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CONVEX SETS IN $\mathbb{R}^2$ AS AFFINE IMAGES OF SOME FIXED SET IN $\mathbb{R}^3$

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1. Introduction

In this paper we describe and prove the following result: Every convex and compact set in $\mathbb{R}^2$ is the union of two affine images of a fixed set in $\mathbb{R}^3$. This will follow from the following consideration: Take a convex and compact set in $\mathbb{R}^2$ and divide it into two parts by joining two furthest points in it with a straight line. (In the sequel we shall refer to such parts as "half-sets"). We shall show that every half-set is an affine image of a fixed set in $\mathbb{R}^3$, and clearly this will imply the above result.

Suppose that we restrict our attention to the class of convex, compact and symmetric sets in $\mathbb{R}^2$. There is a one to one correspondence between these sets and the class of half-sets in the plane, and in fact the two classes can be topologically identified if given suitable Hausdorff metrics. In this sense we claim, by the above result, that the convex compact and symmetric sets in $\mathbb{R}^2$ can be realized as affine images of a fixed set in $\mathbb{R}^3$.

The motivation to study this problem arose in the study of a differential inclusion

$$\frac{dz}{dt} \in G(z(t)), \; z(t) \in \mathbb{R}^n.$$

(See Aubin and Cellina [1] for a detailed discussion concerning differential inclusions). Such differential inclusions are strongly connected with control systems

$$z(t) = g(z,u), \; z(t) \in \mathbb{R}^n, \; u(t) \in U \subset \mathbb{R}^m.$$

For such systems we get a differential inclusion by defining

$$G(z) = \{g(z,u) : u \in U\}$$
(see, e.g. Boltyanskii [3] and Stassinopoulos and Vinter [6] for a demonstration of this approach). A study of single valued representations of set valued mappings, in a very general framework, is given by Ioffe [4]. In particular it is proved there that if \( G(\cdot) \) is convex-valued, then a single valued representation \( g(\cdot, \cdot) \) exists which is linear in \( u \). However, in the general context of the treatment in [4] the control set \( U \) is a subset of an infinite dimensional space (essentially the set of all continuous selections \( g(\cdot) \) of \( G(\cdot) \)). Our result implies that if \( G(\cdot) \subset \mathbb{R}^2 \) is convex and compact valued then it has a double-valued linear representation with a control set in \( \mathbb{R}^3 \), namely

\[
G(z) = [a_1(z) + A_1(z)(U)] U [a_2(z) + A_2(z)(U)]
\]

where \( a_1(\cdot) \) and \( a_2(\cdot) \) are single valued and \( A_1(z), A_2(z) \) are \( 2 \times 2 \) matrices.

2. Notations, terminology and preliminary results

For a function \( f:X \to Y \) we denote graph \( f = \{(x,f(x)): x \in X\} \) and for a set \( S \subset \mathbb{R}^n \) we denote \( \text{conv} S = \{\text{the convex hull of } S\} \). We shall consider a special kind of convex and compact sets in \( \mathbb{R}^2 \) which are defined as follows:

**Definition 2.1.** We call a set \( K \subset \mathbb{R}^2 \) an half-set if and only if there is a continuous and concave function \( f: [-1,1] \to \mathbb{R}^1 \) such that

\[
f(-1) = 0, \quad f(1) = 0 \quad \text{and} \quad K = \text{conv}(\text{graph } f).
\]

We shall call \( f \) the function corresponding to \( K \). If the function \( f \) is also piecewise linear we call \( K \) a polygonal half-set.
We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$ for either $n = 2$ or $n = 3$. Let $\mathbf{x}$ and $\mathbf{y}$ be an orthonormal basis in $\mathbb{R}^2$ and denote a point in it by $(x,y)$.

Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be an orthonormal basis in $\mathbb{R}^3$. The subspace spanned by $\{\mathbf{x}\}$ will be called the $x$-axis, and we shall use similar terminology for the subspaces spanned by $\{\mathbf{y}\}$ and by $\{\mathbf{z}\}$. The subspace spanned by $\{\mathbf{x}, \mathbf{y}\}$ will be called the $x$-$y$ plane and we shall use similar terminology for the subspaces spanned by $\{\mathbf{y}, \mathbf{z}\}$ and by $\{\mathbf{z}, \mathbf{x}\}$. We shall need the following simple Lemma:

**Lemma 2.2:** Let $K \subset \mathbb{R}^2$ be convex and compact. Then there is a $k \in \mathbb{R}^2$ and two half-sets $K_1$ and $K_2$ such that

$$K = k + A_1(K_1) \cup A_2(K_2)$$

where $A_1$ and $A_2$ are $2 \times 2$ matrices.

