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FALL QUARTER

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CONTINUUM PHYSICS AND PARTIAL DIFFERENTIAL EQUATIONS

Seminar Abstracts
Volume 1 Fall Quarter

The Fall Quarter calendar included two workshops, (IMA preprints 98 and 115) and the seminars whose abstracts are presented here. We attempted to limit the seminars to two a week, not always successfully. They were informal and many were of an expository character. They exhibit some evidence of the brio of the participants and the high level of adrenalin which characterizes this program.

Haim Brezis
Constantine Dafermos
Jerry Ericksen
David Kinderlehrer

Scientific Committee
On Leray's problem of steady Navier-Stokes flow past a body in the plane

Charles Amick

Let \( K \subset \mathbb{R}^2 \) be a compact set, set \( \mathcal{U} = \mathbb{R}^2 - K \), and \( \Gamma = \partial \mathcal{U} \). We consider the problem of finding a solution \((p,u)\) of the steady Navier-Stokes equation in the exterior domain \( \mathcal{U} \):

\[
\begin{align*}
- \nu \Delta u + (u \cdot \nabla) u &= - \nabla p, \\
\nabla \cdot u &= 0, \\
\n\Gamma &= \partial \mathcal{U}, \\

u &= 0 \text{ on } \Gamma
\end{align*}
\]

with

\( u(x) + u_0 \) as \( |x| \to \infty \).

Here \( \nu > 0 \) is the given kinematic viscosity, \( u_0 \in \mathbb{R}^2 - \{0\} \) is a given constant vector, \( p \) denotes the pressure and \( u = (u_1,u_2) \) the velocity field. A natural way to proceed was suggested by Leray in 1933: for each large \( R > 0 \), let \((p^R,u^R)\) denote a solution of (1) - (3) in \( \mathcal{U}_R = \mathcal{U} \cap \{|x| < R\} \) with \( u^R = u_0 \) on \( \{|x| = R\} \). It was shown then that

\[
\int_{\mathcal{U}_R} |\nabla u^R|^2 < \text{const.},
\]

and so as \( R \to \infty \) a suitable subsequence converges on compact sets to a solution \((p^L,u^L)\) of (1) - (3), but the behavior of \( u^L \) at infinity remained a mystery. Indeed, it was not obvious that \( u^L \) was bounded at infinity or that \( u^L \) was even non-trivial. For a solution of (1)-(2) in \( \mathcal{U} \) or \( \mathcal{U}_R \), the vorticity

\[
\omega = \frac{\partial}{\partial x_2} u_1 - \frac{\partial}{\partial x_1} u_2
\]

satisfies a maximum principle: \(- \nu \Delta \omega + u \cdot \nabla \omega = 0\), and the total head pressure \( \psi = p + 1/2|u|^2 \) satisfies a one-sided principle: \(- \nu \Delta \psi + u \cdot \nabla \psi = -\nu \omega^2\). Gilbarg and Weinberger exploited these properties in two papers, and were able to give a number of results:

(A) \( p^L(x) \) has a limit, say \( 0 \), as \( |x| \to \infty \);

(B) \( \omega^L(x) = O(|x|^{-3/4}) \) at \( \infty \);

(C) there exists a constant vector \( u_\infty \) such that
\[ 2\pi \int_0^\infty |u(r,\theta) - u_\infty|^2 d\theta \to 0 \text{ as } r \to \infty. \]

It is not known whether \( u_\infty = u_0 \). In this note we state some new results for \((p^L, u^L)\).

**THEOREM 1.** (i) The velocity field \( u^L \) is non-trivial; that is, \( u^L \) vanishes at only isolated points of \( \omega \).

(ii) \( u^L(x) \to u_\infty \) as \( |x| \to \infty \).

(iii) If \( u_\infty \neq 0 \), then the vorticity and total-head approach their limits 0 and \( \frac{1}{2} |u_\infty|^2 \), respectively, at an exponential rate away from the direction \( u_\infty \).

The main new ingredient needed to prove this theorem is a maximum principle for the quantity \( \gamma = \psi - \psi_\omega \), where \( \psi \) denotes the streamfunction: \( u = (u_1, u_2) = (\psi x_2, -\psi x_1) \). One finds from (1)-(2) that \( -\nu \Delta \gamma + (u+q) \cdot \nabla \gamma = 0 \) where

\[ q = (-2u^2 + 2\psi(-u_2, u_1))/(\nu^2 + \psi^2). \]

One must also use properties of various physical quantities along level curves of \( \omega \) or \( \gamma \).

