Thin Plates with Rapidly Varying Thickness, and Their Relation to Structural Optimization

by

Robert V. Kohn
Courant Institute of Mathematical Sciences
New York, NY 10012

and

Michael Vogelius
Department of Mathematics and
Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

This research was partially supported by NSF grant DMS-8312229 (RVK), ONR grants N00014-83-K-0536 (RVK) and N00014-85-K-0169 (MV), the Sloan Foundation, and the Institute for Mathematics and its Applications.

Introduction

There is a close relationship between problems of structural optimization and the analysis of media with microstructure. The optimal design of variable thickness plates is a case in point: for certain problems, plates with "stiffeners" formed by rapid thickness variation can be stronger per unit volume than any traditional, uniform or slowly varying plates. To resolve such a design problem one must introduce a "generalized plate model," representing the overall effect of a microstructure of stiffeners on the behavior of the plate.

One idea would be to substitute a rapidly varying thickness function into the fourth-order equation of Kirchhoff plate theory and perform some kind of "homogenization". There is, however, a physically more correct approach: it appeals directly to three-dimensional linear elastostatics on thin, rapidly-varying, plate-like domains. There are two small parameters -- the mean thickness ε and the length scale of thickness variation δ -- and one can study the asymptotics of the solution as they both tend to zero. This was the focus of our recent papers [13,14]. We showed that it makes a difference which parameter tends to zero faster. Use of the Kirchhoff plate equation with a rapidly varying thickness corresponds to the case $\varepsilon \ll \delta$. The other extreme, $\delta \ll \varepsilon$, corresponds to averaging the effect of the thickness variation first, then applying Kirchoff theory to the resulting anisotropic plate. Intermediate between these is a third case, $\varepsilon \sim \delta$, which has no such simple interpretation. For applications to optimal design it is natural to ask which alternative gives the strongest structure, and that was the focus of our most recent paper [15].

The present article is an expository review of this work and its relevance to optimization. Special attention is focused on plates with "one family of stiffeners," for which the theory is relatively complete. Much remains to be done for more general thickness variation; various open questions will be indicated as we proceed, and especially in section 6. We shall refer only to the most recent relevant articles, without any attempt at a complete survey of the extensive literature. More references on homogenization and plate theory can be found in [6,13,24] and an extensive bibliography on structural optimization is given in [3]. Recent surveys on plate optimization include [2] and [20].

1. An Optimal Design Problem

Kirchhoff plate theory models the behavior of symmetric, variable-thickness plates under transverse loads. It specifies the vertical displacement w_0 as the solution of an elliptic equation

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left(M_{\alpha\beta\gamma\delta} \frac{\partial^2 w_0}{\partial x_\gamma \partial x_\delta} \right) = F \tag{1.1}$$

on the midplane domain ω, with appropriate boundary conditions at the plate edges $\partial\omega$. The tensor $M_{\alpha\beta\gamma\delta}$ relates bending moment to midplane curvature; it depends on the plate's thicknss 2h and on the constant elastic moduli $B_{ijk\ell}$ of the material from which the plate is made, through the formula

$$M_{\alpha\beta\gamma\delta} = \frac{2}{3} h^3 \tilde{B}_{\alpha\beta\gamma\delta} , \tag{1.2}$$

where

$$\tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - B_{\alpha\beta 33}\, B_{\gamma\delta 33} \,/\, B_{3333} \,.$$

(The Hooke's law tensor $B_{ijk\ell}$ is assumed to satisfy the usual symmetries $B_{ijk\ell} = B_{jik\ell} = B_{k\ell ij}$, and to have the midplane as a plane of elastic symmetry.) For an isotropic material, $\tilde{B}$ is given by

$$\tilde{B}_{1111} = \tilde{B}_{2222} = E/(1-\nu^2)$$

$$\tilde{B}_{1122} = \tilde{B}_{2211} = E\nu/(1-\nu^2)$$

$$\tilde{B}_{1212} = \tilde{B}_{1221} = \tilde{B}_{2112} = \tilde{B}_{2121} = E/2(1+\nu) \;,$$

where ν devotes Poisson's ratio and E is Young's modulus. The right side of (1.1) is the load per unit midplane area.

For simplicity, we shall discuss only plates that are clamped at the edges; this means that

$$w_0 = \frac{\partial w_0}{\partial_n} = 0 \quad \text{at } \partial\omega \,.$$

The principle of minimum energy gives an alternate characterization of w_0 as the minimizer of

$$\text{(1.3)} \qquad \frac{1}{2}\int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w}{\partial x_\gamma \partial x_\delta} - \int_\omega Fw$$

in the Sobolev space $\overset{\circ}{H}{}^2(\omega)$. The compliance L is the work done by the load,

$$L = \int_\omega F w_0 = \int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w_0}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w_0}{\partial x_\gamma \partial x_\delta} .$$

By (1.3), it has the variational characterization

(1.4)
$$\begin{aligned} L &= -\int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w_0}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w_0}{\partial x_\gamma \partial x_\delta} + 2\int_\omega F w_0 \\ &= \max_{w \in \overset{\circ}{H}{}^2(\omega)} \left(-\int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w}{\partial x_\gamma \partial x_\delta} + 2\int_\omega F w\right) . \end{aligned}$$

For a given load F, we think of $L = L(h)$ as a functional of the thickness h. It represents an overall measure of the plate's rigidity under F. Therefore it is natural to consider the problem of <u>optimization for minimum compliance</u>: we seek to minimize $L(h)$ among all plates with prescribed volume and specified minimum and maximum thickness, <u>i.e.</u> among all h such that

(1.5) $$h \in L^\infty(\omega) , \; h_{min} \le h \le h_{max} \text{ and } \int_\omega h\,dx = c .$$

It is now widely recognized that for some choices of F and h_{max}/h_{min} <u>this optimal design problem will have no solution</u>. The difficulty is easy to understand physically. We anticipate that formation of "stiffeners" by means of an oscillatory thickness could improve the strength of the plate. Since tall, thin beams are stronger than short, fat ones, the strength should increase as the stiffener width tends to zero. If there is no optimum scale for the oscillation, then there will be no optimal h. (A more precise version of this argument will be presented in section 2.)

Numerical manifestations of this phenomenon have been observed in

[1,9]. For certain loads F and sufficiently large ratios h_{max}/h_{min}, numerical methods for minimizing L(h) are seen to display instabilities. The computed solutions become strongly mesh-dependent, with "stiffeners" (oscillations of the thickness between h_{min} and h_{max}) forming on the same scale as the mesh size.

