \int_{\mathbb{E}^n} y(x) dx = \int_{\mathbb{E}^n_k} R(p) dp. \]  \hfill (2.5)

Then the second boundary value problem for the Monge-Ampère equation

\[ \det (u_{ij}) = \frac{y(x)}{R(Du)} \]  \hfill (2.6)

has a generalized convex solution \( u(x) \) and this solution is unique to within an additive constant.
As we noted in Subsection 2.1 this Theorem is proved by author in [7].

The proof of the statement of uniqueness to within an additive constant is based on the same ideas and facts as the proof of the uniqueness of convex generalized solutions of the Dirichlet problem for equation (2.6) (see [8], Chapter IV, §23).

The proof of the assertion of existence of generalized solutions made in Theorem 1 can be proved by the approximation of convex polyhedra. First we replace equation (2.6) by the equation

\[ \omega(R,u,e) = \mu(e) \quad (2.7) \]

which can now be considered for all convex functions defined in \( E^n \). In (2.7) \( \mu(e) \) is any non-negative, completely additive set function subordinated to the condition

\[ \mu(E^n) = \int_{X_k(E^n)} R(p) \, dp \quad (2.8) \]

If we consider the polyhedron case, then set functions \( \mu(e) \) and prescribed convex cone \( K \) must satisfy the following necessary Assumptions.

**Assumption A.** The convex cone \( K \) is a non-degenerate convex polyhedral angle. We can assume in addition that the origin \( 0' \) of \( \mathbb{R}^n \) lies inside \( X_k(\mathbb{R}^n) \) and therefore the equation of \( K \) has the form

\[ z = k(x) + b \quad (2.9) \]

where \( k(x) \) is a convex continuous piecewise linear homogeneous (of degree 1) function in \( E^n \), satisfying the conditions \( k(0) = 0 \); \( k(x) > 0 \) if \( x \neq 0 \), and \( b \) is any constant

**Assumption B.** The set function \( \mu(e) \) satisfies the following condition: there
exists a finite system of points $A_1, A_2, \ldots, A_m$ and a finite system of non-negative numbers $\nu_1, \nu_2, \ldots, \nu_m$ such that

$$\mu(A_i) = \nu_i$$

(2.10)

and

$$\mu(e) = \sum_{A_i \in e} \mu(A_i)$$

(2.11)

for every subset $e$ of $E^n$.

Theorem 1 can be reduced to the following Theorem for the case of convex polyhedra.

**Theorem 2.** If the polyhedral angle $K$ satisfies Assumption $A$ and the set function $\mu(e)$ satisfies Assumption $B$ and if

$$\mu(E^n) = \int_{\partial K} R(p) \, dp,$$

(2.12)

then there exist convex polyhedra

$$S_u : z = u(x), \quad x \in E^n$$

such that the vertices of $S_u$ are projected only in the points $A_1, A_2, \ldots, A_m$; $K$ is the asymptotic cone of $S_u$ and

$$\omega(K, u, A_i) = \nu_i, \quad i = 1, 2, \ldots, m.$$  

(2.13)

Moreover if $u_0(x)$ is a convex function defining one of such polyhedra, then all others can be obtained by the formula

$$u(x) = u_0(x) + C,$$

(2.14)

where $C$ is an arbitrary constant.
Remarks. 1) Theorem 1 can be obtained by approximation of convex polyhedra which are constructed with help of Theorem 2. This is the traditional way now.

In the theory of convex bodies and hypersurfaces the approximation method by polyhedra was developed by Minkowski, Alexandrov, Pogorelov and others (see [9], [6], [10], [11], [2], [8]). Therefore we note only that the conditions of Theorem 1 provide to obtain the sequence of convex polyhedra

\[ S_{U_j} : z = u_j(x), x \in \mathbb{E}^n \]

such that: a) \( u_j(0) = 0 \), \( j = 1, 2, \ldots, m \); b) all functions \( u_j(x) \) satisfy the Lipschitz condition of degree 1 with the common constant; c) \( \omega(R, u_j, e) \) converge weakly to the function \( \mu(e) \) and

\[ \lim_{j \to \infty} \omega(R, u_j, E^n) = \int_{\chi_k(E^n)} R(p) dp. \]

From convex polyhedra \( u_j(x) \) it is possible to extract a convergent subsequence whose limit is a solution of the second boundary value problem for equation (2.7) (or equation (2.6)) with prescribed non-degenerate convex asymptotic cone. This completes the proof of Theorem 1.

Finally in this Subsection we present a new variant of the proof of Theorem 2 based on a variational method which up to now was not used in non-compact problems related to infinite convex hypersurfaces.

Proof of Theorem 2. We denote by \( W^r(A_1, A_2, \ldots, A_m, K) \) the set of all convex polyhedra, whose vertices are projected only in the points \( A_1, A_2, \ldots, A_m \) and whose asymptotic cones coincide with prescribed non-degenerate convex polyhedral angle \( K \). Let \( T \) be the set of convex polyhedra \( P : z = u(x) \) such that
a) \( T \) is a subset of \( W^+(A_1, A_2, \ldots, A_m, K) \);

b) If \( u(x) \in T \), then the inequalities

\[
0 < \omega(R, u, A_1) < \nu_1
\]

hold for all \( i = 2, 3, \ldots, m \) and

\[
\omega(R, u, A_1) = \int \chi_k(\mathcal{E}^n) K(p) \, dp - \frac{1}{2} \sum_{i=2}^m \omega(R, u, a_i)
\]

(2.16)

c) \( u(A_1) = a_1 \),

(2.17)

where \( a_1 \) is an arbitrary fixed constant.

The set \( T \) is not empty, because the convex cone \( K \) with the vertex \((A_1, a_1)\) belongs to \( T \).

The system of real numbers

\[
\xi_1 = u(A_1), \quad \xi_2 = u(A_2), \ldots, \quad \xi_m = u(A_m),
\]

(2.18)

taken from every convex polyhedron \( u(x) \in W^+(A_1, A_2, \ldots, A_m, K) \), defines this polyhedron one-to-one. The metric

\[
d(u(x), v(x)) = \left\{ \sum_{i=1}^m \left\{ \frac{1}{2} \left[ u(A_i) - v(A_i) \right]^2 \right\}^{1/2} \right\}^{1/2}
\]

(2.19)

introduced in \( W^+(A_1, A_2, \ldots, A_m, K) \) shows that the mapping (2.17) isometrically maps \( W^+(A_1, A_2, \ldots, A_m, K) \) on some closed convex subset of the Euclidean space \( \mathbb{R}^m = \{ \xi = (\xi_1, \xi_2, \ldots, \xi_m) \} \). Since \( W^+(A_1, A_2, \ldots, A_m, K) \) can be identified with a corresponding closed subset of \( \mathbb{R}^m \), then \( T \) is also a closed subset of \( \mathbb{R}^m \).

From (2.15-17) it follows that \( T \) is a bounded subset of \( \mathbb{R}^m \). Thus \( T \) is a compact set in \( \mathbb{R}^m \).

The function \( f : T \to \mathbb{R} \) acting by the formula

\[
f(u) = \sum_{i=1}^m u(A_i)
\]

(2.20)
is continuous in $T$. Hence
\[
\inf_{T} f(u) = f_0 > -\infty
\]
and there exists the polyhedron $u_0(x) \in T$ such that
\[
f(u_0(x)) = f_0. \tag{2.21}
\]
Now we prove that $u_0(x)$ is the desired solution of the second boundary value problem, satisfying the condition (2.17).