**Proof:** Let $p_1$ and $p_2$ be points such that

$$|p_1 - p_2| = \max\{|q_1 - q_2|, \ q_1,q_2 \in K\}.$$

Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible affine transformation which is a translation followed by a rotation and an expansion of the $x$-axis such that $\phi(p_1) = (-1,0)$ and $\phi(p_2) = (1,0)$ and denote $K' = \phi(K)$. We define

$$\gamma(x) = \max\{y:(x,y) \in K'\}$$

$$\sigma(x) = -\min\{y:(1-x,y) \in K'\}$$

then $\gamma(\cdot)$ and $\sigma(\cdot)$ are continuous and concave functions which vanish at $x = \pm 1$ and by (2.2)

$$K' = \operatorname{conv}(\operatorname{graph} \gamma) \cup \mathbb{R}_\pi \operatorname{conv}(\operatorname{graph} \sigma)$$
where $R_\pi$ is the rotation of $\pi$ radians about the origin in $\mathbb{R}^2$. This implies (2.1) and concludes the proof of the Lemma.

Our aim in the rest of this section is to prove the following claim: Given a finite collection \( \{K_1, K_2, \ldots, K_n\} \) of polygonal half-sets then there is a convex and compact set $U \subset \mathbb{R}^3$, which may depend on the collection \( \{K_1, \ldots, K_n\} \), such that each $K_i$ is an affine image of $U$. We consider now the following basic construction:

Let $K_1, K_2$ and $K_3$ be three polygonal half-sets in the $x$-$y$ plane in $\mathbb{R}^3$. To $K_i$ corresponds the function $\gamma_i$, and we assume

\begin{align*}
(2.3) & \quad \gamma_2(x) > 2\gamma_1(x) \quad \text{for all } -1 < x < 1 \\
(2.4) & \quad \gamma_3(x) > 2\gamma_2(x) \quad \text{for all } -1 < x < 1.
\end{align*}

We shall denote a clockwise rotation of $\alpha$ radians about the $x$-axis in $\mathbb{R}^3$ by $\rho_\alpha$. We now apply a rotation $\rho_\alpha$ on $\mathbb{R}^3$, which takes the set $K_3$ in the $x$-$y$ plane onto a half-set $\tilde{K}_3$ in a plane $\Sigma_3 \subset \mathbb{R}^3$. Similarly we apply a rotation $\rho_{\alpha+\theta}$ which takes $K_2$ onto $\tilde{K}_2$ in a plane $\Sigma_2 \subset \mathbb{R}^3$, and a rotation $\rho_{\alpha+2\theta}$ which takes $K_1$ onto $\tilde{K}_1$ in $\Sigma_1 \subset \mathbb{R}^3$. All the $\Sigma_i$'s contains the $x$-axis, the angle between $\Sigma_1$ and $\Sigma_2$, as well as that between $\Sigma_2$ and $\Sigma_3$ is $\theta$, and the angle between $\Sigma_3$ and the $y$-axis is $\alpha$. We assume that $\alpha$ and $\theta$ are such that

\begin{align*}
(2.5) & \quad 0 < \alpha, \theta, \alpha + \theta, \alpha + 2\theta < \frac{\pi}{2}; \quad \frac{\sin(\alpha + \theta)}{\sin \alpha} < 2; \quad \frac{\sin(\alpha + 2\theta)}{\sin(\alpha + \theta)} < 2.
\end{align*}

The boundary of $\tilde{K}_1$ in $\mathbb{R}^3$ consists of two parts: The line segment

\[ ((x,0,0); \ -1 < x < 1) \]

and the image under $\rho_{\alpha+(3-i)\theta}$ of the graph of $\gamma_i$, which is denoted by $\Gamma_i^*$ and can be parametrized as follows:
(2.6) \[ \{(\xi, \eta, \rho, \phi) : -1 < \xi < 1\} = \{(\xi, \eta, \rho, \phi) : -1 < \xi < 1\}. \]

We fix \( \xi \) and join \( \Gamma_2(\xi) \) and \( \Gamma_1(\xi) \) with a straight line, which lies in a plane parallel to the \( y - z \) plane. In this plane this segment has a slope