Mathematically, the point is that L(h) is not weak* lower semicontinuous on the space (1.5) of admisible h's. There will surely be a minimizing sequence $\{h_n\}$ which approaches the optimal behavior, and (after passage to a subsequence) it will have a weak* limit h_∞. But he compliance can jump up in the limit, and in that case h_∞ will not be an optimum.

Clearly there is something unsatisfactory about the formulation of a design problem that has no solution. One way out is to <u>restrict</u> the design space by imposing a pointwise or integral bound on $|\nabla h|$ (cf. [5]). The other, we think more natural alternative is to <u>extend</u> the design space by allowing plates with stiffeners or rapidly varying thickness [4,10,17]. This entails introduction of a class $\mathcal{D}$ of "generalized plate-thicknesses" and an extension $\tilde{L}$ of L to $\mathcal{D}$ such that

(1.6a) For each $\tilde{h} \in \mathcal{D}$ the generalized compliance $\tilde{L}(\tilde{h})$ is realizable by a limit of ordinary plates. In other words, there exists a sequence $\{h_n\}$ satisfying (1.5) for which $\tilde{L}(\tilde{h}) = \lim_{n\to\infty} L(h_n)$.

(1.6b) The functional $\tilde{L}$ attains its minimum value on $\mathcal{D}$.

The first condition assures that the stiffeners have been modelled correctly, and hence that the underlying problem has not been altered. In particular, it implies that inf L = inf $\tilde{L}$. The second condition says that the class $\mathcal{D}$ of generalized thickness variations is "large enough". It promises that nothing would be gained (for this design problem) by considering further extensions of the design space.

The new problem of minimizing $\tilde{L}$ on $\mathcal{D}$ is sometimes called a <u>full relaxation</u> of the original design problem. (The reader is warned, however, that this term is used slightly differently in the calculus of variations, for example in [12].) An extension to some intermediate class of plate models satisfying (1.6a) but not (1.6b) could be called a <u>partial relaxation</u>. Finding a partial relaxation requires the correct modelling of a particular class of plates with rapidly varying thickness. Finding a full relaxation is more difficult: it requires understanding just which types of stiffeners or rapidly varying thicknesses can occur in an optimal structure. This remains in general an unsolved problem, but the easier case of plates with a "single family of stiffeners" is fairly well in hand. We shall discuss it in the next section.

As if finding a relaxation of the original design problem were not trouble enough, there is also the further difficulty of its relation to three-dimensional elasticity. This will be treated in sections 3 and 4, where we describe a class of three-dimensional "plates" with rapidly-varying thickness which are correctly modelled by homogenization of the Kirchhoff plate equation (1.1). The analysis shows, however, that use of the Kirchhoff theory above represents a loss of information: plates with more rapid thickness variation require a different model. Section 5 discusses the implications of this for structural optimization.

Though our discussion of the need for relaxation has focused on questions of existence, the relaxed problem is as important for computation as it is for the theory. Even partial relaxation may be advantageous for numerical use. Numerical minimization of a fully relaxed $\bar{L}$ will be free of the instabilities experienced using L; also, experience suggests that $\bar{L}$ will have fewer local minima than L. Finally, since $\bar{L}$ is known to achieve its minimum, one can obtain qualitative information about extremal designs by studying the first-order optimality conditions for $\bar{L}$.

2. Rapid Variation and Relaxation of the Compliance Functional

In order to relax the design problem, we must consider how rapid variations in h affect the compliance. There is a general theory of homogenization of periodic structures, which addresses precisely this sort of question [6,24]. It characterizes the vertical midplane displacement $\tilde{w}_0$ - in the limit as the length scale of the oscillation tends to zero - a the solution of (1.1) with a new, effective rigidity $\tilde{M}_{\alpha\beta\gamma\delta}$. The limiting compliance is correspondingly $\int_\omega F\tilde{w}_0$.

The simplest case is that of a plate made from an isotropic material using "stiffeners in the x_2 direction." This means that h is a function of x_1 only, independent of x_2. We obtain oscillations on a length scale δ by taking the particular form

$$h_\delta = H(x_1,\ x_1/\delta) \tag{2.1}$$

where $H(x_1,\eta_1)$ is periodic in the second variable with period 1, and sufficiently smooth in the first variable. If w_δ is the solution of

(1.1) with $h = h_\delta$, then it is an exercise in homogenization to see that w_δ tends, as $\delta \to 0$, to the solution $\tilde{w}_0$ of

$$(2.2) \qquad \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left(\tilde{M}_{\alpha\beta\gamma\delta} \frac{\partial^2 \tilde{w}_0}{\partial x_\gamma \partial x_\delta} \right) = F$$

with

$$(2.3) \qquad \begin{aligned}
\tilde{M}_{1111} &= \frac{2}{3} \frac{E}{1-\nu^2} \, \overline{H(x_1,\cdot)^{-3}}^{\,-1} \\
\tilde{M}_{2222} &= \frac{2}{3} E \, \overline{H(x_1,\cdot)^3} + \frac{2}{3} \frac{E\nu^2}{1-\nu^2} \, \overline{H(x_1,\cdot)^{-3}}^{\,-1} \\
\tilde{M}_{1122} &= \tilde{M}_{2211} = \frac{2}{3} \frac{E\nu}{1-\nu^2} \, \overline{(H(x_1,\cdot)^{-3})}^{\,-1} \\
\tilde{M}_{1212} &= \tilde{M}_{2112} = \tilde{M}_{1221} = \tilde{M}_{2121} = \frac{E}{3(1+\nu)} \, \overline{H(x_1,\cdot)^3} \, .
\end{aligned}$$

Here $\overline{H(x_1,\cdot)^3}$ denotes the average of the periodic function $H(x_1,\cdot)^3$ with respect to its second variable, and similarly for $\overline{H(x_1,\cdot)^{-3}}$. If $H = H(x_1)$ is independent of η_1, i.e. if there is no rapid variation, then (2.3) naturally agrees with (1.2). The convergence of w_δ towards $\tilde{w}_0$ is in the weak topology on $\overset{\circ}{H}{}^2$; it follows that the compliances converge

$$(2.4) \qquad L(h_\delta) = \int_\omega F w_\delta \longrightarrow \int_\omega F \tilde{w}_0 = \tilde{L} \, ,$$

and also that $w_\delta \to \tilde{w}_0$ uniformly on ω.