If our assertion is incorrect, then
\[
\omega(K, u_0, A_s) < \mu_s \tag{2.22}
\]
at least at one point $A_s$ ($s = 2, 3, \ldots, m$), because
\[
\omega(K, u_0, A_s) < \mu_s
\]
from the definition of the set $T$. Now consider the convex polyheron
\[
\tilde{P} : z = \tilde{u}(x) \in W^+(A_1, A_2, \ldots, A_m, K)
\]
such that
\[
\tilde{u}(A_1) = a_1, \quad \tilde{u}(A_2) = u_0(A_2), \ldots,
\]
\[
\tilde{u}(A_{s-1}) = u_0(A_{s-1}), \quad \tilde{u}(A_s) = u_0(A_s) - \epsilon,
\]
\[
\tilde{u}(A_{s+1}) = u_0(A_{s+1}), \ldots, \tilde{u}(A_m) = u_0(A_m), \quad \epsilon > 0
\]
where $\epsilon > 0$ is a sufficiently small number for which the inequality
\[
\omega(K, \tilde{u}, A_s) < \mu_s \tag{2.23}
\]
holds. Since
\[ \omega(R, \tilde{u}, a_i) = \int_{\chi_k(E^n)} R(p) \, dp \]  
(2.24) 

and

\[ \omega(R, u, A_k) < \omega(R, u_0, A_k) \]

for \( k = 1, 2, \ldots, s-1, s+1, \ldots, m \), then from (2.23-24) it follows that \( \tilde{u}(x) \in T \). 

But

\[ f(u(x)) = \frac{1}{m} \sum_{i=1}^{m} \tilde{u}(A_i) = \frac{1}{m} \sum_{i=1}^{m} u_0(A_i) - \varepsilon = \]

\[ = f_0 - \varepsilon < f_0. \]

The last inequality is incompatible with equality (2.21). Hence \( u_0(x) \) is the desired solution of the second boundary value problem for the equation (2.13).

Now let convex polyhedra \( P_1 \) and \( P_2 \) be the graphs of convex solutions \( u_1(x), u_2(x) \in W^+(A_1, A_2, \ldots, A_m, \mathbb{R}) \) of the equation (2.13), satisfying the additional condition

\[ u_1(A_1) = u_2(A_1) = a_1. \]

If \( u_1(x) \) and \( u_2(x) \) are different functions, then according to Lemma proved in [6], Chapter IX, there exists at least one point \( A_j \), where \( j \) is one of integers \( 1, 2, \ldots, m \), such that the polyhedral angles \( V_1 \subset P_1 \) and \( V_2 \subset P_2 \) are projected in the point \( A_j \) and

\[ V_1 \supset V_2 \quad \text{and} \quad V_1 \setminus V_2 \neq \emptyset. \]

Hence

\[ \text{mes} \, x_{u_1}(A_j) > \text{mes} \, x_{u_2}(A_j). \]
Thus

\[ u_j = \int_{X_{u_1}^{u_1}(A_j)} R(\rho) d\rho > \int_{X_{u_2}^{u_2}(A_j)} R(\rho) d\rho = u_j . \]

Therefore our assumption is incorrect. The proof of Theorem 2 is completed.

2.3. The second boundary value problem for general Monge-Ampere equations.

In this subsection we investigate the second boundary value problem for the Monge-Ampere equation

\[ \det(u_{ij}) = f(x,u,Du) \quad (2.25) \]

in the class of convex generalized solutions. We divide the presentation into a few separate parts.

A) The main Assumptions.

A.1 Admissible convex cones. A nondegenerate convex cone in \( \mathbb{R}^{n+1} \) is called admissible, if the equation of \( K \) is

\[ z = k(x) \quad (2.26) \]

where \( k(x) \) is a continuous, homogeneous of the order 1, convex function in \( \mathbb{R}^n \) satisfying the conditions

a) \( k(0) = 0 \),

b) \( k(x) > 0 \) for all \( x \neq 0 \) in \( \mathbb{R}^n \).

It is sufficient to consider only admissible convex cones (see the final part of Section 1). We denote by \( K^* \) the normal image of a convex cone \( K \), i.e.

\[ K^* = \chi_{k}(\mathbb{R}^n) . \quad (2.27) \]
For every admissible convex cone $K$ the set $K^*$ is closed, bounded and convex in the space $\mathbb{R}^n$ and the point $0^*$ is an interior point of $K^*$, where $0^*$ is the origin of $\mathbb{R}^n$.

A.2 The properties of the function $f(x,u,\mu)$ and its derivatives.

The function $f(x,u,\mu)$ is continuous in $E^n \times \mathbb{R} \times \mathbb{R}^n$ together with its derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial \mu_i}$, $(i=1,2,\ldots,n)$ and

$$f(x,u,\mu) > 0,$$
$$\frac{\partial f(x,u,\mu)}{\partial u} > 0$$

in $E^n \times \mathbb{R} \times \mathbb{R}^n$.

We also assume that the inequalities

$$|\frac{\partial f}{\partial u}| < \frac{C_U}{|x|^{n+2+\alpha}},$$

$$|\frac{\partial f}{\partial \mu_i}| < \frac{C_1}{|x|^{n+\alpha}}, \quad (i=1,2,\ldots,n)$$

hold for all $(x,u,\mu) \in E^n \times \mathbb{R} \times K^*$ with $|x| > m_U$, where $\alpha = \text{const.} > 0$, $C_U = \text{const.} > 0$, $C_1 = \text{const.} > 0$, $m_U = \text{const.} > 1$.

A.3 Estimators and their properties.

For every admissible convex cone $K$ there exist two functions $\lambda_K(x,u)$ and $\phi_K(x,u)$, depending only on prescribed cone $K$, such that:

a) $\lambda_K(x,u)$ and $\phi_K(x,u)$ are positive and continuous in $E^n \times \mathbb{R}$ and increase with respect to $u$ for every fixed $x \in E^n$;
b) the inequalities

$$\lambda_k(x, u) \leq f(x, u, p) \leq \phi_k(x, u)$$  \hspace{1cm} (2.32)

hold for all $x \in E^n$, $u \in K$, $p \in K^*$.

The functions $\lambda_k(x, u)$ and $\phi_k(x, u)$ are called estimators.

B) The statement of the main Theorem and the scheme of its proof.

**Theorem 3.** Let $K$ be an admissible convex cone described in Assumption 1 and let $\lambda_k(x, u)$, $\phi_k(x, u)$ be estimators satisfying Assumption 2 (see part A of Subsection 2.3).

If there exist two numbers

$$-\infty < a_k < b_k < +\infty$$

such that

a) $\int_{E^n} \phi_k(x, k(x)) + \gamma)dx < +\infty$  \hspace{1cm} (2.33)

for all $\gamma \in [a_k, b_k]$;

b) $\int_{E^n} \phi_k(x, k(x)) + a_k)dx < \text{mes } K^*$;  \hspace{1cm} (2.34)

c) $\inf_{\gamma \in K^*} \int_{E^n} \lambda_k(x, (x, \gamma) + b_k)dx > \text{mes } K^*$  \hspace{1cm} (2.35)

where $z = k(x)$ is the equation of the cone $K$ and $(x, \gamma) = \sum_{i=1}^{n} \gamma_i x_i$, $x = (x_1, \ldots, x_n) \in E^n$, $\gamma = (\gamma_1, \ldots, \gamma_n) \in K^*$, then equation (2.25) has at least one generalized solution $u(x)$ with asymptotic cone $K$ and

$$a_k \leq u(0) \leq b_k$$  \hspace{1cm} (2.36)
The scheme of proof.

Since we consider unbounded convex functions on the entire space $E^n$, then the application of fixed point principles requires to construct the special functional space, where the second boundary value problem can be investigated. In this special functional space we study some modification of the Monge-Ampere operator $\det(u_{ij})$ and its inverse, which is induced by the original equation (2.25) and by prescribed admissible convex cone $K$. The final part of the proof for Theorem 3 is based on the application of Schauder principle to the inverse of the modified Monge-Ampere equation.

C) The functional space of the second boundary value problem. We denote by $C^{0}(E^n)$ the set of all continuous functions $u : E^n \to K$ and by $C^{0,1}(E^n)$ the subset of $C^{0}(E^n)$ consisting of all Lipschitz functions $u : E^n \to K$, i.e. $u(x) \in C^{0,1}(E^n)$ if and only if

$$L(u) = \sup_{x,y \in E^n} \frac{|u(y) - u(x)|}{|y - x|} < + \infty,$$

where $|y-x| = \text{dist}(x,y)$ in $E^n$.