(2.7) \[ a(\xi) = \frac{\Gamma_2 y(\xi) - \Gamma_1 y(\xi)}{\Gamma_2 z(\xi) - \Gamma_1 z(\xi)} \]

and by the condition \( \sin(\alpha + 2\theta) < 2 \sin(\alpha + \theta) \) in (2.5) and by (2.3) it follows that \( a(\xi) \) is a positive number. Similarly, joining \( \Gamma_2(\xi) \) to \( \Gamma_3(\xi) \) we obtain a straight line segment in a plane parallel to the \( y - z \) plane and a slope

(2.8) \[ b(\xi) = \frac{\Gamma_3 y(\xi) - \Gamma_2 y(\xi)}{\Gamma_3 z(\xi) - \Gamma_2 z(\xi)} \]

which is a positive number too. We want to show that the slopes \( b(\xi) \) are larger than the slopes \( a(\xi) \) is a uniform manner, namely: There is a constant \( \beta \) such that

(2.9) \[ \sup_{|\xi|<1} a(\xi) < \beta < \inf_{|\xi|<1} b(\xi) \]

Geometrically this means that the straight line through \( \Gamma_2(\xi) \) in a plane parallel to the \( y - z \) plane and which has a slope \( \beta \), is below the two line segments which connect \( \Gamma_2(\xi) \) to \( \Gamma_1(\xi) \) and to \( \Gamma_3(\xi) \) and which lie in this same plane. Further, this implies that the following projection

(2.10) \[ \pi_\beta(x,y,z) = (x,0,z - \frac{1}{\beta} y) \]

which projects \( \mathbb{R}^3 \) onto the \( x - z \) plane along lines parallel to the \( x - y \) plane and having a slope \( \beta \), projects the set \( K_1 \cup K_2 \cup K_3 \) onto \( \lambda \cdot \rho_\pi/2(K_2) \), which is
some scalar multiple of the half-set $K_2$ rotated by $\frac{\pi}{2}$ angle to the x-z plane. This is the interpretation of (2.9) which is important to us and which we shall use in the sequel.

**Lemma 2.3.** Let $K_1, K_2, \gamma, \Gamma_i, 1 < i < 3$, $a(\xi)$ and $b(\xi)$ be as in (2.3)-(2.8). Then there is a constant $\beta$ satisfying (2.9).

**Proof:** We begin with the set $K_1 \cup K_2 \cup K_3$ in $\mathbb{R}^3$ and apply to $\mathbb{R}^3$ a rotation $\rho_{-(\alpha+\theta)}$ of angle $-(\alpha + \theta)$ about the x-axis. Thus $\Sigma_2$ is transferred to the x-y plane, and the planes $\Sigma_3$ and $\Sigma_1$ are transferred to $\Sigma_3'$ and $\Sigma_1'$, which make angles $-\theta$ and $\theta$ with the x-y plane respectively. The curves $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are transferred to the curves $\Gamma_1'$, $\Gamma_2'$ and $\Gamma_3'$. Analogous to the definition of $a(\xi)$ and $b(\xi)$ in (2.7) and (2.8) we define

$$a'(\xi) = \frac{\Gamma_{2y}(\xi) - \Gamma_{1y}(\xi)}{\Gamma_{2z}(\xi) - \Gamma_{1z}(\xi)}, \quad b'(\xi) = \frac{\Gamma_{3y}(\xi) - \Gamma_{2y}(\xi)}{\Gamma_{3z}(\xi) - \Gamma_{2z}(\xi)}.$$ 

Now both $a'(\xi)$ and $b'(\xi)$ are negative numbers, and we want to prove that

$$\sup_{|\xi|<1} a'(\xi) < -\frac{1}{\sin \theta} < \inf_{|\xi|<1} b'(\xi).$$

This means, geometrically, that the lines with slopes $-\frac{1}{\sin \theta}$ through $\Gamma_2'(\xi)$ are above the line segments which connect $\Gamma_2'(\xi)$ to $\Gamma_1'(\xi)$ and to $\Gamma_3'(\xi)$.

