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THE n-TH ROOTS OF SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we shall prove the following theorem: Let K be a differential field of characteristic zero. Let φ and ψ be elements of a differential field extension of K such that (i) $\varphi \neq 0$ and $\psi \neq 0$; (ii) φ and ψ satisfy non-trivial linear differential equations with coefficients in K, say, $P(\varphi)=0$ and $Q(\psi)=0$; (iii) $\varphi = \psi^n$ for some positive integer n such that $n \geq \text{ord } P$. Then the logarithmic derivatives of φ and ψ are algebraic over K. (Note that $\varphi'/\varphi = n(\psi'/\psi)$.)

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by William A.Harris,JR. and Yasutaka Sibuya

1. **Introduction:** Let K be a differential field of characteristic zero. We denote by $\mathcal{D}_K$ the ring of linear ordinary differential operators with coefficients in K, that is

$$\mathcal{D}_K = \{ \Sigma_{k=0}^{m} a_k D^k \quad ; \quad a_k \in K \ , \quad m \in N \} \ ,$$

where D denotes differentiation in any differential field extension of K and N is the set of all non-negaive integers. In a previous paper[2] we characterized those functions which together with their reciprocal satisfy linear differential equations:

THEOREM A: *Let* φ *be an element of a differential field extension of* K *such that*

(i) $\varphi \neq 0$;

(ii) $P(\varphi) = 0$ *for some* $P \in \mathcal{D}_K - \{0\}$;

(iii) $Q(1/\varphi) = 0$ *for some* $Q \in \mathcal{D}_K - \{0\}$.

Then, the logarithmic derivative of φ $(= \varphi'/\varphi)$ *is algebraic over* K .

In the present paper, utilizing a similar method we shall prove the following result:

THEOREM B: *Let* φ *and* ψ *be elements of a differential field extension of* K *such that*

(i) $\varphi \neq 0$ *and* $\psi \neq 0$;

(ii) $P(\varphi) = 0$ *for some* $P \in \mathcal{D}_K - \{0\}$;

(iii) $Q(\psi) = 0$ *for some* $Q \in \mathcal{D}_K - \{0\}$;

(iv) $\varphi = \psi^n$ *for some positive integer* n *such that*

(1.1) $$n \geq \text{ord } P \quad .$$

Then $\varphi'/\varphi = n(\psi'/\psi)$ *is algebraic over* K .

Theorem B gives the following two corollaries:

COROLLARY C: *In case* K *is the field of rational functions of* x *with coefficients in* C , *if* φ *and* ψ *satisfy the hypothese of Theorem B, then* $\varphi'/\varphi = n(\psi'/\psi)$ *is an algebraic function of* x .

COROLLARY D: *In case* K *is the quotient field of* C{x} *(the ring of convergent power series in* x *with coefficients in* C), *if* φ *and* ψ *are formal power series in* x *with coefficients in* C , *and if* φ *and* ψ *satisfy the hypothese of Theorem B, then* φ *and* $\psi \in$ C{x} *(i.e.* φ *and* ψ *are convergent).*

The following examples illustrate these results:

Example 1: The power series $\{\cos x\}^{1/n}$ and $J_\nu(x)^{1/n}$ $(n \geq 2)$ do not satisfy any linear ordinary differential equation with polynomial coefficients (cf. also [1]).

Example 2: The divergent power series $(\Sigma_{m=0}^{\infty} m!\ x^m)^{1/n}$ $(n \geq 2)$ do not satisfy any linear ordinary differential equation with coefficients in $C\{x\}$.

2. **A fundamental lemma:** The following two important steps were utilized in our previous paper[2]:

1) Set

$$P = \Sigma_{h=0}^{m+1} p_h D^h, \quad Q = \Sigma_{h=0}^{n+1} q_h D^h , \tag{2.1}$$

where m and $n \in N$; p_h and $q_h \in K$; in particular

$$p_{m+1} \neq 0, \quad q_{n+1} \neq 0 \quad . \tag{2.2}$$

For an element of f of K , let us set

$$\hat{f} = \Sigma_{h=0}^{\infty} \frac{f^{(h)}}{h!} x^h \quad \epsilon \quad K[[x]] \quad ,$$

where $K[[x]]$ is the ring of formal power series in x with coefficients in K . Then, $T(f) = \hat{f}$ defines an injective homomorphism of rings:

$$T : K \longrightarrow K[[x]]$$

such that

$$T(f') = dT(f)/dx \quad .$$

Corresponding to two operators P and Q of (2.1), let us consider two operators

$$\hat{P}=\Sigma_{h=0}^{m+1}\hat{p}_h(d/dx)^h, \quad \hat{Q}=\Sigma_{h=0}^{n+1}\hat{q}_h(d/dx)^h.$$

We assume that φ is an element of a differential field extension of K. Denote this extension by $\tilde{K}$. Then, $P(\varphi) = 0$ and $Q(1/\varphi) = 0$ imply, respectively, that the formal power series

$$\hat{\varphi} = \Sigma_{h=0}^{\infty}\frac{\varphi^{(h)}}{h!}x^h \quad \epsilon \quad K[[x]]$$

satisfies $\hat{P}(\hat{\varphi}) = 0$ and $\hat{Q}(1/\hat{\varphi}) = 0$.