Let $U_1 \subset U_2 \subset \ldots \subset U_m \ldots$ be the sequence of $n$-balls:

$$U_m : |x| < m,$$

$m = 1, 2, 3, \ldots$. We denote by $\|u\|_m$ the number $\sup_{U_m} |u(x)|$ and by $\|u\|_A$ the number

$$\|u\|_A = |u(0)| + \sup_{m=1}^{\infty} \frac{\|u\|_m}{m^{2+\alpha}},$$

where $\alpha = \text{const.} > 0$ (see Assumption 2).

Let $A$ be the subset of $C^{0}(E^n)$ consisting of all functions $u(x)$ such that
Clearly

\[ \|u\|_A < +\infty. \]

\[ \|u\|_A < |u(0)| + L(u) \sum_{m=1}^{\infty} \frac{1}{m^{2+\alpha}} < +\infty \]

for every \( u(x) \in C_{0,1}^1(\mathbb{R}^n) \). On the other hand \( \|x_1 + \cdots + x_n\|_A = +\infty \). Thus \( A \) is a proper non-empty subset of \( C_{0}^1(\mathbb{R}^n) \).

The functional \( \|u\|_A \) is a norm of the set \( A \). Clearly \( A \) is a Banach space with respect to the norm. We denote by \( A_1 \) the subspace of \( A \) consisting of all functions \( u(x) \in C_{0,1}^1(\mathbb{R}^n) \).

Now we introduce the equivalence relation \( \Gamma \) in \( A \) setting \( u(X) \Gamma v(x) \)

if and only if \( u(x) - v(x) = \text{const} \in \mathbb{R}^n \). Clearly the factor space \( B = A/\Gamma \)

is again a Banach space with respect to the induced norm

\[ \|\xi\|_B = \|\tilde{u}\|_A \]

where \( \xi \) is any element of \( B \) generated by the class of functions \( \{u(x) + q\} \),

where \( u(x) \in A \) and \( q \) is any real number. We use the notation \( u(x) = u(x) - u(0) \).

We call the function \( u(x) \in A \) a basic representative of the element \( \xi \in B \), if \( u(0) = 0 \). We shall use the notation \( \xi_u \) for elements of the space \( B \) generated by basic representatives \( u(x) \in A \). Clearly \( \tilde{u}(x) \) is a basic representative for every \( u(x) \in A \).

Every convex function \( u(x) \) defined in \( \mathbb{R}^n \), whose graph has an admissible asymptotic cone, is an element of \( A_1 \). Therefore \( u(x) \in A \). This statement follows directly from the compactness of \( x_u(\mathbb{R}^n) \) in the space \( P^n \).
Let $T_k$ be a subset of $\mathcal{B}$ consisting of all elements $\xi \in \mathcal{B}$ such that $\xi = \xi_u$, where $u(x)$ is any convex function defined in $E^N$ and satisfying two conditions:

a) $u(\theta) = 0$;

b) the asymptotic cone of $u(x)$ is a fixed admissible cone $K$, considered in the second boundary value problem for equation (2.25).

The set $T_k$ is not empty, since $\xi_k \in T_k$, where $z = k(x)$ is the equation of the admissible convex cone $K$ mentioned above. The following lemmas are correct for any admissible convex cones $K$.

**Lemma 1.** $T_k$ is a convex subset of $\mathcal{B}$.

**Lemma 2.** $T_k$ is a closed subset of $\mathcal{B}$.

**Lemma 3.** $T_k$ is a compact subset of $\mathcal{B}$.

**Proof of Lemma 1.**

Let $\lambda$ and $\mu$ be any positive numbers such that $\lambda + \mu = 1$. If $\xi_f$ and $\xi_y$ are any elements of $T_k$ generated by the basic representatives $f(x)$ and $y(x)$, then functions $f(x)$ and $y(x)$ are convex in $E^N$, have one and the same asymptotic cone $K$, which is admissible, and $f(\theta) = y(\theta) = 0$.

Clearly the element $\lambda \xi_f + \mu \xi_y$ is generated by the function $\lambda f(x) + \mu y(x)$. Since $\lambda f(x) + \mu y(x)$ is convex and

$$\lambda f(\theta) + \mu y(\theta) = 0,$$

then

$$\lambda \xi_f + \mu \xi_y = \xi_{\lambda f + \mu y}.$$

(2.40)

If we prove that $K$ is the asymptotic cone of $\lambda f(x) + \mu y(x)$, then from this fact and from equality (2.40) it follows that $T_k$ is a convex set in $\mathcal{B}$. Clearly the statement concerning the asymptotic cone of $\lambda f(x) + \mu y(x)$ is suf-
efficient to be established only for $C^2$ convex functions $f(x)$ and $g(x)$, because the case of general convex functions can be obtained by the simple approximation of corresponding $C^2$ convex functions.

Thus our convex functions $f(x)$ and $g(x)$, introduced in the beginning of the proof of Lemma 1, are twice differentiable in $\mathbb{E}^n$.

Clearly

$$\nabla (\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g$$

(2.41)

at any point $x_0 \in \mathbb{E}^n$. From (2.41) it follows that

$$x(\gamma) = \lambda x(\gamma_1) + \mu x(\gamma_2),$$

where $\gamma, \gamma_1, \gamma_2$ are correspondingly the normal images of tangent hyperplanes of the graphs of the functions $\lambda f(x) + \mu g(x), f(x)$ and $g(x)$ at the points $(x_0, \lambda f(x) + \mu g(x)), (x, f(x)), (x, g(x))$.

Since $x(\gamma_1), x(\gamma_2)$ are points of the convex set $x_k(\mathbb{E}^n)$, then

$$x(\gamma) \in x_k(\mathbb{E}^n).$$

(2.42)

If $K_1$ is the asymptotic cone of the convex function $\lambda f(x) + \mu g(x)$, then from (2.42) it follows that

$$x_k'(\mathbb{E}^n) \subseteq x_k(\mathbb{E}^n),$$

(2.43)

where $z = k'(x)$ is the equation of the asymptotic cone $K'$, whose vertex is at the point $\theta \in \mathbb{E}^n$. From (2.43) we obtain

$$0 < k'(x) < k(x)$$

(2.44)

for all $x \in \mathbb{E}^n$. 

Now we prove that

$$k'(x) = k(x)$$

for all $x \in \mathbb{R}^n$. Let $l$ be any axis in $\mathbb{R}^n$, passing through the point $\theta$, and $s$ be the Cartesian coordinate in $l$ such that

$$|s| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

for any point $x = (x_1, x_2, \ldots, x_n) \in l$.

Let $\hat{k}'(s), \hat{k}(s), \hat{f}(s), \hat{g}(s)$ be functions, which are generated by $k'(x)$, $k(x)$, $f(x)$, $y(x)$ on the axis $l$.

Then

$$k'(s) = \begin{cases} 
-k_1s & \text{if } s < 0; \\
k_2s & \text{if } s > 0
\end{cases}$$

and

$$k(s) = \begin{cases} 
-k_2s & \text{if } s < 0; \\
k_2s & \text{if } s > 0
\end{cases}$$

where $0 < k_1 < k_1$ and $0 < k_2 < k_2$.

Thus we obtain the following chain of equalities

$$k'_2 + k'_1 = \int_{-\infty}^{+\infty} \left( \lambda \frac{d^2f}{ds^2} + \mu \frac{dy}{ds^2} \right) ds =$$

$$\lambda \int_{-\infty}^{+\infty} \frac{d^2f}{ds^2} ds + \mu \int_{-\infty}^{+\infty} \frac{dy}{ds^2} ds =$$
\[
= \lambda (k_2 + k_1) + \mu (k_2 + k_1) = k_2 + k_1 , \text{ because } \lambda > 0 ; \mu > 0 \text{ and } \lambda + \mu = 1.
\]
Thus \( k'_2 + k'_1 = k_2 + k_1 \).