In another paper, Gilbarg and Weinberger considered solutions \((p, u)\) of (1)-(3) with finite Dirichlet norm (5). (The Leray solution \((p^L, u^L)\) being such a case.) They showed that \( u(x) = o(\log |x|) \) at infinity, and that if \( u \in L_\infty(\omega) \), then (C) holds for some \( u_\infty \). A variation of the arguments for Theorem 1 gives

**THEOREM 2.** If \((p, u)\) satisfies (1)-(3) and \( \nu u \in L_2(\omega) \), then \( u \in L_\infty(\omega) \), and parts (ii) and (iii) of Theorem 1 hold.
MOLECULAR STRUCTURE

AND

THE FINITE DEFORMATION AND OSCILLATIONS OF A RUBBER CORD

by

Millard F. Beatty
Professor
Department of Engineering Mechanics
University of Kentucky
Lexington, KY 40506

and

Senior Fellow
Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, MN 55455

ABSTRACT

It was shown by some experiments [1,2] done at the turn of the century that the pitch of the sound generated by plucking a sufficiently stretched rubber cord is very nearly constant. This phenomenon is analyzed for three models of rubber materials, and the results are compared with experimental data [3]. It is shown that the virtually constant frequency phenomenon is a molecular network, finite extensibility effect that is controlled mainly by the molecular chain number. Some additional experimental results [4,5] based upon finite deformation analysis of other phenomena are mentioned briefly.
References


Convergence of solutions of nonlinear elliptic systems: an example of quantization.

H. Brezis

The lecture describes a joint work with J.M. Coron to appear in the Archive for Rational Mech. Anal. Our original motivation comes from a question raised by J. Serrin: let \( \Gamma_n \) be a sequence of curves in \( \mathbb{R}^3 \) such that \( \Gamma_n \to 0 \) and let \( \Sigma_n \) be a sequence of surfaces of constant mean curvature one, spanned by \( \Gamma_n \); does \( \Sigma_n \) converge to a sphere? Our results are the following:

1) Let \( \Sigma_n \) be any large bubble spanned by \( \Gamma_n \) and obtained via the construction of Brezis-Coron (see Comm. Pure Appl. Math, 1984). Then indeed \( \Sigma_n \) converges to a sphere of radius one passing through \( 0 \).

2) In general, if we assume that the areas of the \( \Sigma_n \) remain bounded, then \( \Sigma_n \) converges to a finite union of spheres of radius one.

The proof relies on the analysis of the behaviour of a sequence \((u^n)\) of solutions of the elliptic system:

\[
\begin{cases}
\Delta u^n = 2u^n_x \wedge u^n_y & \text{on } \Omega \\
u^n = \gamma^n & \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 < 1\} \) and \( \gamma^n \to 0 \) on \( \partial \Omega \). It turns out that in general

\[
u^n(z) = \sum_{i=1}^{p} \omega_i \left( \frac{z - a_i}{\varepsilon_n} \right)
\]

where \( p \) is a fixed integer, \( a_i \in \Omega \) and \( \varepsilon_i \to 0 \) as \( n \to \infty \); \( \omega_i \) is a solution of the system

\[
\begin{aligned}
\Delta \omega &= 2\omega_x \wedge \omega_y \\
\int |\nabla \omega|^2 &< \infty, \omega(\infty) = 0.
\end{aligned}
\]
Moreover all solutions of (*) have been completely classified. In other words the sequence \((u^n)\) picks a finite number of points in \(\Omega\) where a singularity develops by concentration; elsewhere \(u^n \to 0\) in a uniform sense.

It is expected that other problems in physics and mechanics have similar features because of their invariance under scaling (or under some more general group action).
A NEW ELECTROMECHANICAL COUPLING PHENOMENON
IN THE ELECTROOPTIC CERAMIC PLZT7/65/35

by

Peter J. Chen
Sandia National Laboratories
Albuquerque, New Mexico 87185

A new electromechanical coupling phenomenon has been detected in the
electrooptic ceramic PLZT/65/35 with the aid of a dual beam laser inter-
ferometer system. Specifically, when a low amplitude dc voltage is applied
across a virgin specimen of the material the specimen will deflect in the direc-
tion opposite to the positive direction of the voltage with no measurable accom-
panying axial strain. The observation remains valid when the polarity of the
voltage is switched. When ac voltages are applied, the vibrational patterns and
the sense of the mechanical displacements depend on the frequency ranges of the
applied voltages. In particular, they change when the specimen goes through
those specific mechanical resonances with nodal rings only, and not those with
nodal rings and nodal diameters.

The existence of "non classical" resonances has been observed in many
ferroelectric and electrooptical materials.
SYMMETRY AND BIFURCATION FOR THE TRACTION PROBLEM
IN NONLINEAR ELASTOSTATICS

David Chillingworth

This is a report on joint work with J.E. Marsden and Y.-H. Wan [1982-83],
which should be consulted for full details and related references.
We consider a body $B$ (compact 3-manifold in $\mathbb{R}^3$ with smooth boundary) in
equilibrium under body forces $b_0$ and surface traction $\tau_0$ and ask: what
equilibrium configurations $\phi$ exist when the load $\lambda_0 = (b_0, \tau_0)$ is perturbed to
$\lambda = (b, \tau)$? We are able to give a fairly full answer in geometric terms for
those $\phi$ close to the orbit $G|_B$ of the initial configuration $I_B$ under the
action (via rigid rotations in $\mathbb{R}^3$) of the symmetry group $G$ of $\lambda_0$; note
$G = S^1$ or $SO(3)$ according as $\lambda_0$ is parallel or zero, and $G$ is trivial
otherwise. This fills out and considerably extends the local analyses of the
problem carried out by Stoppelli [1958].