While applying a backwards rotation $\rho_{(\alpha+\theta)}$ all these lines with a slope $-\frac{1}{\sin \theta}$ will be transferred to lines through $\Gamma_2(\cdot)$, all having a common positive slope $\beta$ in planes parallel to the x-y plane. Moreover, these lines will be below the line segments connecting $\Gamma_2(\xi)$ with $\Gamma_1(\xi)$ and with $\Gamma_3(\xi)$, hence (2.9) will be satisfied for this value of $\beta$. Therefore, in order to conclude the proof of the Lemma, it only remains to prove (2.11). For $\Gamma_1'$, $\Gamma_2'$ and $\Gamma_3'$ we have the parametrization:
\[(\Gamma^i_{1x}(\xi), \Gamma^i_{1y}(\xi), \Gamma^i_{1z}(\xi) = (\xi, \gamma_1(\xi)\cos \theta, \gamma_1(\xi)\sin \theta)\]
\[(\Gamma^i_{2x}(\xi), \Gamma^i_{2y}(\xi), \Gamma^i_{2z}(\xi) = (\xi, \gamma_2(\xi), 0)\]
\[(\Gamma^i_{3x}(\xi), \Gamma^i_{3y}(\xi), \Gamma^i_{3z}(\xi) = (\xi, \gamma_3(\xi)\cos \theta, -\gamma_3(\xi)\sin \theta)\]

and the slopes \(a'(\xi)\) and \(b'(\xi)\) are given by
\[a'(\xi) = \frac{\gamma_2(\xi) - \gamma_1(\xi)\cos \theta}{-\gamma_1(\xi)\sin \theta}; \quad b'(\xi) = \frac{\gamma_3(\xi)\cos \theta - \gamma_2(\xi)}{-\gamma_3(\xi)\sin \theta},\]
\[a'(\xi) = -\frac{1}{\sin \theta} \left[ \frac{\gamma_2(\xi)}{\gamma_1(\xi)} - \cos \theta \right]; \quad b'(\xi) = -\frac{1}{\sin \theta} \left[ \cos \theta - \frac{\gamma_2(\xi)}{\gamma_3(\xi)} \right]\]

which, in view of (2.3) and (2.4), implies (2.9). As discussed above, this completes the proof of the Lemma. 

\[\square\]

Lemma 2.4. Let \(n > 1\) and let \(\{K_0, K_1, \ldots, K_n, K_{n+1}\}\) be a collection of polygonal half-sets. Then there is a convex and compact set \(U \subseteq \mathbb{R}^3\), depending on the collection \(\{K_0, \ldots, K_{n+1}\}\), such that for every \(1 < i < n\), \(K_i\) is an affine image of \(U\).

Proof: We claim that we can assume, without loss of generality, that
\[(2.12) \quad \gamma_{i+1}(x) > 2\gamma_i(x) \quad \text{for all} \quad -1 < x < 1\]
\[\quad \text{for every} \quad 0 < i < n.\]

Indeed the function \(\gamma_i(*)\) corresponding to \(K_i\) is piecewise linear, hence its slopes at the points \(x = +1\) and \(x = -1\) are finite and nonzero.

Therefore, by choosing appropriate positive constants \(\lambda_i\) and defining
\[\gamma_i'(x) = \lambda_i \gamma_i(x), \quad 0 < i < n + 1\]
the inequalities \(\gamma_{i+1}'(x) > 2\gamma_i'(x)\) for all \(-1 < x < 1\) will hold for every \(0 < i < n\). Thus, as we are merely interested in affine images of sets in \(\mathbb{R}^3\), we can assume (2.12).
We construct the set \( \tilde{K}_{n+1} \) in \( \mathbb{R}^3 \) as follows: Let \( \rho_{\pi/4} \) be a rotation of \( \frac{\pi}{4} \) radians about the x-axis, applied to \( \mathbb{R}^3 \). Then define

\[
\tilde{K}_{n+1} = \rho_{\pi/4}(K_{n+1}).
\]

Similarly, \( \rho_{\pi/4 + i\theta} \) is a rotation of \( \frac{\pi}{4} + i\theta \) radians about the x-axis, applied to \( \mathbb{R}^3 \), and we define

\[
\tilde{K}_{n+1-i} = \rho_{(\pi/4 + i\theta)}(K_{n+1-i}), \quad i = 0, 1, \ldots, n+1.
\]

We denote by \( \Gamma_{n+1-i} \) the curve in \( \mathbb{R}^3 \) which is the image under \( \rho_{\pi/4 + i\theta} \) of the graph of \( \gamma_{n+1-i}(\cdot), \quad 0 < i < n+1 \). We fix now some \( m, 1 < m < n \), and consider the sets \( K_{m-1}, K_m, K_{m+1} \). We assume that \( \theta > 0 \) in sufficiently small so that the following holds

\[
\frac{\pi}{4} + (n + 1)\theta < \frac{\pi}{2}; \quad \frac{\sin[\frac{\pi}{4} + (m+1)\theta]}{\sin[\frac{\pi}{4} + m\theta]} < 2 \quad \text{for} \quad m = 1, \ldots, n.
\]