Since \( 0 < k'_2 < k_2 \) and \( 0 < k'_1 < k_1 \), then

\[
k'_2 = k_2 \text{ and } k'_1 = k_1.
\]

Hence

\[
k'(x) = k(x)
\]

for all \( x \in \mathbb{E}^n \), because \( \xi \) is an arbitrary axis in \( \mathbb{E}^n \) passing through \( \theta \).

From (2.46) it follows that

\[ K' = K \]

and the proof of Lemma 1 is completed.

**Proof of Lemma 2.**

Let \( T_k \) be the closure of \( T_k \) in the space \( \mathbb{B} \). If \( \xi \in T_k \), then there exist the elements \( \xi_1, \xi_2, \ldots, \xi_q, \ldots \) of the set \( T_k \) such that

\[
\lim_{q \to \infty} \| \xi - \xi_q \|_B = 0.
\]

The basic representatives \( u_q(x) \) of \( \xi_q \) are convex functions on \( \mathbb{E}^n \) with one and the same asymptotic cone \( K \), which is admissible, and \( u_q(\theta) = 0 \), \( q = 1, 2, 3, \ldots \).

Clearly

\[
\lim_{q \to \infty} \| u_q(x) - u(x) \|_A = 0,
\]

where \( u(x) \) is the basic representative of \( \xi \in T_k \). Hence \( u_q(x) \) converges uniformly to \( u(x) \) in every closed ball \( U_m : |x| < M \).
Thus $u(x)$ is a convex function with asymptotic cone $K$ and $\xi \in T_k$. The proof of Lemma 2 is completed.

**Proof of Lemma 3.**

Let $\xi$ be any element of $T_k$ and $u(x)$ be the basic representative of $\xi$. Then $u(0) = 0$ and convex function $u(x)$ satisfies the Lipschitz condition in $\mathbb{E}^n$ with the constant $d_U = \text{diam } K^*$. Thus

$$
\|\xi\|_B = \|u(x)\|_A = \sum_{m=1}^{\infty} \frac{\|u(x)\|_m}{m^{2+\alpha}} < \\
< d_U \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} = \text{const.} < + \infty.
$$

Let $\{u_y(x)\}$ be the collection of basic representatives of $\{u_y(x)\}$. From the last estimate it follows that there exists a sequence of convex functions $u_y(x)$ which is subsequence of $\{u_y(x)\}$ and which is uniformly convergent to some convex function $u_0(x)$ in every ball $U_m : |x| < m$. Clearly $u_0(0) = 0$ and $K$ is the asymptotic cone of $u_0(x)$. Hence

$$
\xi \in T_k.
$$

If we establish that

$$
\lim_{q \to \infty} \|u_y(x) - u_0(x)\|_A = 0,
$$

then the proof of Lemma 3 will be completed.

Let $\varepsilon > 0$ be any number. We fix a positive integer $m_U$ such that

$$
\sum_{m=m_U}^{\infty} \frac{1}{m^{1+\alpha}} < \frac{\varepsilon}{4\text{diam } K^*}
$$

Then

$$
\|u_y - u_0\|_A = \sum_{m=1}^{m_U-1} \frac{\|u_y - u_0\|_m}{m^{2+\alpha}} + \sum_{m=m_U}^{\infty} \frac{\|u_y - u_0\|_m}{m^{2+\alpha}}
$$

$$
< \frac{\varepsilon}{4\text{diam } K^*}.
$$
\[-25-\]

\[
+ \sum_{m=m_0}^{\infty} \frac{\|u_{\gamma_q} - u_0\|_m}{m^{2+\alpha}} < \\
\left< (m_0-1) \|u_{\gamma_q} - u_0\|_m + 2\text{diam } K^* \sum_{m=m_0}^{\infty} \frac{1}{m^{1+\alpha}} \right> < \left< (m_0-1) \|u_{\gamma_q} - u_0\|_m + \frac{\varepsilon}{2} \right>.
\]

Since \(\lim u_{\gamma_q} - u_0\|_m = 0\), then there exists a positive integer \(Q_0\) such that

\[(m_0-1) \|u_{\gamma_q} - u_0\|_m < \frac{\varepsilon}{2}\]

for \(q > Q_0\). Thus \(\|u_{\gamma_q} - u_0\|_A < \varepsilon\) for \(q > Q_0\), where \(\varepsilon > 0\) is any given number.

Lemma 3 is proved.

D) The proof of Theorem 3.

Now we return to the proof of Theorem 3, which is the main existence Theorem of §2.

Let \(\xi\) be any element of the set \(I_K\) and let \(u(x)\) be its basic representative. We consider the collection of convex functions

\[u_a(x) = u(x) + a\]  

(2.47)

where \(a \in (-\infty, +\infty)\). The function

\[F_a(x) = f(x, u_a(x), \nabla u_a(x)) = \]

\[= f(x, u(x) + a, \nabla u(x))\]
is non-negative for all \( x \in \mathbb{E}^n \). Let \( z = k(x) \) be the equation of the cone \( K \). Then

\[
u_a(x) = u(x) + a < k(x) + a
\]  

(2.49)

for all \( x \in \mathbb{E}^n \). From Assumption A.3 it follows that

\[
F_{u_a}(x) < \phi_k(x,u(x) + a) < \\
\phi_k(x,k(x) + a)
\]

(2.50)

for all \( x \in \mathbb{E}^n \) and all real values of \( a \).

From the conditions of Theorem 3 it follows that

\[
\int_{\mathbb{E}^n} F_{u_a}(x) dx < \int_{\mathbb{E}^n} \phi_k(x,k(x) + a) dx < + \infty
\]

for all \( a \in [a_k, b_k] \).

From Assumption A.3 we obtain the inequality

\[
F_{u_a}(x) > \inf_{\gamma \in K^*} \lambda_k(x, (x, \gamma) + a)
\]

(2.51)

for all \( x \in \mathbb{E}^n \) and all \( x \in \mathbb{R}^n \).

Now we introduce the function

\[
\psi(a) = \int_{\mathbb{E}^n} F_{u_a}(x) dx
\]

(2.52)

in \([a_k, b_k]\). Since

\[
\psi(a) = \int_{\mathbb{E}^n} f(x,u(x) + a, \nu u(x)) dx
\]

(2.53)
for \( a \in [a_k, b_k] \), then from Assumptions A.2, A.3 and the conditions of Theorem 3 it follows that \( \psi(a) \) is continuous, \( \psi'(a) \) exists on \([a_k, b_k]\), and

\[
\psi(a_k) \leq \int_{E_n} \phi_k(x, k(x) + a_k) \, dx < \text{mes } K^* \tag{2.54}
\]

and

\[
\psi(b_k) > \inf_{k^*} \int_{E_n} \lambda_k(x, (x, y) + b_k) \, dx > \text{mes } K^*
\]

hold. Since

\[
\psi'(a) = \int_{E_n} \frac{\partial f(x, u(x) + a, u(x))}{\partial u} \, dx > 0,
\]

then from (2.54) it follows that there exists only one number \( a^* \in [a_k, b_k] \)

such that

\[
\psi(a^*) = \int_{E_n} F_{u_{a^*}}(x) \, dx = \text{mes } K^*. \tag{2.55}
\]

Now we consider the second boundary value problem for equation

\[
\det (z_{ij}) = F_{u_{a^*}}(x) \tag{2.56}
\]

with prescribed asymptotic cone \( K \).

Since all conditions of Theorem 1 are fulfilled, then this boundary value problem has only one convex generalized solution \( z(x) \) satisfying the condition \( z(0) = a^* \) and having \( K \) as the asymptotic cone of its graph.

Let \( \tilde{u}(x) = u_{a^*}(x) - a^* \) and \( \tilde{Z}(x) = z(x) - z(0) \) be the basic representatives of the elements \( \xi_u \) and let \( \eta_z \) belong to the convex set \( T_k \). Clearly the second boundary value problem for equation (2.56) and prescribed admissible convex cone \( K \) generate some operator
\[ G : T_k + T_k \]

such that \( \eta_z = G(\xi_u) \).

The further considerations are devoted to the establishment of continuity and compactness of the operator \( G \) on the convex compact set \( T_k \). They will permit to apply the Schauder fixed point Theorem to the operator \( G : T_k + T_k \).