Observe that (2.2) implies $1/\hat{p}_{m+1} \in K[[x]]$ and $1/\hat{q}_{n+1} \in K[[x]]$. Therefore

(2.3) $$y = \hat{\varphi}/\varphi = 1 + \Sigma_{h=1}^{\infty}\frac{\varphi^{(h)}}{\varphi\, h!}x^h$$

satisfies the differential equation

(2.4) $$y^{(m+1)} + \Sigma_{h=0}^{m}\frac{\hat{p}_h}{\hat{p}_{m+1}}y^{(h)} = 0,$$

and $u = \varphi/\hat{\varphi}$ satisfies the differential equation

(2.5) $$u^{(n+1)} + \Sigma_{h=0}^{n}\frac{\hat{q}_h}{\hat{q}_{n+1}}u^{(h)} = 0.$$

In this manner, the general case was reduced to the case of formal power series.

2) Let k be a field. We denote by $k[y_1,\dots,y_p]$ the ring of polynomials in p indeterminates $y_1, \dots, y_p$ with coefficients in k. For $F \in k[y_1,\dots,y_p]$ we set

$$w(F) = \deg_t F(\alpha_1 t,\alpha_2 t^2,\dots,\alpha_p t^p) \quad , \tag{2.6}$$

where we regard $F(\alpha_1 t,\alpha_2 t^2,\dots,\alpha_p t^p)$ as a polynomial in t whose coefficients are polynomials in $\alpha_1,\dots,\alpha_p$. The following lemma was the fundamental algebraic tool in the proof of Theorem A.

LEMMA E: *Let* n *and* p *be positive integers, and let* $F_1,\dots,F_p \in k[y_1,\dots,y_p]$. *Assume that*

(i) $w(F_p - y_p^n) < np$;

(ii) $w(F_j) = nj \quad (j = 1, \dots, p-1)$;

(iii) $w(F_j(y_1,\dots,y_j,0,\dots,0) - y_j^n) < nj \quad (j=1,\dots,p-1)$.

Then, the system of algebraic equations

$$F_j(y_1,\dots,y_p) = 0 \qquad (j = 1, \dots, p) \tag{2.7}$$

admits only a finite number of solutions in any field extension of k *and these solutions are algebraic over* k.

3. A lemma on differential equations: Let k be a field of characteristic zero, and let $k[[x]]$ be the ring of formal power series in x with coefficients in k. Then a linear ordinary differential equation

(3.1) $$y^{(m+1)} + \Sigma_{h=0}^{m} a_h(x) y^{(h)} = 0 \qquad (a_h \in k[[x]])$$

admits a canonical basis of m+1 solutions of the form:

(3.2) $$f_j = \frac{1}{j!} x^j + \Sigma_{h=m+1}^{\infty} f_{j,h} x^h \qquad (j=0,1,\dots,m),$$

where $f_{j,h} \in k$.

Let $\tilde{k}$ be a field extension of k . If a formal power series

(3.3) $$\varphi = \Sigma_{h=0}^{m} c_h x^h$$

with coefficients in $\tilde{k}$ satisfies (3.1), then we have

(3.4) $$\varphi = \Sigma_{j=0}^{m} (j!) c_j f_j \qquad .$$

Let us consider another linear ordinary differential equation:

(3.5) $$u^{(p+1)} + \Sigma_{h=0}^{p} b_h(x) u^{(h)} = 0 \qquad (b_h \in k[[x]]).$$

This equation also has a canonical basis of $p+1$ solutions of the form:

(3.6) $$g_j = \frac{1}{j!} x^j + \Sigma_{h=p+1}^{\infty} g_{j,h} x^h \qquad (j=0,1,\dots,p),$$

where $g_{j,h} \in k$.

As in our previous paper[2], in order to prove Theorem B it suffices to prove the following result:

LEMMA F: *There are at most a finite number of solutions of the form:*

$$(3.7)\qquad \begin{cases} y = f_0 + \Sigma_{j=1}^{m}\alpha_j f_j & (\alpha_j \in \tilde{k}), \\ u = g_0 + \Sigma_{j=1}^{p}\beta_j g_j & (\beta_j \in \tilde{k}) \end{cases}$$

of (3.1) *and* (3.5) *respectively such that*

$$(3.8)\qquad y = u^n \quad ,$$

if

$$(3.9)\qquad n \geq m+1 \quad .$$

For these solutions, the coefficients $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_p$ *are algebraic over* k .