Now the assumptions of Lemma 2.3 hold for \( \{K_{m-1}, K_m, K_{m+1}\} \) and there are numbers \( \beta_m > 0 \) such that

\[
\sup_{|\xi| < 1} \frac{\Gamma_{my}(\xi) - \Gamma_{(m-1)y}(\xi)}{\Gamma_{mz}(\xi) - \Gamma_{(m-1)z}(\xi)} < \beta_m < \inf_{|\xi| < 1} \frac{\Gamma_{(m+1)y}(\xi) - \Gamma_{my}(\xi)}{\Gamma_{(m+1)z}(\xi) - \Gamma_{mz}(\xi)}
\]

for \( m = 1, \ldots, n \). We define

\[
V = \bigcup_{i=1}^{m+1} \tilde{K}_i.
\]

By the geometrical interpretation we gave to (2.13) it follows that the projection \( \pi_{\beta_m} \) (recall (2.10)) projects \( V \) onto \( u_m\rho_{\pi/2}(K_m) \), some scalar multiple of the rotation of the half-set \( K_m \) to the x-z plane. Therefore there are linear transformations \( \phi_m : \mathbb{R}^3 \to \mathbb{R}^2 \) such that

\[
K_m = \phi_m(V), \quad m = 1, \ldots, n.
\]
It is easy to verify that for every set $S$ and every linear transformation $\phi$ the following holds

$$\phi(\text{conv } S) = \text{conv}(\phi(S)).$$

Since $K_m$ in (2.14) is convex, it follows that the convex and compact set

$$U = \text{conv } V$$

satisfies

$$(2.15) \quad K_m = \phi_m(U), \quad m = 1, \ldots, n,$$

which concludes the proof of the Lemma.

3. The Main Result.

We shall state and prove here our main result, namely:

**Theorem 3.1.** There is a convex and compact set $U \subset \mathbb{R}^3$ such that every convex and compact set in $\mathbb{R}^2$ is the union of two affine images of $U$.

For the proof of Theorem 3.1 we shall need one more Lemma. We denote by $B(x_0, r)$ the closed ball of radius $r$ with center $x_0$ in $\mathbb{R}^3$.

**Lemma 3.2.** There is a constant $\rho_0 > 0$ such that for every convex and compact set $U \subset \mathbb{R}^3$, which has a nonempty interior, there is an affine transformation $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$B(0, \rho_0) \subset \phi(U) \subset B(0, 1).$$

The proof of the Lemma is displayed in section 4.
Proof of Theorem 3.1. Let \( X \) denote the metric space of compact subsets in the closed unit ball in \( \mathbb{R}^3 \), endowed with the Hausdorff metric. Then \( X \) is a complete metric space. (For a verification of this statement consult Kelly and Weiss [5] page 237). We shall denote the Hausdorff metric in \( X \) by \( h(\cdot,\cdot) \).

Let \( K \) be a polygonal half set in \( \mathbb{R}^2 \) and denote by \( F_K \) the family \( \{U\} \) of all convex and compact sets in \( \mathbb{R}^3 \) with the following properties:

(i) \( K \) is an affine image of \( U \).

(ii) \( B(0,\rho_0) \subset U \subset B(0,1) \), where \( \rho_0 \) is the constant in Lemma 3.2.

For every \( K \) there are clearly sets \( U \) which satisfy just condition (i). Thus, by Lemma 3.2, \( F_K \) is nonvoid for every \( K \). We claim that \( F_K \) is closed in \( X \). To prove this, let \( U_1 \in F_K \) and

\[
\lim_{i \to \infty} h(U_i, U_0) = 0,
\]

and we have to show that \( U_0 \in F_K \). Since \( U_1 \in F_K \) there are \( k_1 \in \mathbb{R}^2 \) and \( 2 \times 3 \) matrices \( A_i \) such that

\[
K = k_1 + A_i(U_i)
\]

for every \( i > 1 \). Since \( 0 \in U_1 \) and \( K \) is bounded, (3.2) implies that the sequence \( \{k_1\}_{i=1}^{\infty} \) is bounded and contains a convergent subsequence, which we again denote by \( \{k_i\}_{i=1}^{\infty} \) and

\[
\lim_{i \to \infty} k_i = k_0.
\]