The existence Theorem 3 is the final result of these investigations.

**Lemma 4.** The functional \( a^* : T_k + K \) is continuous.

**Proof.** Let \( \xi \) be any element of \( T_k \) and \( u(x) \in A \in A \) be a basic representative of \( \xi \). Then \( u(x) \) is a convex function, whose graph has \( K \) as its asymptotic cone, and \( u(0) = 0 \).

The real number \( a^* = a^*(u) \) is the root of the equation

\[ \int_{\mathbb{E}^n} f(x, u(x) + a^*(u), \nabla u(x)) dx = dx = \text{mes } K^* . \tag{2.57} \]

We proved above that this equation has only one root \( a^* = a^*(u) \in [a_k, b_k] \).

Now it should be proved that \( a^*(u_q) \) converges to \( a^*(u) \) if

\[ \lim_{q \to \infty} \| u_q - u \|_{A} = 0 \]

and \( \xi_{u_q} \), \( \xi_u \in T_k \).

From (2.57) it follows that

\[ \int_{\mathbb{E}^n} f(x, u(x) + a^*(u), \nabla u(x)) dx = \]

\[ \int_{\mathbb{E}^n} f(x, u_q(x) + a^*(u_q), \nabla u_q(x)) dx = \text{mes } K^* . \]
Hence

\[
\int_{\mathbb{R}^n} \left[ u(x) - U_q(x) \right] + [a^*(u) - a^*(u_q)] \left( \int_0^1 \frac{\partial f}{\partial u} \left| v_t \right| \, dt \right) \, dx = \\
= - \sum_{i=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right) \left( \int_0^1 \frac{\partial f}{\partial u} \left| v_t \right| \, dt \right) \, dx,
\]

where \( v_t(x) = (1-t)u_q(x) + tu(x) + \)

\( + (1-t) \, a^*(u_q) + ta^*(u) \)

and

\[
\frac{\partial v_t(x)}{\partial x_i} = (1-t) \frac{\partial u_q(x)}{\partial x_i} + t \frac{\partial u(x)}{\partial x_i},
\]

\( 0 < t < 1, \ i = 1, 2, \ldots, n. \)

Since \( a^*(u) \) and \( a^*(u_q) \) do not depend on \( x \), then from \( (2.58) \) we obtain

\[
a^*(u) - a^*(u_q) = \\
\int_{\mathbb{R}^n} \left( u(x) - U_q(x) \right) \left( \int_0^1 \frac{\partial f}{\partial u} \left| v_t \right| \, dt \right) \, dx \\
= - \int_{\mathbb{R}^n} \left( \int_0^1 \frac{\partial f}{\partial u} \left| v_t \right| \, dt \right) \, dx \\
- \sum_{i=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right) \left( \int_0^1 \frac{\partial f}{\partial u} \left| v_t \right| \, dt \right) \, dx
\]

Since \( \frac{\partial f}{\partial u} \) is positive and continuous in \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^* \) (see Assumption A.2), then for any compact set \( Q \) in \( \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^* \) there exists a constant \( h(Q) > 0 \) such that
\[
\frac{\partial f(x,u,\nu)}{\partial u} > h(Q) > 0 .
\]

For our purpose it is sufficient to consider the compact set

\[ Q_0 = U_1 \times [\delta_1, \delta_2] \times K^* , \]

where \( U_1 \) is the unit ball \(|x| < 1\) in \( \mathbb{E}^n \), and the numbers \( \delta_1, \delta_2 \) are determined by conditions

\[ \delta_1 < \nu_t(x) < \delta_2 . \]

The finite values of \( \delta_1, \delta_2 \) depend only on \( \|u(x)\|_1 \), numbers \( a_k, b_k \) and the integer \( N_1 > 0 \) such that

\[ \|u_q(x) - u(x)\|_A < 1 \]

if \( q > N_1 \). If \( x \in U_1 \), then the point

\[(v_t(x), \nabla v_t(x)) \in [\delta_1, \delta_2] \times K^* \]

for all \( t \in [0,1] \), because \( \nabla v_t(x) = (1-t)\nabla u(x) + t\nabla u_q(x) \in K^* \).

Here we take into account that \( \nabla u(x) \) and \( \nabla u_q(x) \) are points of the convex set \( K^* \). Thus we obtain the inequality

\[
\int_{\mathbb{E}^n} \left( \int_0^1 \frac{\partial f}{\partial u} \, dt \right) dx > h(Q_0) \text{ mes } Q_0 > 0
\]

(2.59)

*) Since \( u(x), u_q(x) \) and \( v_t(x) \) are convex functions, the notation \( \nabla u(x) \) etc. is used also for supporting hyperplanes if there does not exist a tangent hyperplane of the graph of \( u(x) \) at the point \( (x, u(x)) \).
for all \( q > N_1 \).

According to Assumption A.2 the inequalities

\[
0 < \frac{\partial f}{\partial u} < \frac{c_u}{|x|^{n+2+\alpha}}
\]

hold for all \( x \in \mathbb{R}^n \) with \( |x| > m_0 \), \( u \in \mathbb{R} \) and \( p \in K^* \), where \( \alpha = \text{const.} \) \( > 0 \) and \( m_0 = \text{const.} \) \( > 1 \); without losing generality we can assume that \( m_0 \) is any positive integer more than \( (2^{1/2+\alpha} - 1)^{-1} \). Let

\[
I_1 = \left| \int_{\mathbb{R}^n} \left[ (u(x) - u_q(x)) \cdot \frac{1}{|\partial f/\partial u|} \right] dx \right| v_t.
\]

Then

\[
I_1 < \int_{|x| < m_0} \{ |u(x) - u_q(x)| \cdot \frac{1}{|\partial f/\partial u|} \} v_t dx + \int_{|x| > m_0} \{ |u(x) - u_q(x)| \cdot \frac{1}{|\partial f/\partial u|} \} v_t dx.
\]

Let

\[
C_2 = \sup_{|x| < m_0} \frac{\partial f}{\partial u} \rightarrow \infty.
\]

Clearly \( C_2 \) depends only on \( \|u(x)\|_{m_0}, K^* \) and numbers \( a_k, b_k \). Since \( u(x) \) and \( u_q(x) \) are basic representatives of elements of the set \( T_k \), then

\[
u(\theta) = u_q(\theta) = \theta
\]

and

\[
|u(x)| < |x| \cdot \text{diam} \ K^*,
\]

\[
|u(x)| < |x| \cdot \text{diam} \ K^*.
\]

From (2.60 - 2.63) it follows that
\[ I_1 < C_2 \| u(x) - u_q(x) \|_{m_0} + \]
\[ + \frac{C_U \sigma_{n-1}}{2^{2+\alpha}} \sum_{m=m_0}^{\infty} \| u(x) - u_q(x) \|_{m+1} \left( \frac{1}{m^{2+\alpha}} - \frac{1}{(m+1)^{2+\alpha}} \right), \]

where \( \sigma_{n-1} \) is the area of the unit sphere \( S^{n-1} \). Since

\[ \| u(x) - u_q(x) \|_{m_0} < m_0^{2+\alpha} \| u(x) - u_q(x) \|_A \]

and

\[ \frac{1}{(m+1)^{2+\alpha}} > \frac{1}{m^{2+\alpha}} - \frac{1}{(m+1)^{2+\alpha}} \]

for \( m > m_0 > (2^{2+\alpha} - 1)^{-1} \), then

\[ I_1 < \left[ C_2 m_0^{2+\alpha} + \frac{C_U \sigma_{n-1}}{2^{2+\alpha}} \right] \| u(x) - u_q(x) \|_A, \quad (2.64) \]

where the integer \( m_0 \) satisfies the inequality

\[ m_0 > (2^{2+\alpha} - 1)^{-1} \]

and constants \( C_U \) and \( C_2 \) are independent of \( q \).