To see the advantage of rapid thickness variation, we consider oscillatory perturbations of a smoothly varying $\bar{H}$:

$$H_\lambda(x_1,\eta_1) = \overline{H}(x_1) + \lambda\phi(\eta_1) , \qquad \int_0^1 \phi(\eta_1)d\eta_1 = 0 .$$

If $h_{min} < \bar{H} < h_{max}$ then the same will be true for H_λ when λ is small enough. Calculation gives that

$$\frac{d}{d\lambda}\overline{H_\lambda^3} = 0 , \quad \frac{d^2}{d\lambda^2}\overline{H^3} = 6\,\overline{H}\,\overline{\phi^2}$$

$$\frac{d}{d\lambda}(\overline{H_\lambda^{-3}})^{-1} = 0 , \quad \frac{d^2}{d\lambda^2}(\overline{H_\lambda^{-3}})^{-1} = -12\,\overline{H}\,\overline{\phi^2}$$

at $\lambda = 0$. We see from (2.3) that a small, oscillatory perturbation always decreases $\tilde{M}_{1111}$ and $\tilde{M}_{1122}$, while it increases $\tilde{M}_{1212}$. In the physical range $0 < \nu < \frac{1}{2}$, $0 < E$, it also increases $\tilde{M}_{2222}$. This is entirely reasonable, since stiffeners should resist twisting and lengthwise bending, but should be rather weak under bending in the orthogonal direction.

We assert that for suitable loads F, the compliance is decreased by such perturbations. Indeed, let $\tilde{M}_{\alpha\beta\gamma\delta}$ be the effective rigidity of the (oscillatory) perturbed geometry, $M_{\alpha\beta\gamma\delta}$ that of the (smooth) $\overline{H}$, and suppose that the solution w_λ of (2.2) satisfies

$$\left|\frac{\partial^2 w_\lambda}{\partial x_1^{\,2}}\right|^2 << \left|\frac{\partial^2 w_\lambda}{\partial x_2^{\,2}}\right|^2 + \left|\frac{\partial^2 w_\lambda}{\partial x_1 \partial x_2}\right|^2 .$$

Then the perturbed compliance $\tilde{L}$ satisfies

$$\tilde{L} = -\int_\omega \tilde{M}_{\alpha\beta\gamma\delta} \frac{\partial^2 w_\lambda}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w_\lambda}{\partial x_\gamma \partial x_\delta} + 2 \int_\omega F w_\lambda$$

$$< - \int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w_\lambda}{\partial w_\alpha \partial x_\beta} \frac{\partial^2 w_\lambda}{\partial x_\gamma \partial x_\delta} + 2 \int_\omega F w_\lambda$$

$$\leq L(\bar{H}) \ ,$$

and therefore the oscillations have improved the compliance. Intuitively, deformations of the desired type are expected (at least away from the edge of the plate) whenever F oscillates rapidly enough with respect to x_2.

The preceding discussion is easily extended to a "single family of stiffeners" with smoothly varying profile and direction. This corresponds to choosing

$$h_\delta = H(x;\ e(x)\cdot x/\delta) \ , \ |e(x)|^2 = 1 \ ,$$

in place of (2.1). The stiffeners are then orthogonal to the field of unit vectors $e(x)$; their profile is determined by $H(x_1, x_2; \eta)$, which should be periodic in the real variable η with period 1. Since homogenization is local, the effective rigidity at any $x \in \omega$ will be given by (2.3) in the orthogonal coordinate system which takes the x_1 axis parallel to $e(x)$. The case of axially symmetric plates with circumferential stiffeners is an especially natural one. It was treated in [9,10], and rapid thickness variation was found to be advantageous for loads of the form $F = \cos k\theta$ when $k \geq 4$.

Returning for simplicity to plates with stiffeners in the x_2 direction (i.e. h a function of x_1 alone), we formulate a relaxation by following the ideas of [10]. The space $\mathcal{D}$ of "generalized thickness variations" consists of pairs $\tilde{h} = (g,\theta)$. The first element is restricted by $h_{min} \leq g(x_1) \leq h_{max}$; it represents the "minimum height of the stiffeners." The second is constrained by $0 \leq \theta(x_1) \leq 1$, and plays the role of the stiffener density. Both g and θ are assumed to be measurable, but not necessarily continuous. The rigidity tensor $\tilde{M}_{\alpha\beta\gamma\delta}$ corresponding to $\tilde{h}$ is defined by (2.3) with

$$H(x_1;\eta_1) = \begin{cases} h_{max} & 0 < \eta_1 \leq \theta(x_1) \\ g(x_1) & \theta(x_1) < \eta_1 \leq 1 , \end{cases}$$

and the corresponding compliance $\tilde{L}(\tilde{h})$ is obtained by solving (2.2).

This defines at least a partial relaxation, by virtue of (2.4). We believe that it is the full relaxation, though this has not yet been established with mathematical rigor. Optimality conditions for the analogous relaxation of axisymmetric plates are presented in [10]; they determine the circumstances under which "stiffeners" occur in an optimal structure.

The situation becomes much more complicated if more general thickness variation is allowed. For "two or more families of stiffeners" the effective rigidity can only be expressed in terms of the solutions of certain fourth-order equations on the period cell (see e.g. [13]). For a "single family of stiffeners" with unspecified direction, there is the possibility of the direction itself becoming oscillatory. And there are more general geometries to consider, analogous to he composites of rank two or more considered in [16], see also [23]. One route to a full relaxation involves seeking "optimal bounds" for the effective rigidity of a plate in terms of h_{min}, h_{max}, and the mean thickness $\bar{h}$; another involves the study of lower semicontinuity and the calculus of variations. Both approaches have successfully treated related problems [12,17,19,21] and it seems reasonable to hope for further progress soon.

3. An Apparent Physical Contradiction.

The primary justification for the plate equation (1.1) is that it follows from the equations of linear elasticity on the three-dimensional plate domain in the limit as the thickness tends to zero. Specifically, let Ω_ε denote the plate domain

$$\Omega_\varepsilon = \{(x_1,x_2,x_3): \ (x_1,x_2) \in \omega \ , \ |x_3| < \varepsilon\, h(x_1,x_2)\} \ ,$$

and let $\underline{u}^\varepsilon$ be the linearly elastic displacement, satisfying:

$$(3.1) \qquad \begin{aligned} &\operatorname{div} \sigma(\underline{u}^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon \\ &\sigma(\underline{u}^\varepsilon)\cdot\underline{n}^\varepsilon = \tfrac{1}{2}\,\varepsilon^3(0,0,F|n_3^\varepsilon|) \quad \text{for } |x_3| = \pm\,\varepsilon h \\ &\underline{u}^\varepsilon = 0 \quad \text{for } (x_1,x_2) \in \partial\omega \ . \end{aligned}$$