We consider the matrices \( A_i \) as elements of \( \mathbb{R}^6 \). Since each \( U_i \) contains the ball \( B(0,\rho_0) \) and \( K \) and \( k_i \) in (3.2) are bounded, it follows that the sequence \( \{A_i\}_{i=1}^{\infty} \) is bounded in \( \mathbb{R}^6 \). Therefore it contains a convergent subsequence which we again denote by \( \{A_i\}_{i=1}^{\infty} \) and
(3.4) \[ A_i \to A_0 \text{ as } i \to \infty, \text{ in } R^6. \]

We denote the Hausdorff metric for compact sets in \( R^2 \) by \( d(\cdot, \cdot) \), namely if \( Q_1 \) and \( Q_2 \) are compact sets in \( R^2 \) then

\[ d(Q_1, Q_2) = \max \{ \max_{x \in Q_1} \{ \min_{y \in Q_2} |x-y| \}, \max_{x \in Q_2} \{ \min_{y \in Q_1} |x-y| \} \}. \]

Then

(3.5) \[ d(A_i(U_i), A_0(U_0)) < d(A_i(U_i), A_i(U_0)) + d(A_i(U_0), A_0(U_0)). \]

Let \( \| \cdot \| \) be the operator norm for 2x3 matrices as linear operators from \( R^3 \) to \( R^2 \). Then by (3.4) there is a bound for the sequence \( \{ \| A_i \| \}_{i=1}^{\infty} \), thus (3.1) implies that

(3.6) \[ d(A_i(U_i), A_i(U_0)) \to 0 \text{ as } i \to \infty. \]

The boundedness of \( U_0 \) and (3.4) imply that

(3.7) \[ d(A_i(U_0), A_0(U_0)) \to 0 \text{ as } i \to \infty. \]

Thus it follows from (3.5), (3.6) and (3.7) that

\[ A_i(U_i) \to A_0(U_0) \text{ as } i \to \infty \]

in the Hausdorff metric \( d(\cdot, \cdot) \). This combined with (3.2) and (3.3) yields

\[ K = k_0 + A_0(U_0) \]

for some \( k_0 \in R^2 \) and a 2x3 matrix \( A_0 \), showing that \( U_0 \in F_K \) and proves the closedness of \( F_K \).

The space \( X \) is a compact metric space (for a verification of this claim see Blaschke [2], Chapter 2, section 18, page 62). We claim that there is a set \( U \) such that
(3.8) \[ U \in \bigcap \{ F_K : K \text{ is a polygonal half-set} \}. \]

Indeed, by Lemmas 2.4 and 3.2, the family of closed subsets of \( X \)

\[ \{ F_K : K \text{ is a polygonal half-set} \} \]

has the finite intersection property, therefore, by the compactness of \( X \), the intersection in (3.8) is nonempty and \( U \) is chosen to be any set in it.

Now let \( K_0 \) be any half set. Then there is a sequence of polygonal half-sets \( \{ K_i \}_{i=1}^{\infty} \) such that

(3.9) \[ d(K_i, K_0) \to 0 \text{ as } i \to \infty. \]

For every \( i \) the equality

(3.10) \[ K_i = h_i + B_i(U) \]

where \( h_i \in \mathbb{R}^2 \) and \( B_i \) is a \( 2 \times 3 \) matrix, holds. Arguing as above in the proof of the closedness of \( F_K \), we conclude that

\[ h_i \to h_0, \quad B_i \to B_0 \text{ as } i \to \infty, \]

where \( h_0 \in \mathbb{R}^2 \) and \( B_0 \) is a \( 2 \times 3 \) matrix, and

\[ h_i + B_i(U) \to h_0 + B_0(U) \]

in the Hausdorff metric \( d(\ldots) \). This, in view of (3.9) and (3.10), yields

\[ K_0 = h_0 + B_0(U). \]

We have thus proved that every half-set \( K \) is an affine image of a fixed convex and compact set \( U \) in \( \mathbb{R}^3 \). By Lemma 2.2, every convex and compact set in \( \mathbb{R}^2 \) is the union of the affine images of two half-sets. These two facts imply that every convex and compact set in \( \mathbb{R}^2 \) is the union of two affine images of a fixed \( U \) in \( \mathbb{R}^3 \), concluding the proof of the Theorem.
4. Proof of Lemma 3.2.