Let

\[ I_2 = \left| \int_{\mathbb{R}^n} \left( \sum_{j=1}^{n} \left| \frac{\partial u(x)}{\partial x_j} - \frac{\partial u_q(x)}{\partial x_j} \right| \right) \left( \int_{0}^{1} \frac{\partial f}{\partial u} \mathrm{d}t \right) \mathrm{d}x \right|. \]

Clearly

\[ I_2 < \sum_{j=1}^{n} \| \frac{\partial u(x)}{\partial x_j} - \frac{\partial u_q(x)}{\partial x_j} \|_A \| \int_{0}^{1} \frac{\partial f}{\partial u} \mathrm{d}t \| \mathrm{d}x. \]

According to Assumption A.2 the inequalities
\[ \left| \frac{\partial f}{\partial u_i} \right| < \frac{C_1}{|x|^{n+\alpha}}, \quad i = 1, 2, \ldots, n \] (2.66)

hold for all \( x \in \mathbb{R}^n \) with \( |x| > m_0 \), \( u \in \mathbb{R} \), and \( p \in K^* \). The functions \( \frac{\partial u_i}{\partial x_i} \) converge to \( \frac{\partial u(x)}{\partial x_i} \) almost everywhere in \( \mathbb{R}^n \) and

\[ \forall u_i(x) \in K^*, \quad \forall u(x) \in K^* \] (2.67)

for all \( x \in K^* \).

From (2.65-67) it follows that

\[ I_2 \leq \left( \sup_{1 \neq i} \int_{|x| = m} \left| \frac{\partial f}{\partial u_i} \right| v_t \right) \cdot \int_{|x| = m} \left| \frac{\partial u}{\partial x_i} \right| - \left| \frac{\partial u_i}{\partial x_i} \right| \, dx + \] (2.68)

\[ + 2C_1 n \text{ diam } K^* \int_{|x| > m} \frac{dx}{|x|^{n+\alpha}}, \]

where \( m > m_0 \) is an arbitrary positive integer. Since

\[ \int_{|x| > m} \frac{dx}{|x|^{n+\alpha}} = \left( \frac{\sigma_{n-1}}{\alpha n^\alpha} \right) \]

then we can find \( m^* > 0 \) such that for every integer \( m > \max \{m_0, m^*\} \) the inequality

\[ 2C_1 n \text{ diam } K^* \frac{\sigma_{n-1}}{\alpha m^\alpha} < \frac{\varepsilon}{2} \] (2.69)

holds, where \( \varepsilon > 0 \) is a given positive arbitrary number. We fix some integer \( m > \max \{m_0, m^*\} \). Then

\[ \sup_{1 \neq i} \int_{|x| = m} \left| \frac{\partial f}{\partial u_i} \right| v_t \]

\((i = 1, 2, \ldots, n)\), where the constant \( C_m \) depends only on \( m, a_k, b_k \) and \( \text{diam } K^* \).

Really \( \frac{\partial f}{\partial u_i} \) are continuous functions according to Assumption \( A_2 \) and we consider the supremum of \( \left| \frac{\partial f}{\partial u_i} \right| \) in the compact set:
\[ |x| < m, a_k = |x| \cdot \text{diam } K^* \leq u \leq b_k + |x| \text{diam } K^* , \]

Since \( u(x) \) and \( u_q(x) \) are convex functions, then

\[
\lim_{q \to \infty} \int_{|x| < m} \left| \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right| \, dx = 0
\]

Therefore we can find \( N_2 \) such that

\[
\sup_{i=1}^{n} \int_{|x| < m} \left| \frac{\partial f}{\partial x_i} \right| \, dt \int_{|x| < m} \left| \frac{\partial u(x)}{\partial x_i} - \frac{\partial u_q(x)}{\partial x_i} \right| \, dx < \frac{\varepsilon}{2} .
\]

if \( q > N_2 \).

Thus

\[ l_2 < \varepsilon , \quad (2.70) \]

if \( q > N_2 \).

Now from (2.59), (2.64), (2.70) it follows that

\[
\lim a^* (u_q) = a^*(u)
\]

if \( \|u_q - u\|_A \to 0 \). Lemma 4 is proved.

\textbf{Lemma 5.} The operator \( G : T_k \to T_k \) is continuous.

\textbf{Proof.} Let the sequence \( \varepsilon_q \in T_k \) converge to the element \( \varepsilon_0 \in T_k \) in the space \( B \). We should prove that

\[
\lim_{q \to \infty} \| \eta_q - \eta_0 \|_B = 0 , \quad (2.71)
\]

where \( \eta_q = G(\varepsilon_q) \) and \( \varepsilon_0 = G(\varepsilon_0) \).

Since \( T_k \) is compact subset of \( B \) (see Lemma 3), then there exists a subsequence \( \eta_{q_i} \) convergent to some element \( \bar{\eta} \in B \), i.e.
\[
\lim_{j \to \infty} \| u_j - u_0 \|_B = 0.
\]

Since \( T_k \) is closed in \( B \), then \( u \in T_k \). It is well known that the set functions \( \omega(l, v_j, e) \) converge weakly to the set function \( \omega(l, v, e) \) in \( \mathbb{E}^n \), where \( v_j(x) \) and \( v(x) \) are representatives of \( n_{q_j} \) and \( n \) satisfies the conditions

\[
v_{q_i}(z) = a^*(u_{q_i}); \quad v(z) = a^*(u_0)
\]

On the other hand all the functions

\[
F_{u_{q_j}} + a(x) = f(x, u_{q_j} + a^*(u_{q_j}), \nu u_{q_j})
\]

are non-negative and satisfy the inequality

\[
F_{u_{q_j}} + a(x) \leq \psi_k(x, k(x) + b_k)
\]

for all \( z \in \mathbb{E}^n \), where \( z = k(x) \) is the equation of admissible convex cone \( K \) (see Assumption A.1), prescribed for all functions \( u_{q_j}(x) \).

Note that the following facts are fulfilled:

a) \( \lim_{j \to \infty} \| u_j - u_0 \|_A = 0 \);

*) \( \omega(l, u, e) \) is the \( R \)-curvature of a convex function \( u(x) \) with \( R(p) = 1 \) for all \( p \in \mathbb{P}^n \).

**) The functions \( u_{q_j}(x) \) and \( u_0(x) \) are basic representatives of the elements \( e_{q_j} \) and \( e_0 \).
b) $\forall \eta_j (x) \in K^*$ for all $x \in \mathbb{E}^n$ and all integers $q_j$;

c) $\frac{\partial u_{q_j}}{\partial x_i}$ converges to $\frac{\partial u_0}{\partial x_i}$ almost everywhere in $\mathbb{E}^n$, $(i=1,2,...,n)$;

d) $\psi_M (x, k(x) + b_k)$ is a non-negative summable function in $\mathbb{E}^n$;

e) Estimates (2.59) are correct for the functions $u_{q_j}(x)$ and $u_0(x)$.

Now we use the well known Lebesgue Theorem and obtain

$$\omega(1,v,e) = \lim_{q_j \to \infty} \omega(1,u_{q_j},e) = \lim_{q_j \to \infty} \int_{\mathbb{E}^n} F u_{q_j} + a^* (u_{q_j}) dx = \int_{\mathbb{E}^n} F u_0 + a^* (u_0) dx,$$

where $e$ is any Borel subset of $\mathbb{E}^n$. Note that we used continuity of the functional $a^*: T_0 + R$ in these equalities.

Thus $v(x)$ is a convex generalized solution of the equation

$$\text{det} \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) = F u_0 + a^* (u_0)(x)$$

and the function $v_0(x)$ is also a convex generalized solution of the equation

$$\text{det} \left( \frac{\partial^2 v_0}{\partial x_i \partial x_j} \right) = F u_0 + a^* (u_0)(x).$$

Since

$$v(0) = v_0(0) = a^*(u_0)$$

and the admissible convex cone $K$ is the asymptotic cone for both functions $v(x)$ and $v_0(x)$, then $\overline{v}(x) = v_0(x)$ for all $x \in \mathbb{E}^n$.