Here $\sigma(\underline{u}^\varepsilon)$ denotes the stress $B_{ijk\ell}\, e_{k\ell}$ associated to the strain $e_{k\ell} = \frac{1}{2}(\partial u_k^\varepsilon/\partial x_\ell + \partial u_\ell^\varepsilon/\partial x_k)$, and $\underline{n}^\varepsilon$ is the outward unit normal to Ω_ε. Using either the dual variational principles [18] or a direct asymptotic expansion of u^ε [11], one can show that the (rescaled) energy converges as $\varepsilon \to 0$ to that given by the Kirchhoff model:

$$\lim_{\varepsilon\to 0} \varepsilon^{-3}\Big(\frac{1}{2}\int_{\Omega_\varepsilon} \sigma(\underline{u}^\varepsilon)\cdot e(\underline{u}^\varepsilon) - \frac{1}{2}\varepsilon^3 \int_{x_3=\pm\varepsilon h} F|n_3^\varepsilon|u_3^\varepsilon\Big)$$

$$(3.2) \qquad = \frac{1}{2}\int_\omega M_{\alpha\beta\gamma\delta} \frac{\partial^2 w_0}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 w_0}{\partial x_\gamma \partial x_\delta} - \int_\omega F\, w_0 ,$$

where w_0 solves (1.1).

The convergence (3.2) holds provided that h and F are fixed and sufficiently smooth. However, we have seen that relaxing the design problem requires introduction of "generalized plate models" based on rapidly varying thicknesses. Since this appears to violate the hypotheses on which (3.2) is based, it is important to ask:

(Q1) What is the relation between the homogenized plate equation (2.2), its relaxed compliance $\bar{L}$, and solutions of the equations of three-dimensional elasticity?

This leads naturally to a second, much harder question:

(Q2) Consider linear elasticity on the three dimensional plate domain $|x_3| < \varepsilon\, h_\varepsilon(x_1,x_2)$, $(x_1,x_2) \in \omega$, with vertical load $\varepsilon^3 F$ per unit midplane area. Can one characterize (in a simple way) the lowest limiting point that may be obtained for the compliances, as $\varepsilon \to 0$, by letting h_ε vary with ε subject to $h_{min} \le h_\varepsilon \le h_{max}$, $\int_\omega h_\varepsilon = c$?

These issues are the focus of our recent work [13,14,15] which will be summarized in the next two sections. Our answer to the first question is that (2.2) correctly models plates whose thickness varies on a length scale that is large compared with the mean thickness but small compared with the plate diameter. We do not have a complete answer to (Q2); however, we are able to compute the effect of thickness variation on a length scale comparable to or shorter than the mean thickness. These lead to different results than those predicted by the Kirchhoff theory.

4. Summary of our generalized plate models.

The first important step towards answering the questions Q1 and Q2 is to understand the influence of rapid thickness variation on the (rescaled) compliance of a "three-dimensional plate",

$$\frac{1}{2}\int_{\partial_{\pm}\Omega_\varepsilon} F|n_3^\varepsilon|u_3^\varepsilon = \varepsilon^{-3}\Big(\varepsilon^3\int_{\partial_{\pm}\Omega_\varepsilon} F|n_3^\varepsilon|u_3^\varepsilon - \int_{\Omega_\varepsilon} \sigma(\underline{u}^\varepsilon)\cdot e(\underline{u}^\varepsilon)\Big) ,$$

where $\partial_{\pm}\Omega_\varepsilon$ denotes the upper and lower surfaces of the plate. We shall assume that the thickness is locally periodic with period $\delta = \delta(\varepsilon)$; in other words

$$(4.1) \qquad \Omega_\varepsilon = \{(\underset{\sim}{x},x_3):\ \underset{\sim}{x} \in \omega,\ |x_3| < \varepsilon H(\underset{\sim}{x};\underset{\sim}{x}/\delta)\} ,$$

where $H(\underset{\sim}{x};\underset{\sim}{\eta})$ is periodic in its second variable with period 1. Though the dependence of δ on ε could in principle be arbitrary, it is convenient to choose

$$(4.2) \qquad \delta = \varepsilon^a , \quad 0 < a < \infty .$$

We are convinced that (4.1) and (4.2) represent no less of generality toward the problem of characterizing the limiting compliances.

The asymptotics of $\underline{u}^{\varepsilon}$ as $\varepsilon \to 0$ can be studied using the method of multiple scales [13,14]. (A closely related problem, involving smoothly varying thickness but rapidly varying elastic moduli, was analyzed simultaneously and independently by Caillerie [8].) The main conclusion is that the energy of the elastic displacement $\underline{u}^{\varepsilon}$ approaches that of a fourth-order equation

(4.3)
$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(M_{\alpha\beta\gamma\delta}\frac{\partial^2 w}{\partial x_\gamma \partial x_\delta}\right) = F \quad \text{in } \omega$$

$$w = \frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\omega .$$

Since the compliance is -2 times the energy, the compliances also converge. The solution w of (4.3) is the limiting vertical displacement; therefore $M_{\alpha\beta\gamma\delta}$ represents an <u>effective rigidity tensor</u>, relating midplane curvature to bending moments in the limit as $\varepsilon \to 0$. It depends not only on $H(\underset{\sim}{x};\underset{\sim}{\eta})$ - corresponding to the geometry of the variation - but also on the choice of scaling, specifically on whether $a < 1$, $a = 1$, or $a > 1$. Formulae are given in [13] for each of the rigidities $M^{a<1}$, $M^{a=1}$, and $M^{a>1}$, in terms of the solutions of certain periodic boundary-value problems.

The tensor $M^{a<1}$ has a simple interpretation: it is precisely what one obtains by homogenizing the plate equation (1.1). Thus for "one family of stiffeners" and an isotropic elastic law, $M^{a<1}$ is given by (2.3). In particular, this analysis yields an answer to our first question (Q1):

(A1) <u>Rapid variation of the thickness of the "three-dimensional plate is correctly modelled by homogenizing the Kirchhoff plate equation (only) if it occurs on a length scale larger than the average thickness.</u>

There is an equally simple interpretation for $M^{a>1}$, involving homogenization of a rough boundary. Consider the system of elasticity on the domain (4.1) with ε fixed, as $\delta \to 0$. The limiting displacement will solve a new elasticity problem on the "smoothed" plate domain

$$\{(\underset{\sim}{x}, x_3): \ \underset{\sim}{x} \in \omega \ , \ |x_3| < \varepsilon \max_{\underset{\sim}{\eta}} H(\underset{\sim}{x};\underset{\sim}{\eta})\}$$

with non-constant, effective elastic moduli $B^{\prime}_{ijk\ell}(\underset{\sim}{x})$ (see the appendix of [13] for details). Applying Kirchhoff plate theory to this "smoothed" structure corresponds to taking the limit $\varepsilon \to 0$, and it yields precisely the rigidity tensor $M^{a>1}$. (This calculation remains at present merely formal. The homogenization of rough boundaries was made rigorous for a scalar equation in [7], but the system of elasticity presents additional difficulties.)