In the model adopted, equilibrium states under load $\lambda$ are the points $\phi$
in a suitable configuration space $C$ which are critical points of a smooth func-
tion $V_\lambda: C \to \mathbb{R}$ given by

$$V_\lambda(\phi) = W(\phi) - \lambda \cdot \phi$$

where $W(\phi)$ is a stored energy function of the form $\int W(x, F(x))dV$ (with $F =$
deformation gradient $D\phi$) and $\lambda \cdot \phi$ denotes

$$\int_B b(x) \cdot \phi(x)dV + \int_{\partial B} \tau(x) \cdot \phi(x)dA.$$ 

Differentiation of $V_\lambda$ and the Divergence Theorem yield the usual equations of
nonlinear elasticity.

Basic geometrical hypotheses are:
(1) $W$ is invariant under the action of $SO(3)$ on $C$ (so $V_{\lambda_0}$ is
$G$-invariant);
(2) the orbit $G|_B$ is a nondegenerate minimum critical orbit for $V_{\lambda_0}$. 

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These follow from physical/analytical assumptions of (1) material frame-indifference, (2) strong ellipticity for \( \mathbf{W} \), and stability for the linearized problem.

As a first step, the Implicit Function Theorem shows there is a smooth manifold \( M \) close to \( G \) along which \( DV_\xi \) annihilates vectors in \( C \) which are 'normal' to \( G \) (using some \( \mathbf{G} \)-invariant tangential and 'normal' coordinate system around \( G \)). Thinking of \( M \) as the graph of a section \( \sigma_\xi \) of the normal bundle of \( G \) in \( C \), we note that all our sought-for critical points of \( V_\xi \) must lie on \( M \), and easily show by the Chain Rule:

**Lemma.** Let \( z \in G \). Then \( \sigma_\xi(z) \) is a critical point of \( V_\xi \) if and only if \( z \) is a critical point of \( f_\xi = V_\xi \circ \sigma_\xi : G \rightarrow \mathbb{R} \).

Thus we must next compute critical points of \( f_\xi \). Topological considerations alone give at least **two** if \( \mathbf{G} = S^1 \), **four** if \( \mathbf{G} = SO(3) \). For closer analysis we put \( \xi = \xi_0 + \lambda \xi_1 + \lambda^2 \xi_2 + \ldots \) and write \( \sigma_\lambda, f_\lambda \) for \( \sigma_\xi, f_\xi \). We find

\[
f_\lambda(z) = f_0(z) - \lambda(\xi_1 \cdot z + Df_0(z) \cdot u(z)) \]
\[
+ \lambda^2 (-\xi_2 \cdot z - \xi_1 \cdot u(z) + \frac{1}{2}D^2f_0(z)(z)^2) \]
\[
+ O(\lambda^3)
\]

where \( u(z) \in N_z \) is the coefficient of \( \lambda \) in \( \sigma_\lambda(z) \), namely the solution of the linearized elasticity problem

\[
D^2f_0(z)(u,v) - \xi_1 \cdot v = 0 \quad \text{for all} \quad v \in N_z
\]

where \( N_z \) is the normal space to \( G \) at \( z \). Deleting \( f_0(z) \) (= constant on \( G \)) and \( Df_0(z) \) (=0 on \( G \)), collecting up terms and dividing by \( (-\lambda) \) we are finally led to study the critical points on \( G \) of
\( \hat{f}_\lambda(z) = \lambda_1 \cdot z + \lambda(\lambda_2 \cdot z + B(z)) + O(\lambda^2) \)

where \( B(z) = \sqrt{2} \lambda_1 \cdot u(z) = \sqrt{2} U^2 f_0(z) \cdot u(z)^2 \), called the 'Betti form' for the load \( \lambda_0 + \lambda \epsilon_1 + \ldots \) on \( G \mid B \).

Any critical points of \( \lambda^*_1 : z + \lambda_1 \cdot z \) which are nondegenerate will persist for \( |\lambda| \) small. For \( G \) trivial (typical \( \lambda_0 \)) this is the case for \( z = I_B \). For \( G = S^1 \) (parallel \( \lambda_0 \)) either \( \lambda^*_1 \) has two nondegenerate critical points on \( G \mid B \), or \( \lambda^*_1 | G \mid B \) is constant if \( \lambda_0, \lambda_1 \) are specially related: cf. Bharatha & Levinson [1978]. For \( G = SO(3) \) (\( \lambda_0 = 0 \)) there are different possibilities according to five types of \( \lambda \) corresponding (in elasticity terms) to the existence of one or more axes of equilibrium for \( \lambda \): see Chillingworth, Marsden & Wan [1982, 1983] for details.