Thus (2U.71) is correct. Lemma 5 is proved.
Now we can finish the proof of Theorem 3. Since $T_k$ is a compact set in $B$, then $G(T_k)$ is also compact in $B$. Moreover $G(T_k)$ $T_k$. Hence the operator $G$ has at least one fixed point $\xi \in T_k$. But the function $u(x) + a^*(\tilde{u}(x))$ is the representative for both elements $\xi$ and $G(\xi)$. Therefore $u(x)$ is the desired solution of the second boundary value problems for equation

$$\det(u_{ij}) = f(x,u,Uu).$$

Theorem 3 is proved.

§3. The maximum principle and estimates of solutions for elliptic Euler-Lagrange equations.

The concept and techniques of the global theory of convex hypersurfaces and functions can successfully be applied to the Dirichlet problem for elliptic Euler-Lagrange equations. In the present Section we establish the sharp interlocked necessary and sufficient conditions of two-sided estimates for solutions of such equations. These conditions are not subjected to a number of essential traditional limitations (see for example [12], [13], [14]). The interesting special cases of such Euler-Lagrange equations arise in elasticity-plasticity and global differential geometry problems (see [15], [16], [17]).

3.1 The main Assumptions. Let $F(x,u;p)$ be a $C^2$-funciton in $\Omega \times R \times P^n$, where $\Omega$ is a bounded open domain in $\mathbb{E}^n$, and $\partial \Omega$ is a closed $C^2$-hypersurface. It is well known that any $C^2$-function $u(x)$ minimizing (maximizing) the multiple integral

$$I(u) = \int_{\Omega} F(x,u,Uu)dx$$

is necessarily a solution of the Euler-Lagrange equation

$$\sum_{i=1}^{n} \frac{d}{dx_i} \left( \frac{\partial F}{\partial p_i} \right) - \frac{\partial F}{\partial z} = 0.$$
This equation can also be written in the following form

\[ \sum_{i,k=1}^{n} \frac{\partial^2 F}{\partial \mu_i \partial \mu_k} u_{ik} + \sum_{i=1}^{n} \frac{\partial^2 F}{\partial \mu_i \partial u} u_i + \]

\[ + \sum_{i=1}^{n} \frac{\partial^2 F}{\partial \mu_i \partial x_i} \frac{\partial F}{\partial z} = 0. \]  

(3.3)

Equation (3.3) is elliptic if and only if the quadratic form

\[ \sum_{i,k=1}^{n} \frac{\partial^2 F(x,u,p)}{\partial \mu_i \partial \mu_k} \xi_i \xi_k \]  

(3.4)

is positive (negative) definite in \( \Omega \times \mathbb{R} \times p^n \). The Dirichlet problem for wide classes of Euler-Lagrange equations is investigated by many authors (see [12], [18], [13], [16], [15], and other papers *).

Assumption A.4. Let \( F(x,u,p) \) be a \( C^2 \)-function in \( \Omega \times \mathbb{R} \times p^n \) and let the quadratic form (3.4) is positive definite in \( \Omega \times \mathbb{R} \times p^n \). Then there exist functions \( \phi(p), \phi(x,u), R_1(p), R_2(p), y_1(x,u), y_2(x,u) \) satisfying the following conditions:

1) \( \phi(p) \) is a strictly convex \( C^2 \)-function in \( p^n \) *) and \( \lambda(x,u) \) is positive and the inequality

\[ \lambda(x,u) \det \left( \frac{\partial^2 \phi(p)}{\partial \mu_i \partial \mu_j} \right) \leq \det \left( \frac{\partial^2 F(x,u,p)}{\partial \mu_i \partial \mu_j} \right) \]  

(3.5)

holds for every point \( (x,u,p) \in \Omega \times \mathbb{R} \times p^n \);

2) The functions \( R_1(p), R_2(p) \) are positive and locally summable with degree \( n \) in \( p^n \); the functions \( y_1(x,u), y_2(x,u) \) are non-negative in \( \Omega \times \mathbb{R} \) and the inequalities
\[-39-\]
\[
- \frac{y_1(x,u)}{R_1(p)} < 0(x,u,p) < \frac{y_2(x,u)}{R_2(p)}
\]

(3.6)

hold for every point \((x,u,p) \in \Omega \times R \times \mathbb{R}^n\), where

\[
D(x,u,p) = - \frac{\partial^2 F(x,u,p)}{\partial u \partial \bar{u}} \rho_i - \frac{n}{i} \frac{\partial^2 F(x,u,p)}{\partial v_i \partial \bar{u}} \rho_i - \frac{n}{i} \frac{\partial^2 F(x,u,p)}{\partial x_i \partial \bar{u}} + \frac{\partial F(x,u,p)}{\partial u}
\]

(3.7)

3) The function

\[
\psi_1(x,u) = \frac{n}{n} \frac{y_1(x,u)}{\lambda(x,u)}
\]

(3.8)

and

\[
\psi_2(x,u) = \frac{n}{n} \frac{y_2(x,u)}{\lambda(x,u)}
\]

(3.9)

are non-decreasing with respect to \(u\) for every fixed \(x \in \Omega\).


We consider the Dirichlet problem

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial F}{\partial u} \right) - \frac{\partial F}{\partial u} = 0
\]

(3.10)

\[
u_{\partial \Omega} = h(X) \in C(\partial \Omega).
\]

(3.11)

In Subsections 3.2 and 3.3 it is sufficient to consider only open bounded domains with continuous \(\partial \Omega\).

*) A convex \(C^2\)-function \(\phi(p)\) is called strictly convex, if \(\det \left( \frac{\partial^2 \phi(p)}{\partial p_i \partial p_j} \right) > 0\) in \(\mathbb{R}^n\).
Let \( M = \sup_{\Omega} h(x) \), \( m = \inf_{\Omega} h(x) \). Clearly \(-\infty < m < M < +\infty\).

**Theorem 4.** Let \( u(x) \in W^1_2(\Omega) \cap C(\overline{\Omega})^* \) be a solution of the Dirichlet problem (3.10-11) and let all conditions of Assumption 4 be fulfilled.

Then the inequalities

\[
\int_{\Omega} \psi_1(x,m)dx < \int_{K^*} R_1^n(q)d\nu, \tag{3.12}
\]

\[
\int_{\Omega} \psi_2(x,M)dx < \int_{K^*} R_2^n(q)d\nu \tag{3.13}
\]

provide the estimates

\[
m - C(R_2,\omega_2) \text{ diam } \Omega < u(x) < M + C(R_2,\omega_1) \text{ diam } \Omega \tag{3.14}
\]

for all \( x \in \overline{\Omega} \), where

\[
\omega_1 = \int_{\Omega} \psi_1(x,m)dx, \omega_2 = \int_{\Omega} \psi_2(x,M)dx \tag{3.15}
\]

and the values of the constants \( C(R_i,\omega_i) \) \( (i=1,2) \) will be described in the end of the proof of Theorem 4.

**Proof.** First of all we explain the notations in the right parts of inequalities (3.12), (3.13). \( K^* \) is the normal image of the asymptotic cone \( K \) of the strictly convex \( C^2 \)-function \( \phi(p) \); \( \chi_\phi : P^n \rightarrow Q^n \) is the tangential mapping, generated by the convex function \( \phi(p) \);

\[
\tilde{R}_i(q) = R_i(\chi_\phi^{-1}(q)), (i=1,2) \tag{3.16}
\]

and \( d\nu \) is the element of the Lebesgue measure in the Euclidean space \( Q^n \).

*) \( W^1_2(\Omega) \) is the Sobolev space with functions having all second generalized derivatives summable with degree \( n \) in \( \Omega \).
Let \( G = \overline{C} \Omega \) and \( \overline{G} \) be the closure of \( G \). Clearly \( \overline{G} = \overline{C} \Omega \), where \( \overline{C} \Omega \) is the closed convex hull of \( \Omega \).

Let \( H_m = \overline{G} \times \{ M \} \) and \( H_m = \overline{G} \times \{ m \} \). We construct the convex body \( Z \) which is the closed convex hull of the set \( H_m \cup H_m \cup S_u \), where \( S_u \) is the graph of \( u(x), x \in \overline{\Omega} \).