The intermediate case a = 1 has, unfortunately, no such simple interpretation. The tensor $M^{a=1}$ depends on auxiliary functions $\underset{\sim}{\phi}^{\alpha\beta}(\underset{\sim}{x};\underset{\sim}{\eta})$, obtained by solving the elastotatic boundary value problem

$$\frac{\partial}{\partial \eta_j} \Sigma_{ij} (\underset{\sim}{\phi}^{\alpha\beta}) = 0 \quad \text{in } Q(\underset{\sim}{x})$$

$$(4.4) \qquad \Sigma_{ij}(\underset{\sim}{\phi}^{\alpha\beta})\nu_j = \begin{cases} \eta_3 \tilde{B}_{i\gamma\alpha\beta}\nu_\gamma & i=1,2 \\ 0 & i=3 \end{cases} \quad \text{on } \partial_+ Q(\underset{\sim}{x}) ,$$

$$\underset{\sim}{\phi}^{\alpha\beta} \text{ periodic in } \underset{\sim}{\eta} .$$

Here $Q(\underset{\sim}{x})$ denotes the rescaled period cell

$$Q(\underset{\sim}{x}) = \{\underset{\sim}{\eta} : |\eta_3| < H(\underset{\sim}{x};\underset{\sim}{\eta})\} ,$$

$\Sigma_{ij}(\underset{\sim}{\phi})$ is the stress $B_{ijk\ell} E_{k\ell}(\underset{\sim}{\phi})$ associated to the strain $E_{k\ell}(\underset{\sim}{\phi}) = \frac{1}{2}(\partial\phi_k/\partial\eta_\ell + \partial\phi_\ell/\partial\eta_k)$ and $\underset{\sim}{\nu}$ is the outward normal to Q; notice that $\underset{\sim}{x}$ enters (4.4) only as a parameter. The formula for $M^{a=1}$ is

$$(4.5) \qquad M^{a=1}_{\alpha\beta\gamma\delta} = \frac{2}{3}\,\overline{h^3}\,\tilde{B}_{\alpha\beta\gamma\delta} - \overline{\int_{-H}^{H} \Sigma_{ij}(\underset{\sim}{\phi}^{\alpha\beta})E_{ij}(\underset{\sim}{\phi}^{\gamma\delta})\,d\eta_3} ,$$

where as usual the overbar denotes an average over (η_1,η_2). In most cases the actual calculation of $M^{a=1}$ must be done numerically, by solving a finite element or finite difference approximation of (4.4).

There is a sense in which the a = 1 model includes the other two. For any $\lambda > 0$, one can apply (4.4) - (4.5) to the λ-periodic function $H_\lambda(\underset{\sim}{x};\underset{\sim}{\eta}) = H(\underset{\sim}{x};\underset{\sim}{\eta}/\lambda)$; this amounts to taking $\delta = \lambda\varepsilon$ in (4.1). The tensor so obtained converges to $M^{a<1}$ (rigorously) as $\lambda \to \infty$, and to $M^{a>1}$ (formally) as $\lambda \to 0$; see [14] for details.

The convergence of three-dimensional elastotatics to our generalized plate model has been proved with mathematical rigor only for the intermediate case a = 1. In [14] we considered a sufficiently regular $H = H(\underset{\sim}{\eta})$, taken for simplicity not to depend on the "slow" variable $\underset{\sim}{x}$. We established, among other things, that the relative energy error is of order $\sqrt{\varepsilon}$ as $\varepsilon \to 0$. The main ingredients of the proof are a pair of integral estimates, an averaging lemma, and lots of integration by parts. The first integral estimate is a version of Korn's inequality for thin domains with mean thickness ε and thickness variation on length scale $\geq \varepsilon$, making explicit the dependence of the "constant" on ε. The second inequality asserts a weak form of Kirchhoff's hypothesis for the elastostatic displacement $\underset{\sim}{u}^{\varepsilon}$, as a consequence of the symmetries of the problem (3.1). The averaging lemmas serve to replace certain rapidly-varying expressions by their mean values; they quantify the rate at which a periodic function converges (weakly) to its mean as the period tends to zero. A convergence proof could probably be given for the $a < 1$ case using similar methods, but new ideas seem required to handle $a > 1$. A particular stumbling block is Korn's inequality for Ω_ε: as $\varepsilon \to 0$, the "constant" blows up faster when $a > 1$ than it does for $a \leq 1$.

5. Comparison of the models for one family of stiffeners.

We have seen that three-dimensional elasticity supports several different models for thin plates with rapidly varying thickness, depending on how the length scale of thickness variation compares with the mean thickness. For applications to structural optimization, it is natural to ask: which scaling gives the strongest structure? A rather complete answer is available in the case of plates with "one family of stiffeners in a specified direction."

One way of comparing the "strength" of two plates is to compare the quadratic forms defined by their rigidities M and $\tilde{M}$. We say that M is weaker than $\tilde{M}$ if $M \leq \tilde{M}$, in other words if

$$M_{\alpha\beta\gamma\delta}\ t_{\alpha\beta}\ t_{\gamma\delta} \leq \tilde{M}_{\alpha\beta\gamma\delta}\ t_{\alpha\beta}\ t_{\gamma\delta}$$

for every symmetric 2×2 tensor $t_{\alpha\beta}$. By the variational characterization (1.4), the weaker plate has the greater compliance under any load. It may happen, of course, that neither plate is weaker than the other; in that case the ordering of the compliances will be load-dependent.

We shall suppose for simplicity that the thickness h is periodic and depends on x_1 alone:

$$(5.1) \qquad h(x) = \varepsilon\ H(x_1/\varepsilon^a)\ .$$

As remarked in section 2, the generalization to slowly varying stiffener geometry or direction - <u>e.g.</u> to axisymmetric plates with circumferential stiffeners - is immediate. Since we wish to compare the different scalings $a < 1$, $a = 1$, and $a > 1$, it is convenient to fix the choice of H in (5.1). The corresponding effective rigidities $M^{a<1}$, $M^{a=1}$, and $M^{a>1}$ are as discussed in section 4; notice that all three plates have the same average thickness $\varepsilon\bar{H}$.

Surprisingly, the ordering of the strengths of these structures depends on the three-dimensional elastic material used to make them. For plates made from an isotropic elastic material using one family of stiffeners, the $a > 1$ scaling is weakest and the $a < 1$ scaling strongest:

(5.2) $M^{a>1} \leq M^{a=1} \leq M^{a<1}$ <u>for an isotropic elastic law.</u>

The left inequality in (5.2) holds even for anisotropic materials:

(5.3) $M^{a>1} \leq M^{a=1}$ <u>in general</u> (for one family of stiffeners).