For type \( 0 \) (no axis of equilibrium) there are precisely four nondegenerate critical points on \( SO(3) \), which gives Stoppelli's Theorem (Stoppelli [1958]). For type \( 1 \) (one axis of equilibrium) there are two nondegenerate critical points and a circle \( C \). Under generic assumptions we find that if \( \lambda_1 \) varies with one or two parameters \( c \) or \( (c_1, c_2) \) we typically obtain a bifurcation diagram as indicated in the accompanying picture: the numbers 2, 4 refer to the number of critical points of \( \lambda^* + \lambda g \) on \( C \). Note how the number can jump back and forth between 2 and 4 (so the total number for \( f_\lambda \) jumps between 4 and 6) as \( \lambda \to 0 \) for fixed small \( c \neq 0 \). This is unobserved in the analysis of Stoppelli, where \( (\lambda, c) \) - space is not considered.

For types 2 and 4 the geometry becomes higher-dimensional and less explicit. (Type 3 is special, relating to parallel loads \( \lambda_1 \), and is technically similar to type 1.) Generically, the maximum numbers of nondegenerate critical points of \( f_\lambda \) for \( |\lambda| \) small (giving solutions to the original problem) are as in the table:
Extensions of this method to handle pressure boundary conditions and symmetries of the body, as well as an abstract formulation of the whole scheme, are given in Wan & Marsden [1983]. Formal perturbation methods (Signorini [1930], Grioli [1962], Truesdell & Noll [1965]) are interpreted in the light of this geometrical treatment by Marsden and Wan [1983], and re-expressed in the abstract setting by Chillingworth [in preparation].

REFERENCES


We will give an overview of the general theory of one-dimensional materials with fading memory, emphasizing that they exhibit instantaneous elastic response. We will compare the equations of motion for such a material with the equations of motion of elastic materials. We will probe the dissipative action of memory response by means of acceleration waves. This will motivate the proposition that motions starting out from smooth and "small" initial states remain smooth for all time and eventually decay to equilibrium while motions that originate from smooth but "large" initial states develop singularities in finite time. We will describe briefly the project of prolonging the solutions beyond the point of explosion and will discuss the stabilizing role of memory effects in that context.
We give an interpretation of the Cooper Nuttall rule: liquids with a large spreading coefficient $S$ spread better than liquids with a small $S$. This is based on a static calculation of the final thickness $e(S)$ reached by a droplet in "dry" spreading (i.e., when exchanges through the vapor are negligible). The thickness $e$ results from a balance between long range forces (tending to increase $e$) and the $S$ parameter (tending to decrease $e$). The resulting $e(S)$ is always a decreasing function of $S$, thereby explaining the Cooper Nuttall rule. On the other hand the dynamics of a spreading droplet is remarkably insensitive to $S$: all the free energy $S$ is "burned" in a precursor film, and does not show up in macroscopic flow.

The wetting properties of liquids against a solid surface are important for many practical purposes. A vast amount of knowledge has been collected by Zisman and his coworkers /1/, mainly for the case of partial wetting, where contact angles can be measured, and related to interfacial energies. Situations of complete wetting are equally important, but have suffered from a certain lack of experimental observables. In the present paper, we shall concentrate on the special case of complete, "dry", spreading, where the vapor pressure of the liquid is negligible, and the solid-gas interface does not carry any absorbed layer. The essential physical parameter controlling this process is the spreading coefficient, $S$, introduced long ago by Cooper and Nuttall /2/.

$$S = \gamma_{SO} - \gamma_{SL} - \gamma$$  \hspace{1cm} (1)

where $\gamma_{SO}$ and $\gamma_{SL}$ are the interfacial energies between solid/air and solid/liquid, while $\gamma$ is the surface tension of the liquid. We are interested
here in cases of positive $S$, and we shall try to understand first the final state of a spreading droplet (section II). It was commonly stated that the final state is a monomolecular film: but we shall see that this need not be true, especially at small $S$, where the final state should be a relatively thick "pancake". In section III we review some major features of the dynamics of the dry spreading. Finally, certain special cases (polymers, superfluids) are presented in section IV.

References


QUADRATIC FUNCTIONALS AND PARTIAL REGULARITY

by

Mariano Giaquinta
University of Florence
Florence, Italy

In this talk I shall describe some results concerning the partial regularity of minimizers of the simple quadratic variational integral

\[(1) \quad F[u,\Omega] = \int_{\Omega} A_{ij}(x,u) D_{\alpha} u^i D_{\beta} u^j \, dx\]

I shall also discuss some geometric applications, and mainly I shall outline some methods used to prove these results.

For more complete information on the regularity theory for minimizers of variational integrals and for solutions of related nonlinear elliptic systems I refer to my book.

Here I shall restrict myself to the functional (1) since it already shows some of the difficulties and permits me to present in a simple situation some of the methods used.

A more complete description of this lecture will appear as an IMA preprint.
"On free boundaries"

Hans Lewy

This free boundary problem consists of finding, or at least establishing some regularity for, the boundary of the domain of existence of a solution of a second order elliptic p.d.e. if on this boundary or a part of it \( u \) and first derivatives coincide with a given function \( \psi \) and its derivatives. \( \psi \) being smooth in a neighborhood of the boundary.