We denote by \( S(x_0, r) \) the closed disc of radius \( r \) with center \( x_0 \) in \( \mathbb{R}^2 \). We first prove a claim analogous to Lemma 3.2, but concerned with sets in \( \mathbb{R}^2 \): Let \( V \) be a convex and compact set \( \mathbb{R}^2 \), which has a nonempty interior in \( \mathbb{R}^2 \). Then there is an affine transformation \( \psi: \mathbb{R}^2 \to \mathbb{R}^2 \) such that

\[
S(0, \nu_0) \subset \psi(V) \subset S(0,1)
\]

where \( \nu_0 > 0 \) is some constant which does not depend on \( V \).

To prove this claim let \( p_1, p_2 \in V \) be such that

\[
|p_1 - p_2| = \max\{|q_1 - q_2|: q_1, q_2 \in V\},
\]

and let \( \psi_1: \mathbb{R}^2 \to \mathbb{R}^2 \) be an affine transformation which is a translation followed by a rotation and an expansion of the x-axis such that

\[
\psi_1(p_1) = (-1,0), \quad \psi_1(p_2) = (1,0)
\]

and denote

\[
V_1 = \psi_1(V).
\]

Let \( \lambda \) be defined by

\[
\lambda = \max\{|y|: (x,y) \in V_1\}
\]

and \( \psi_2: \mathbb{R}^2 \to \mathbb{R}^2 \) the linear transformation

\[
\psi_2(x,y) = (x, \frac{y}{\lambda}), \quad \text{and let}
\]

\[
V_2 = \psi_2(V_1).
\]

Then clearly
(4.1) \[ V_2 \subset S(0, \sqrt{2}). \]

Also \( V_2 \) contains a triangle with vertices \( T_1 T_2 T_3 \) where

\[ T_1 = (-1,0), \quad T_2 = (+1,0) \]

and \( T_3 \) is either on the line segment \( \{(x,y): y = 1, |x| < 1\} \) or on the line segment \( \{(x,y): y = -1, |x| < 1\} \). In any case it contains some disc \( S((x_0, y_0), 0.1) \), where \( |x_0| < 1, |y_0| < 1 \). If \( \psi_3: \mathbb{R}^2 \to \mathbb{R}^2 \) is the affine transformation

\[ \psi_3(x,y) = (x - x_0, y - y_0) \]

then, recalling (4.1) and \( |x_0| + |y_0| < 2 \) we get

\[ S(0, 0.1) \subset \psi_3(V_2) \subset S(0,10). \]

Thus defining \( \psi_4(x,y) = (\frac{1}{10} x, \frac{1}{10} y) \) and

\[ \psi = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1 \] we obtain

\[ S(0, 0.01) \subset \psi(V) \subset S(0,1) \]

proving our claim with \( v_0 = 0.01 \).

Now let \( s_1, s_2 \in U \) be such that

\[ |s_1 - s_2| = \max\{|t_1 - t_2|: t_1, t_2 \in U\} \]

and let \( \phi_1: \mathbb{R}^3 \to \mathbb{R}^3 \) be an affine transformation which is a translation followed by a rotation and an expansion of the z-axis such that

\[ \phi_1(s_1) = (0,0,-1), \quad \phi_1(s_2) = (0,0,1). \]

We denote
\[ U_1 = \phi_1(U) \]

and \( U_1 \) is bounded between the two planes \( z = 1 \) and \( z = -1 \) (namely \( (x,y,z) \in U_1 \) implies \( |z| < 1 \)). We consider the \( R^2 \) space which is the plane spanned by \( \{\hat{x},\hat{y}\} \) and let

\[ V = \{(x,y,0)\} \cap U_1, \]

Then \( V \subset R^2 \) is convex, compact and with nonempty interior. Also \((0,0) \in V\) since \((0,0,0) \in U_1\). Thus by our claim in the beginning of the proof there is an affine transformation \( \psi: R^2 \to R^2 \) such that

\[ S(0,v_0) \subset \psi(V) \subset S(0,1). \]

Let \( \psi \) be given as follows

\[ \psi(u) = Au + a \]

where \( u \in R^2 \) and \( A: R^2 \to R^2 \) a matrix. Then since \((0,0) \in V\) it follows from (4.3) and (4.2) that

\[ |a| < 1. \]

We extend \( A \) to a linear transformation \( L \) on all \( R^3 \) by defining

\[ L((x,y,0)) = A((x,y)) \]
\[ L((0,0,1)) = (0,0,1) \]

and we define the affine transformation \( \phi_2: R^3 \to R^3 \)

\[ \phi_2((x,y,z)) = L((x,y,z)) + a \]

where \( a \) is as in (4.2) and (4.4).