However, the right inequality in (5.2) can fail for some anisotropic laws and some choices of the profile H:

(5.4) $M^{a>1} \lesseqgtr M^{a=1}$ <u>in general</u> (even for one family of stiffeners).

The proofs, presented in [15], make use of variational principles for each of the three effective rigidity tensors $M^{a<1}$, $M^{a=1}$, and $M^{a>1}$.

More quantitative comparison requires numerical calculation of the effective rigidity tensors. Figure 1 shows the rescaled cross-section (the graph of H) for one of the examples presented in [13]. By cutting along the horizontal midline and glueing together opposite ends of the stiffeners we get a new cross-section with the same amount of material, Figure 2. Tables 1 and 2 list the effective rigidities for these geometries, using an isotropic material with Young's modulus E = 1.0 and Poisson's ration $\nu = 0.25$. The cases $a < 1$ and $a > 1$ were done using explicit formulas, which are easily derived as in [13]. Calculating $M^{a=1}_{\alpha\beta\gamma\delta}$ requires finding the energies of a pair of two-dimensional cell problems, one involving plane strain and the other antiplane shear. This was done using the FEARS finite element code, developed at the University of Maryland. (Since the cross-section in figure 2 is not of the form (5.1), the analysis of [13,14,15] does not strictly speaking apply. However, the arguments presented there are easily modified to include this case.)

The data in tables 1 and 2 naturally satisfy (5.2). More interesting is the observation that $M^{a=1}$ and $M^{a>1}$ are quite close, while $M^{a<1}_{1212}$ is much greater than $M^{a=1}_{1212}$ in each case. It is not surprising that figure 2 is much stiffer than figure 1; we understand that N. Olhoff and his collaborators are currently studying the use of geometries such as that in figure 2 for compliance optimization.

Our assertion (5.4) concerning the anisotropic case is based on an explicit counterexample: for an elastic law of the form

$$B_{iiii} = \lambda + 2\mu$$

(5.5) $$B_{iijj} = \lambda \qquad i \neq j$$

$$B_{1212} = \mu \quad , \quad B_{1313} = B_{2323} = \mu'$$

with μ' sufficiently large, and for a thickness profile of the form

(5.6) $$H(\eta_1) = h_0(1 + \sigma\cdot\cos 2\pi\eta_1)$$

with h_0 and σ sufficiently small, we showed that

(5.7) $$M^{a<1}_{1111} < M^{a=1}_{1111} \text{ and } M^{a<1}_{2222} < M^{a=1}_{2222} \ .$$

Since the inequality

$$M^{a=1}_{1212} \leq M^{a<1}_{1212}$$

holds for the elastic law (5.5) (and more generally whenever $\tilde{B}_{1112} = 0$), this gives an example in which $M^{a=1} - M^{a<1}$ is indefinite. We note that (5.5) is only a minor modification of the isotropic law, which corresponds to the choices

$$(5.8) \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} , \quad \mu = \mu' = \frac{E}{2(1+\nu)} .$$

We hoped at first that this would lead to examples of practical significance. If the difference $M^{a=1}_{1111} - M^{a<1}_{1111}$ were large, then use of the a = 1 plate (with stiffeners in the direction of greatest bending) could be advantageous for some design problems. However, practical numerical examples of (5.7) are exceedingly hard to find. For the geometries of figures 1 and 2, our calculations give $M^{a=1} \leq M^{a<1}$ for the elastic law (5.5). The best example we found - and it's not a good one - is shown in Figure 3. With λ and μ chosen by (5.8) with $E = 1.0$ and $\nu = 0.25$, one obtains easily that

$$(5.9) \qquad M^{a<1}_{1111} = .192901 \times 10^{-2} .$$

(The $a < 1$ model does not depend on the value of μ'.) On the other hand

$$(5.10) \qquad M^{a=1}_{1111} = \frac{64}{45} \int_0^{1/2} \left(\frac{5}{32} - \frac{1}{16}\, t\right)^3 dt - \mathcal{E} ,$$

where $\mathcal{E}$ is the strain energy of the cell problem (5.5) - which depends, of course, on μ'. For the isotropic material with $E = 1.0$ and $\nu = 0.25$, μ' is equal to 0.4; for the calculations reported here we took $\mu' = 5.0$. (Larger values of μ' did not increase the significance of the findings, but only served to increase the numerical error.) The value of $\mathcal{E}$ computed by FEARS was 0.7119×10^{-4}, using 574 elements in a subdivision based on just 1/4 of the unit cell. The code has a built-in error estimator, which says in this case that the energy is off by at most 0.95%. Extensive practical experience with FEARS indicates that the true error exceeds the estimated error by at most 75%. We are thus convinced that the correct value of $\mathcal{E}$ in (5.10) is bounded above by $.7244 \times 10^{-4}$, which yields

$$M^{a=1}_{1111} \geq 0.192951 \times 10^{-2} .$$

In view of (5.9), this gives a numerical example of (5.7) - but hardly one of any practical significance!

The implications of these results for structural optimization are clear: for the design of plates made from an isotropic material using one family of stiffeners with a specified direction, attention may be restricted to the $a < 1$ model (obtained by homogenizing the Kirchhoff plate equation). For plates made from an anisotropic material this is not true, but our numerical experimentation suggests that even so little will be lost in practice by a restriction to Kirchhoff theory.

Care is advised in extending these conclusions to more general situations, such as plates with two or more families of stiffeners. We proved in [15] that $M^{a<1}$ is strongest and $M^{a>1}$ weakest if the elastic law satisfies $B_{\alpha\beta33} = 0$ - which includes the isotropic law with Poisson's ration ν set equal to zero. But in general the relative strengths appear to depend on both the elastic moduli and the form of the thickness variation.

6. Directions for the future.

We indicate some of the areas in which further work is needed.

a) The correctness of our $a > 1$ model as a limit of three-dimensional elasticity has yet to be proved.

b) We still hope for an example (perhaps using two families of stiffeners) where the $a = 1$ model leads to a significantly stronger structure than does homogenization of the Kirchhoff theory.

c) Section 2 described a relaxation of the compliance optimization problem for Kirchhoff theory with "one family of stiffeners in a specified direction." It remains to prove an existence theorem, i.e. to establish that this is the full relaxation. The corresponding problem without the restriction to "one family of stiffeners" is more difficult. A relaxation (partial or full?) has been proposed in [23]; we understand that Lurie and Cherkaev also have made progress in this direction.

d) We have shown that Kirchhoff theory suffices for the optimization of plates made from an isotropic material with one family of stiffeners. This result seems likely to fail, however, for more general geometries. Therefore the best relaxation would be one based not on Kirchhoff theory but instead on three-dimensional elasticity - i.e. on our generalized plated models.

e) We have discussed only the compliance optimization of linearly elastic plates. In plasticity, the analogous problem is to maximize the limit multiplier of a given load. Rapid thickness variation arises naturally in that context, too, and relaxed formulations have been proposed in [22,25]. It remains, however, to prove that the models used there correctly represent the asymptotic behavior of three-dimensional plastic structures. It also remains to check whether a full relaxation has been achieved.