The simplest case of interest is two dimensional:

\[
\Delta u = 0 \quad \text{in a simply connected bounded domain } \omega.
\]

\( \partial \omega \) a connected part of \( \partial \omega \),

\( \psi \) analytic in a neighborhood of \( \partial \omega \).

\( u = \psi, \nabla u = \nabla \psi \) on \( \partial \omega \)

Theorem: \( \partial \omega \) is a curve whose points \( x,y \) are analytic functions of a parameter.

The book by Kinderlehrer & Stampacchia discusses more general such problems in any number of dimensions and in particular replaces analyticity of \( \psi \) by conditions on the smoothness of its derivatives up to some order. In case of an analytic equation and analytic \( \psi \) it relies on a result by Kinderlehrer-Nirenberg which establishes analyticity of the \( C^{1,\alpha} \) boundary by using rather difficult theorems by Nirenberg-Morrey, Morrey, and Friedman. Our topic is more modest: to give a short direct proof of the Theorem: Suppose \( u(x,y) \) is a solution of the equation with analytic coefficients

\[
Lu = \Delta u + au_x + bu_y + cu = 0
\]

in a simply connected domain \( \omega \).

Suppose \( \partial \omega \) a connected rectifiable open part of \( \partial \omega \) and assume that on \( \partial \omega \) \( u \) and first derivatives coincide with \( \psi \) and its first derivatives, \( \psi \)
being analytic in a neighborhood of $a'\bar{u}$. Then $a'\bar{u}$ is analytically parametrizable.

The proof utilizes the analytic extension of $u$ from $\bar{u}$ into a space of complex $x,y$ and studies the extension of $H = u - \psi$ as function $H(z,\zeta)$ of two points $z = x_1 + iy_1$ and $\zeta = x_2 + iy_2$ of $\bar{u}$. We establish with the aid of Riemann's function associated with $L$ the relation

\[
\frac{\partial H}{\partial z}(z,0) = \int_0^z R(z,0,t,\tau)g(t,\tau)dt - \int_0^z R(z,0,z,\tau)g(z,\tau)dt
\]

for $z \in a'\bar{u}$, $\tau \in \bar{u}$, $a'\bar{u}$, $g(t,\tau) = L H(t,\tau)$.

On replacing $\bar{z}$ by $\zeta$ and letting $z$ vary in $\bar{u}$, $a'\bar{u}$ this relation becomes a defining relation for a function $\zeta(z)$, holomorphic for $z$ in $\bar{u}$ near $a'\bar{u}$ and reducing to $\zeta(z) = \bar{z}$ for $z$ on $a'\bar{u}$.

Let $T(\lambda)$ map $\text{Im} \lambda > 0$ conformally on $\bar{u}$ with $a'\bar{u}$ being the image of $\text{Im} \lambda = 0$, $\text{Im} |\lambda| < 1$. Then

\[
T(\lambda) = \{ z, \text{Im} \lambda = 0, |\lambda| < 1 \\
\zeta(T\lambda)), \text{Im} \bar{\lambda} < 0, |\text{Im} \lambda| \text{ small}
\]

is holomorphic on $\text{Im} \lambda = 0$ and gives there the analytic parameterization of $a'\bar{u}$.

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LINEAR OBLIQUE DERIVATIVE PROBLEMS
FOR THE UNIFORMLY ELIPTIC HAMILTON-JACOBI-BELLMAN EQUATION

by
P-L. Lions*

and

N.S. Trudinger°

Abstract

We prove the existence of classical solutions for linear oblique derivative boundary value problems, in particular the Neumann problem, for the uniformly elliptic Hamilton-Jacobi-Bellman equation of stochastic control theory. The methods simultaneously embrace more general fully nonlinear elliptic equations.

* Ceremade
Universite Paris IX -Dauphine
Place de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

° Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601
AUSTRALIA
Dynamic Equilibrium States, Negative Absolute Temperature, 
and the Formulation of Thermostatics

Chi-Sing Man

Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, MN U.S.A.

and

Department of Civil Engineering
The University of Manitoba
Winnipeg, Canada

Abstract

To be consistent with continuum thermodynamics, Gibbsian thermostatics is reformulated so that dynamic nonequilibrium states are admissible as competitors in the variational "Entropy Maximum Principle". Since physical systems with negative absolute temperatures are known to exist, the assumption that absolute temperature be positive, usually adopted a priori in continuum thermodynamics, is abandoned in the new formulation. This talk deals with the following theorem, which is absent in the usual thermostatics but holds for a large class of materials under the new framework: For a body enclosed in a rigid container with adiabatic walls, the equilibrium temperature must be positive. In this talk a proof of the foregoing assertion will be given for the special instance of a fluid of the van der Waals-Korteweg type. The consistency of this theorem with the possibility of negative absolute temperatures in nuclear spin systems will be discussed.