Clearly the part of \( aZ \), which are projected in the open convex domain \( G \) consists of the graphs \( S_{v_1} \) and \( S_{v_2} \), where \( v_1(x) \) is a concave function in \( G \).

Since

\[
m < u|_{\partial \Omega} < M
\]

and

\[
v_2(x) < u(x) < v_1(x)
\]

for all \( x \in G \), then it is sufficient only to establish the inequalities

\[
v_2(x) > m - C(R, \omega_2) \text{ diam } \Omega
\]

and

\[
v_1(x) < M + C(R, \omega_1) \text{ diam } \Omega.
\]

Moreover, it is sufficient to establish only (3.19), because (3.20) can be proved in the same way as (3.19). Our further considerations follow the proof of our analogous Theorems for solutions of the Dirichlet problem for elliptic quasilinear equations with the corresponding changes and developments. (See for example my papers [3], [15], [17]). Clearly the interesting case is only that, when there exist the points \( x \in \Omega \) such that

\[
u(x) < m
\]
We denote by $\Omega_2 \subset \Omega$ the set, where

$$v_2(x) = u(x).$$

Clearly $\Omega_2$ is a Lebesgue measurable set and

$$\text{mes } \Omega_2 = \text{mes } \Omega_2,$$

because $\overline{\Omega}_2 \subset \Omega$, $\text{mes } \Omega = 0$ and

$$\Omega_2 = \overline{\Omega}_2 \setminus \text{an } \Omega.$$

The inequality

$$\frac{1}{n} \left[ \det(u_{ij}) \right] \frac{1}{n} \left[ \det \left( \frac{\partial^2 F}{\partial p_i \partial p_j} \right) \right] \leq$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2 F}{\partial p_i \partial p_j} u_{ij} = \Omega(x, u(x), \forall u(x))$$

hold at any point $x \in \Omega_2$. Applying Assumption A.4 to inequality (3.22) we obtain

$$K_2^n(\Omega u(x)) \det \frac{\partial^2 \phi(\Omega u(x))}{\partial p_i \partial p_j} \det(u_{ij}(x)) \leq$$

$$\leq \frac{y_2(x, u)}{n \chi(x, u)} = \Psi_2(x, u(x)) \leq \Psi_2(x, M)$$

(3.23)

at every point $x \in \Omega_2$.

Since $u(x) \in V_2^n(\Omega) \subset \Omega$, then the following formula

$$\omega(K_2^n \det \left( \frac{\partial^2 \phi}{\partial p_i \partial p_j} \right), v_2, \Omega) =$$

$$= \int_{\Omega_2} K_2^n(\Omega u(x)) \det \left( \frac{\partial^2 \phi(\Omega u(x))}{\partial p_i \partial p_j} \right) \det(u_{ij}(x)) \, dx$$
is established in [3] for the total \( R_2^n \ det(\frac{\partial^2 \psi}{\partial p_i \partial p_j}) \) - curvature of the convex function \( v_2(x) \). Thus from (3.24) and (3.23) it follows that

\[
\omega(\mathcal{R}_2^n \ det(\frac{\partial^2 \phi}{\partial p_i \partial p_j})) < \int \Omega \ \psi_2(x,M) dx = \omega_2 \tag{3.25}
\]

Note that

\[
\int p^n_2 (p) \ det(\frac{\partial^2 \phi}{\partial p_i \partial p_j}) \ dp = \int p^n_2 (q) \ \mathcal{R}_2^n(q) dq = \int \mathcal{R}_2^n(q) dq.
\]

(3.26)

where all notations were explained in the beginning of the proof of Theorem 4.

From Bakelman's Theorem (see [3], [15]), inequality (3.25) and the conditions of Theorem 4 (see inequality (3.13)) it follows the validity of the estimate

\[
m - C(R_2, \omega_2) \ diam \Omega < v_2(x) < u(x) \tag{3.27}
\]

for all \( x \in \Omega \).

Now we describe the way of computing of the constant \( C(R_2, \omega_2) \) (see [3]).

First we consider the function

\[
\tau_{R_2^n}(\rho) = \int_{|p|<\rho} R_2^n(p) \ det(\frac{\partial^2 \phi(p)}{\partial p_i \partial p_j}) \ dp \tag{3.28}
\]

for all \( \rho \in [0, +\infty) \). Since \( \tau_{R_2^n}(\rho) \) is continuous and strictly increasing, then there exists its inverse \( \rho = \tau_{R_2^n}(\tau) \) for all \( \tau \in [0, \mathcal{A}(R_2^n)] \), where

\[
\mathcal{A}(R_2^n) = \int_{k^*} \mathcal{R}_2^n(u) \ dq.
\]
Now $C(R_2, \omega_2)$ is defined by the formula

$$C(R_2, \omega_2) = T_{R_2}(\omega_2),$$

where $\omega_2 = \int \psi_2(x, M)dx < A(R_2)$. 

The estimate

$$u(x) < v_1(x) < M + C(R_1, \omega_1) \text{diam } \Omega$$

can be obtained in the similar way.

The proof of Theorem 4 is completed.

3.3 The important special cases and examples.

A) The geometric maximum principle.

If the function $F$ depends on $x$ and $u$, then $\omega_1 = \omega_2 = 0$. Clearly

$$C(R_1, 0) = C(R_2, 0) = 0.$$ 

Then from (3.14) it follows that

$$m < u(x) < M$$

for all $x \in \overline{\Omega}$.

Therefore Theorem 3 is called the geometric maximum principle.

B) The Euler-Lagrange equations for the functionals

$$I(u) = \int_\Omega [F(u) + f(x, u)]dx.$$ 

The Euler-Lagrange equation for the functionals

$$I(u) = \int_\Omega [F(u) + f(x, u)]dx$$

has the following form

$$\sum_{i,k=1}^{n} \frac{\partial^2 F(u)}{\partial u_i \partial u_j} u_{ik} = \frac{\partial f(x, u)}{\partial u}.$$  

(3.29)
According to Assumption A.4

\[ F(p) = \phi(p) \text{ is a strictly convex function in } p^n ; \phi(x, u) \equiv 1 \text{ and } \]
\[ \frac{\partial^2 f(x, u)}{\partial u^2} > 0 \text{ in } \overline{\Omega} \times \mathbb{R} \text{ (i.e. } f(x, u) \text{ is a convex } C^2 \text{-function of } u \text{ for every}}
\[ \text{fixed } x \in \overline{\Omega}. \text{ Let}
\[ \frac{\partial f(x, u)}{\partial u} = f_u^+(x, u) - f_u^-(x, u)
\]

where \( f_u^+(x, u) > 0 \) and \( f_u^-(x, u) > 0 \) are positive and negative parts of \( \frac{\partial f(x, u)}{\partial u} \).

Thus inequalities (3.12) and (3.13) (see the statement of Theorem 3) can be written in the following form

\[ \frac{1}{n^n} \int_{\overline{\Omega}} L f_u^-(x, u)^n dx < \text{mes} K^* \quad (3.12') \]

and

\[ \frac{1}{n^n} \int_{\overline{\Omega}} L f_u^+(x, u)^n dx < \text{mes} K^* \quad (3.13') \]

Since

\[ \text{a finite positive number if the asymptotic cone of } F(p) \text{ is non-degenerate;}
\]

\[ \text{mes } K^* = + \infty \text{ if this cone degenerate,}
\]

then the inequalities (3.12') and (3.13') are non-trivial restrictions only for the case of a non-degenerate asymptotic cone of \( F(p) \).

The special case

\[ f(x, u) = f(x) \cdot u, \]
which relates to many applications (see [3], [14], [13], [15]), leads to the inequalities

\[
\frac{1}{n^n} \int_{\Omega} f^n(x) dx < \text{mes } K^*.
\]

The last inequalities provide estimates of solutions of the Dirichlet problem for elliptic equations

\[
\sum_{i,j=1}^{n} \frac{\partial^2 F(u)}{\partial p_i \partial p_j} u_{ik} = f(x).
\]
References