REFERENCES

[1] J.-L. Armand and B. Lodier, "Optimal design of bending elements", Int. J. Num. Meth. Engng., 13, 1978, pp. 373-384.

[2] J.-L. Armand, K.A. Lurie, and A.V. Cherkaev, "Optimal control theory and structural design," in Optimum Structural Design, Vol. 2, R.H. Gallagher, E. Atrek, K. Ragsdell, and O.C. Zienkiewicz, eds., J. Wiley and Sons, 1983.

[3] N.V. Banichuk, Problems and methods of optimal structural design. Plenum Press, New York and London, 1983.

[4] M. Bendsøe, These proceedings.

[5] M.P. Bendsøe, "On obtaining a solution to optimization problems for solid, elastic plates by restriction of the design space." J. Struct. Mech., 11, (1983/84), pp. 501-521.

[6] A. Bensoussan, Lions J.-L. and Papanicolaou, G., Asymptotic analysis for periodic structures, North-Holland, Amsterdam, 1978.

[7] R. Brizzi and J.P. Chalot, Homgénéisation de frontière. Thèse, Université de Nice, 1978.

[8] D. Caillerie, "Thin elastic and periodic plates", Math. Methods Applied Sci. 6, 1984, pp. 159-191.

[9] K.-T. Cheng and N. Olhoff, "An investigation concerning optimal design of solid elastic plates," Int. J. Solids Structures, 17, 1981, pp. 305-323.

[10] K.-T. Cheng and N. Olhoff, "Regularized formulation for optimal

design of axisymmetric plates." Int. J. Solids Structures, 18, 1982, pp. 153-169.

[11] P.G. Ciarlet and P. Destuynder, "A justification of the two-dimensional linear plate model" J. Mecanique, 18, 1979, pp. 315-344.

[12] R.V. Kohn and G. Strang, "Optimal design and relaxation of variational problems," to appear in Comm. Pure Appl. Math.

[13] R.V. Kohn and M. Vogelius, "A new model for thin plates with rapidly varying thickness." Int. J. Solids Structures, 20, 1984, pp. 333-350.

[14] R.V. Kohn and M. Vogelius, "A new model for thin plates with rapidly varying thickness II: a convergence proof." Quart. Appl. Math., 43, 1985, pp. 1-22.

[15] R.V. Kohn and M. Vogelius, "A new model for thin plates with rapidly varying thickness III: Comparison of different scalings." Quart. Appl. Math., to appear.

[16] K.A. Lurie and A.V. Cherkaev, "G-closure of some particular sets of admissible material characteristics for the problem of bending of thin plates," J. Opt. Th. Appl., 42, 1984, pp. 305-316.

[17] K.A. Lurie, A.V. Fedorov, and A.V. Cherkaev, "On the existence of solutions to some problems of optimal design for bars and plates.", J. Opt. Theory Appl. 42, 1984, pp. 247-281.

[18] D. Morgenstern, "Herleitung der plattentheorie aus der dreidimensionalen elasticitets theorie." Arch. Rational Mech. Anal., 4, 1959, pp. 145-152.

[19] F. Murat and L. Tartar, "Calcul des variations et

homogénéisation," Ecole d'été d'homogénéisation, Breau sans Nappe, 1983, to be published by Eyrolles.

[20] N. Olhoff and J.E. Taylor, "On structural optimization", J. Appl. Mech., 50, 1983, pp. 1134-1151.

[21] U.E. Raitum, "On optimal control problems for linear elliptic equations," Soviet Math. Dokl. 20, 1979, pp. 129-132.

[22] G.I.N. Rozvany, N. Olhoff, K.-T. Cheng, and J. Taylor, "On the solid plate paradox in structural optimization," J. Struct. Mech., 10, 1982, pp. 1-32.

[23] G.I.N. Rozvany, T.G. Ong, R. Sandler, W.T. Szeto, N. Olhoff, and M.P. Bendsøe, "Least-weight design of perforated plates," preprint.

[24] E. Sanchez-Palencia, Non-homogeneous media and vibration theory, Lecture Notes in Physics, 127, Springer-Verlag, 1980.

[25] C.-M. Wang, G.I.N. Rozvany, and N. Olhoff, Optimal plastic design of axisymmetric solid plates with a maximum thickness constraint, Computers & Structures, 18, 1984, pp. 653-665.

	a<1	a=1	a>1
M_{1111}	.015	.012	.011
M_{1122}	.004	.003	.003
M_{2222}	.334	.334	.334
M_{1212}	.133	.006	.004

Table 1: Effective rigidities for figure 1.

	a<1	a=1	a>1
M_{1111}	.777	.687	.678
M_{1122}	.194	.172	.169
M_{2222}	.851	.845	.844
M_{1212}	.321	.262	.254

Table 2: Effective rigidities for figure 2.