\( S(0,1) \) is a disc in the plane \( z = 0 \), and let \( Q_+ \) be the smallest convex
cone containing \(((0,0,1) + a) \cup S(0,1)\), and let \(Q_\pm\) be the smallest convex cone containing \(((0,0,-1) + a) \cup S(0,1)\). Then also denote
\[
C_+ = \{(x,y,z): \ z > -1\} \cap Q_+ \\
C_- = \{(x,y,z): \ z < 1\} \cap Q_-.
\]
We claim that
\[
\phi_2(U_1) \subset C_+ \cup C_-.
\]
To prove (4.6), let \((x,y,z) \notin C_+ \cup C_-\). If \(z = 0\) then \(x^2 + y^2 > 1\), namely \((x,y,0) \notin S(0,1)\), but by (4.2) we have
\[
\phi_2(U_1) \cap \{(x,y,0)\} \subset S(0,1),
\]
demonstrating that \((x,y,0) \notin \phi_2(U_1)\).

If \(z < 0\) and \((x,y,z) \notin C_+ \cup C_-\), then in particular
\[
(x,y,z) \notin C_+
\]
and assume, to get a contradiction, that
\[
(x,y,z) \in \phi_2(U_1).
\]
Then \(\phi_2(U_1)\) contains the straight line segment which connects \((x,y,z)\) to \((0,0,1) + a\), which by (4.7) intersects the \(R^2\) plane \(\{z = 0\}\) in a point outside \(S(0,1)\). This, however, contradicts (4.2). A similar contradiction occurs if we assume \((x,y,z) \notin C_+ \cup C_-\), \(z > 0\) and \((x,y,z) \notin \phi_2(U_1)\), thus (4.6) is proved.

It is easy to see that there is a ball \(B(0,R)\) such that \(C_+ \cup C_- \subset B(0,R)\), no matter what \(a\) is, as long as it satisfies (4.4). Namely, if we emphasize the dependence of \(C_+\) and \(C_-\) on \(a\) by writing \(C_+(a)\) and \(C_-(a)\), then
\[ \bigcup_{|a| < 1} [C_+(a) \cup C_-(a)] \subseteq B(0, R) \]

for some \( R > 0 \). This together with (4.6) implies that

\[ (4.8) \quad \phi_2 \circ \phi_1(U) \subseteq B(0, R) \]

where \( R \) does not depend on \( U \).

Now let \( q_+ \) and \( q_- \) be defined by

\[ q_+ = \text{conv}\{[(0,0,1) + a] \cup S(0, \nu_0)\} \]

\[ q_- = \text{conv}\{[(0,0,-1) + a] \cup S(0, \nu_0)\} \].

Then by (4.2) and the convexity of \( \phi_2(U_1) \) we have that

\[ (4.9) \quad q_+ \cup q_- \subseteq \phi_2(U_1) \]

It is easy to see that there is an \( \epsilon_0 > 0 \) which does not depend on \( a \) such that \( B(0, \epsilon_0) \) is contained in \( q_+ \cup q_- \) as long as \( a \) satisfies (4.4).

Namely, if we emphasize the dependence of \( q_+ \) and \( q_- \) on \( a \) by writing \( q_+(a) \) and \( q_-(a) \), then

\[ B(0, \epsilon_0) \subseteq \bigcap_{|a| < 1} [q_+(a) \cup q_-(a)] \]

for some \( \epsilon_0 > 0 \). This, together with (4.9), implies that

\[ (4.10) \quad B(0, \epsilon_0) \subseteq \phi_2 \circ \phi_1(U) \]

where \( \epsilon_0 \) does not depend on \( U \). Let \( \phi_3: \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by

\[ \phi_3(x,y,z) = (\frac{1}{R} x, \frac{1}{R} y, \frac{1}{R} z), \]

then (4.8) and (4.10) imply that

\[ B(0, \frac{\epsilon_0}{R}) \subseteq \phi_3 \circ \phi_2 \circ \phi_1(U) \subseteq B(0,1) \]
thus the assertion of the Lemma is proved with $\rho_0 = \varepsilon_0/R$, which is a constant independent on $U$.

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