The complete paper will appear as an IMA preprint.
A Survey of existence results for non convex problems

by Elvira Mascolo

Consider a functional of Calculus of Variations

\[ F(u) = \int_G f(x,u,Du)dx \]

where \( G \) is a open bounded subset of \( \mathbb{R}^N \) with a sufficiently smooth boundary \( \partial G \). Let \( f:(x,s,p) \rightarrow G \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) be a Caratheodory function such that

\[ \lambda_1 |p|^2 - \lambda_2 < f(x,s,p) < \Lambda_1 + \Lambda_2 |s|^2 + \Lambda_3 |p|^2, \]

and consider

(P) \( \inf \{ F(u), \ u \in X, \ \text{with some boundary conditions} \} , \)

with \( X = H^{1,2}(G) = H^1 \) or \( H^{1,\infty}(G) = H^{1,\infty} \) and the boundary conditions are either of Dirichlet or of Neumann type. It is well known that the sequential lower semicontinuity of \( F \) and the compactness of at least one minimizing sequence in the same topology ensure the existence of solutions. On the other hand the convexity of the integrand function with respect to \( p \) is a necessary and sufficient condition for the semicontinuity of \( F \). This result was first discovered by Tonelli [1] in the case \( f \in C^2 \) and \( N=1 \), by Caccioppoli-Scorza Dragnoni [2] in the case \( f \in C^2 \) and \( N=2 \), by Mac-Shane [3], Serrin [4] and later Morrey [5] for all \( N \) and \( f \in C^0 \), by Ekeland-Temam [6] for \( f \) of Caratheodory, independent on \( s \) and by Marcellini-Sbordone [7] in more general case.

Nevertheless the convexity of the integrand function is not a necessary condition for the existence of a solution of problem (P). Moreover, problems of this type are related to elastic problems and it is now well known (see [10], [11], [12], for example) that in this physical context the convexity of \( f \) is not acceptable. To overcome the difficulty that, if \( f \) is not convex in \( p \), it is not possible to apply the "direct methods", new arguments called "relaxation
methods", have been introduced. Consider a problem, of type \((P)\), for which it is not known if there exists a solution or not. A new functional \(\overline{F}\) and a new problem \((\overline{P})\) can be introduced, such that:

\[
\inf (P) = \min (\overline{P})
\]

and the optimal solutions of \((\overline{P})\) are the limit points of \((P)\).

In general, \(\overline{F}\) is the greatest functional less than \(F\) which is lower semicontinuous in the weak topology in which we consider the problem. In some particular case

\[
\overline{F}(u) = F^{**}(u) = \int_G f^{**}(x,u,\partial u)dx
\]

where \(f^{**}\) is the lower convex envelope of \(f\) with respect to \(p\) (see, for example, [6],[7]).

But, we are now interested in finding the conditions on \(f\) to obtain solutions of non convex problems directly.

In one dimension, we recall the results of Ericksen [10], obtained by studying the equilibrium configurations of bars, in soft and hard devices (which correspond to Neumann and Dirichlet problems) and of Aubert-Tahraoui [13] and Marcellini [14] for Dirichlet problem.

Although, the arguments are completely different the Ericksen and Marcellini ones can be generalized to higher dimensions. Consider the Ericksen problem, in the soft device:

\[
\text{(1) Inf} \{ \int_0^1 \phi(v')dx - \sigma(v(1) - v(0)), v \in H^{1,\infty}(0,1), \int_0^1 vdx = 0 \}
\]
with \( \phi \in C^1 \) and \( \phi(t) > |t|^2 - c \).

Let \( \phi^{**} \) be the lower convex envelope of \( \phi \); the relaxed problem of (1), is

\[
\text{Inf } \left\{ \int_0^1 \phi^{**}(v')dx - \sigma(v(1) - v(0)), \ v \in H^1, \int_0^1 v dx = 0 \right\}.
\]

From Jensen's inequality

\[
\int_0^1 \phi^{**}(v')dx - \sigma(v(1) - v(0)) \geq \phi^{**}(v(1) - v(0)) - \sigma(v(1) - v(0))
\]

If \( \overline{\epsilon} \) is a minimum of the function \( \phi(t) = \phi^{**}(t) - \sigma t \), the function is such that \( u' = \overline{\epsilon} \) and \( \int_0^1 u dx = 0 \), is a solution of (2).

On the other hand, \( \overline{\epsilon} \) has to verify

\[
(3) \quad \phi^{**}(\overline{\epsilon}) = \sigma
\]

(3) says that \( u \) verifies the Euler equation of (2). If we define

\[
K = \{ t \in \mathbb{R} : \phi(t) > \phi^{**}(t) \} = ]t_1, t_2[, \text{ we have that there exists } \overline{\sigma} \text{ such that}
\]

\[
\phi_t(t_1) = \phi_t(t_2) = \overline{\sigma} \text{ and } \phi_t^{**}(t) = \overline{\sigma} \text{ in } [t_1, t_2],
\]

where \( \overline{\sigma} \), called the Maxwell stress, is the unique value for which the shaded regions in Fig. 2 have the same area.
Then, for $\sigma \neq \overline{\sigma}$, there is only one $\overline{t}$ such that (3) is verified, and $\overline{t} \in \overline{t}^1, t^2$; thus $u' = \overline{t}$ is also a solution of (1). If $\sigma = \overline{\sigma}$, it is easy to check that the function

$$u = t_1 x_{F_1} + t_2 x_{F_2}$$

where $x_{F_i}$ is the characteristic function of $F_i$, with $F_i \subset [0,1]$, $F_1 \cap F_2 = \emptyset$ such that $\int_0^1 u dx = 0$, is a solution of both problems (1) and (2).