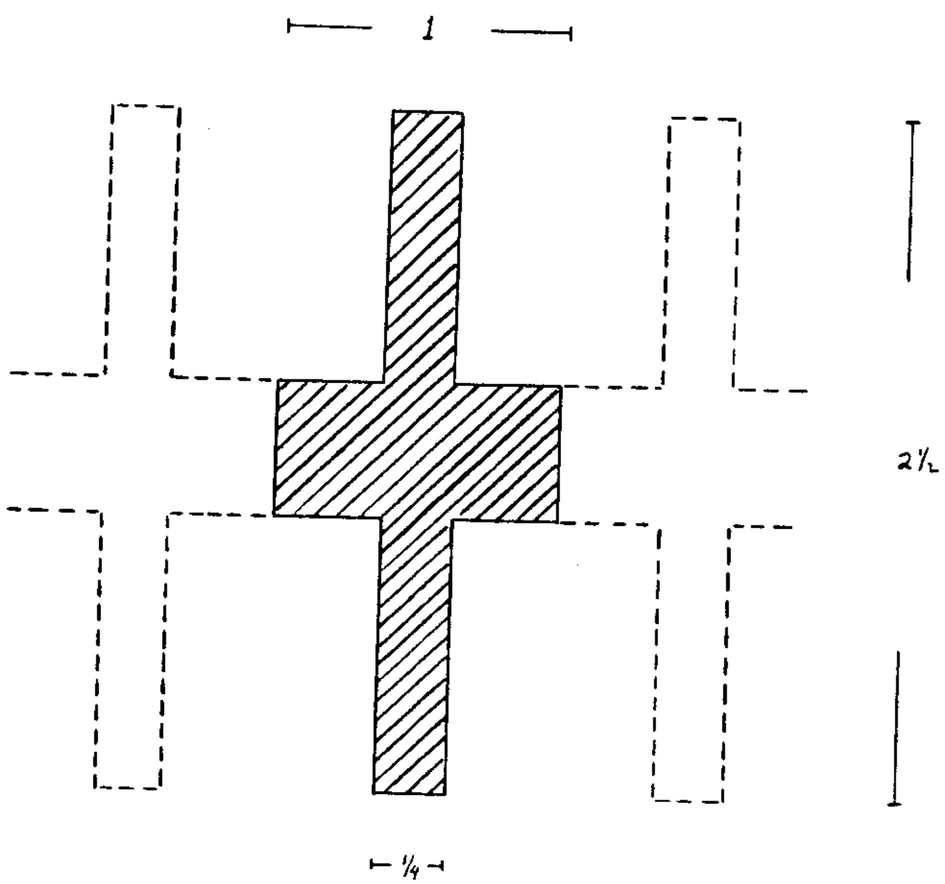

Fig. 1

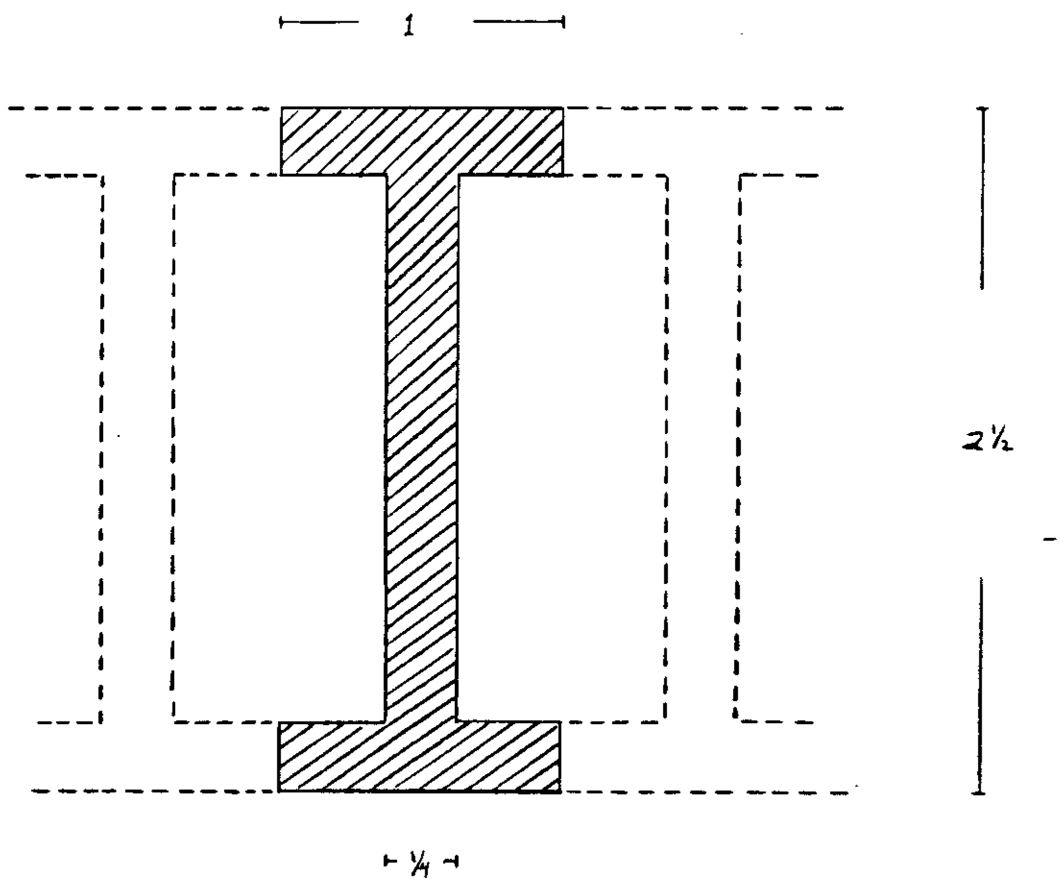

Fig. 2

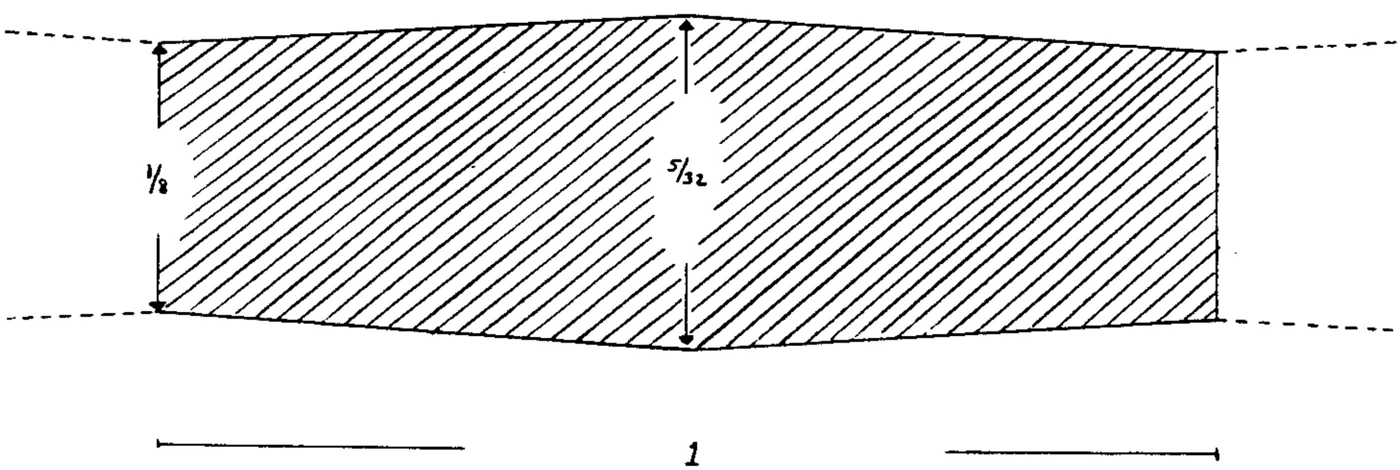

Fig. 3