Proof: Set $y = \Sigma_{h=0}^{\infty}\lambda_h x^h$ and $u = \Sigma_{h=0}^{\infty}\mu_h x^h$, where λ_h and $\mu_h \in \tilde{k}$. Then

$$(3.10)\qquad \begin{cases} \lambda_0 = 1 \ , \qquad \mu_0 = 1 \ , \\ \lambda_j = \frac{1}{j!}\alpha_j \qquad (j=1,\ldots,m), \\ \mu_j = \frac{1}{j!}\beta_j \qquad (j=1,\ldots,p), \\ \lambda_h = f_{0,h} + \Sigma_{j=1}^{m}(j!)f_{j,h}\lambda_j \qquad (h\geq m+1), \\ \mu_h = g_{0,h} + \Sigma_{j=1}^{p}(j!)g_{j,h}\mu_j \qquad (h\geq p+1). \end{cases}$$

In order that

(3.11) $$\Sigma_{h=0}^{\infty}\lambda_h x^h = (\Sigma_{h=0}^{\infty}\mu_h x^h)^n \quad ,$$

it is necessary and sufficient that

(3.12) $$\lambda_h = \sum_{h_1+\cdots+h_n=h,\, h_j\geq 0} \mu_{h_1}\cdots\mu_{h_n} \qquad \text{for } h\geq 0.$$

Let us define $w(F)$ for $F \in k[\mu_1,\ldots,\mu_p]$ in the same way as (2.6), and let us make the following observations:

(1) utilizing the last formulas of (3.10), we regard μ_h $(h\geq p+1)$ as elements of $k[\mu_1,\ldots,\mu_p]$; then

(3.13) $$w(\mu_h) \leq p < h \qquad \text{for} \quad h \geq p+1 \quad ;$$

(2) utilizing $\lambda_h = \Sigma\, \mu_{h_1}\cdots\mu_{h_n}$ for $h = 1, \ldots, m$ (cf. (3.12)), we can regard $\lambda_1,\ldots,\lambda_m$ as elements of $k[\mu_1,\ldots,\mu_p]$; then

(3.14) $$w(\lambda_j) = j \qquad (j=1,\ldots,m);$$

(3) utilizing the formulas for λ_h $(h\geq m+1)$ of (3.10), we can regard λ_h $(h\geq m+1)$ as elements of $k[\mu_1,\ldots,\mu_p]$; then

(3.15) $$w(\lambda_h) \leq m < h \qquad \text{for} \quad h \geq m+1 \quad .$$

Now, regarding (3.12) for $h \geq m+1$ as a system of algebraic equations on $\mu_1, \ldots, \mu_p$, let us consider the system of p equations:

$$(3.16) \qquad \lambda_h - \sum_{h_1+\ldots+h_n=h,\, h_j \geq 0} \mu_{h_1} \cdots \mu_{h_n} = 0$$

where $h = n, 2n, \ldots, pn$. Set

$$(3.17) \qquad F_j(\mu_1, \ldots, \mu_p) = \lambda_{jn} - \sum_{h_1+..h_n=nj} \mu_{h_1} \cdots \mu_{h_n} \qquad (j=1,\ldots,p).$$

Note that, if (3.9) is satisfied, we have

$$(3.18) \qquad w(\lambda_{jn}) < nj \qquad (j=1,\ldots,p) \quad (\text{cf. } (3.15)).$$

Therefore,

$$(3.19) \qquad w(F_j) \leq nj \qquad (j=1,\ldots,p)$$

if (3.9) is satisfied. Furthermore, since

$$F_j(\mu_1, .., \mu_j, 0, .., 0) = \lambda_{jn} - \Sigma \mu_{h_1} \cdots \mu_{h_n}$$

if $\mu_{j+1}=0, \ldots, \mu_p=0$, where the sum is over all n-tuples $(h_1, \ldots, h_n)$ such that $h_1+..+h_n=nj$, $h_j \geq 0$, it follows that

$$(3.20)\quad w(F_j(\mu_1,\ldots,\mu_j,0,\ldots,0) - \mu_j^n) < nj \qquad (j=1,\ldots,p).$$

Thus applying Lemma E to (3.16) we can complete the proof of Lemma F, and thus Theorem B.

References:

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2) Harris,W.A.JR. and Y.Sibuya, The reciprocals of solutions of linear ordinary differential equations, IMA Preprint Series No.72, University of Minnesota, May 1984; to appear in Advances in Mathematics.