For Dirichlet problem the arguments are very similar. In one dimension, a non convex problem has always at least one solution. In the previous arguments, we utilize the fact that set $K$, in which $\phi^{**}$ is strictly less than $\phi$, is a bounded interval (i.e. $\phi$ is convex at infinity) and $\phi^{**}$ is linear on $K$, which is always true in one dimension.

We emphasize that a solution of the non convex problem is a solution of the relaxed, such that $u' \in R - K$ or $u' \in \partial K$. For the Dirichlet problem, we always find a solution with $u' \in \partial K$.

First, we consider Dirichlet problem-type.

Let $f : R^N \rightarrow R$ be continuous and consider

$$(4) \quad \inf \{ \int_G f(Dv) dx, v \in H^{1, \infty}, v = u_0 \text{ on } \partial G \}$$

where $u_0 \in H^{1, \infty}(G)$ verifies the classical "bounded slope conditions" [see, for example [15]].

Define $K = \{ p \in R^N : f(p) > f^{**}(p) \}$ and suppose that:

(i) $K$ is a bounded open subset of $R^N$ and $f^{**}$ is linear on $K$ i.e. there exists $m_i, q \in R$ such that

$$f^{**}(p) = \sum_i m_i p_i + q, p \in K$$

We can see that a solution of
\[
\begin{cases}
Du(x) \equiv K \text{ a.e. in } G \\
u = u_0 \text{ on } \partial G
\end{cases}
\] (5)

is a solution of (4). In fact from the linearity of $f^{**}$ on $K$ and from divergence theorem, we have, for all $v \in H^1_0, v = u_0$ on $\partial G$

\[
\int_G f(Du) \, dx = \int_G f^{**}(Du) \, dx = \int_G \sum_{i=1}^m m_i D_i u \, dx + |G|q
\]

\[
= \int_{\partial G} \sum_{i=1}^m m_i u \cdot v_i \, ds + q|G| < \int_G f^{**}(Dv) \, dx < \int_G f(Dv) \, dx
\]

But (5) is not solvable for all boundary data. In fact, (5) is a special case of the Hamilton-Jacobi equation and for his solvability $u_0$ has to verify a compatibility condition (which is also a necessary condition) (see [16], [17],[18])

This existence results are proved in [17].

To find solutions for every boundary data, consider the relaxed problem of (4)

\[
\text{Inf} \{ \int_G f^{**}(Dv) \, dx, \ v \in H^1_0, v = u_0 \text{ on } \partial G \}.
\] (6)

The idea is to find a solution of (6) such that $Du(x) \in R^{N-K}$ a.e. in $G$, which is also a solution of (4).

We recall that, if $u_0$ verifies the bounded slope condition, with constant $L_0$, there exists at least one solution of (6) with $\|Du\|_\infty < L_0$, see [15].

Consider

\[M_L = \{ u : u \text{ solution of (6) with } \|Du\|_\infty < L \}\]

with $L > L_0; M_L \neq \emptyset$ and suppose it has at least two elements. Let now

\[u_L(x) = \sup \{ u(x), u \in M_L \},\]
we prove:

(a) \( u_L \leq M_L \);
(b) there exists \( \mathcal{I} \) such that for \( L > \mathcal{I} \), the subset of \( G \) in which 
\( Du_L(x) \in K \) has measure zero.

Then for problem (4), under assumption (i), there is a general existence result ([19]). The proof of (b) uses, essentially the fact that \( u_L \in H^{1,\infty} \), more precisely that it is a.e. differentiable.

Consider now a problem, with the integrand function depending also on \( x \), i.e.

\[
\text{Inf} \{ \int_G f(x, Du) dx \mid v \in H^{1,\infty}, v = u_0 \text{ on } \partial G \}
\]

Even if \( f \) is convex in \( p \), there are no existence theorems for (7). Similarly, if we consider the problem in \( H^1 \), there are no \( H^{1,\infty} \)-regularity results.

To apply the previous argument to (7), in \( H^1 \), we need to find at least one solution of the relaxed problem, a.e. differentiable in \( G \), for example in \( H^{1,\infty} \) or \( H^1_{loc} \) \( \infty \).

Consider the general problem

\[
\text{Inf} \{ \int_G g(x, Dv) dx \mid v \in H^1, v = u_0 \text{ on } \partial G \}
\]

where \( g \in C^0(\bar{G} \times \mathbb{R}^n) \) and is convex in \( p \). Suppose that:

(ii) there exists \( t > 0 \) such that, given \( S_t = \{ p \in \mathbb{R}^n : |p| > t \} \).
\( g \) is strictly convex in \( S_t \) and \( g \in C^2(G \times S_t) \)
\( |g_p|, |g_{px}| < \mu (1+|p|), \quad |g_{pp}| < \mu, p \in S_t \).
\( g_{p_i p_j}(x,p) \xi_i \xi_j > |\xi|^2, p \in S_t, x \in \bar{G} \).
We prove, in [20], that under (ii), if \( u_0 \in H^1 \), there exists at least one solution of (8) in \( H^1_{loc}, \omega(G) \). Moreover, if \( u_0 \in H^2, \omega(G) \) and \( \partial G \) is \( C^2 \), (8) has at least one solution in \( H^1, \omega(G) \). We give a sketch of the proofs: We approximate \( u \) with a sequence \( u_n \in C^2 \), strictly convex in \( p \), with \( u_n = g \) in \( S_{t+1} \). Consider the sequence of problems for \( u_n \) and the related sequence of Euler equations. By applying the methods of Ladyzhenskaya and Ural'tseva [21] with a special truncation, we prove that the sequence of solutions \( u_n \) verifies

\[
\sup_{B_R(x_0)} |Du_n|^2 < \frac{c}{R^N} \int_{B_{2R}(x_0)} (1 + |Du_n|^2) \, dx
\]

with \( c \) independent on \( n \). There, we prove that \( u_n \) converges to a solution of (8), which is in \( H^1_{loc}, \omega \). Moreover, by applying the classical barriers (we construct the same barrier for each approximate problem), from (9) and the Caccioppoli inequality, we prove that (8) has at least one solution in \( H^1, \omega(G) \).

We apply these results to the existence of nonconvex problems: Let \( f \) be strictly convex and \( C^2 \) for \( |p| \) sufficiently large and verify suitable growth conditions. If \( f^{**} \) is a special linear function on \( K(x) = \{ p \in R^N : f^{**}(x,p) < f(x,p) \} \), a.e. in \( G \), (7) has at least one solution.

For the Neumann problem, in a special case on \( f(x,p) \) we have a general existence result, without using any regularity results [22].

Consider

\[
\inf \{ F(v) = \int_G \phi(Dv) dx - \int_G fv dx - \int_G yvdS, v \in H^1, \int_G vd = 0 \}
\]

with, \( \phi(p) > c_1 |p|^2 - c_2 \), \( p \in R^N \),

\[
\phi(p) = \phi(p) + \psi(p), \ \ p \in R^N
\]

where \( H^1, \omega(R^N) \) is a strictly convex function \( \psi \) has compact support in \( R^N \) and
\[ \phi(p) = \max \{ \phi(p), 0 \} \quad p \in \mathbb{R}^N. \]

If \( f \in L^2(G) \) and \( y \in H^1 \) with \( \int_G f dx + \int_{\partial G} y dS = 0 \), we have that (10) has at least one solution.

Consider the set

\[ L_J = \{ v \in H^{1, \infty}, \| \nabla v \|_{\infty} < J, \int_G v dx = 0 \} \]

and the sequence of problems.

(11) \[ \inf \{ F^*(v) = \int_G \phi^*(A v) dx - \int_G f v dx - \int_G g v dS, v \in L_J \} \]

We prove that, there exist \( J_0 \), such that for all \( J > J_0 \), (11) has at least one solution \( u_J \) such that \( Du_J(x) \in \mathbb{R}^N - K \) a.e. in \( G \), where \( K = \{ p \in \mathbb{R}^N : \phi^*(p) < \phi(p) \} \). Then, we prove

(a) \( u_J \) is a minimizing sequence of the relaxed problem of (10),

(12) \[ \inf \{ F^*(v), v \in H^1, \int_G v dx = 0 \} \]

and \( u_J \) converges in the weak topology of \( H^1 \), to a solution \( u \) of (12).

(b) \( u_J \) converges to \( u \) also in the strong topology of \( H^1 \). So \( Du(x) \in \mathbb{R}^N - G \) a.e. in \( G \) and \( u \) is a solution of (10).

It is well known that, in some case, it is interesting to find the local minima of problems of Calculus of Variations. We say that \( u_0 \) is a local minimum of (10), if \( u_0 \in H^1 \), \( \int_G u_0 dx = 0 \) and there exists \( \varepsilon > 0 \) such that

\[ F(u_0) < F(v), \quad v \in H^1, \quad \int_G v dx = 0 \quad \| v - u_0 \|_{H^1} < \varepsilon. \]

It is easy to check that for a convex functional, all local minima in the pre-
vious sense, are also minima. This is true, in one dimension, also if \( \phi \) is not convex. Consider problem (1). Suppose \( \sigma > \sigma' \), the equation

\[
\phi_t(t) = \sigma
\]

has three solution \( \alpha_1, \alpha_2, \alpha_3 \). Since \( \phi_{tt}(\alpha_2) < 0 \), a local minimum is a function \( u \) such that

\[
u' = \alpha_1 x F_1 + \alpha_2 x F_2 \quad F_1 C[0,1]
\]

In [23] it is proved that \( u \) consists of only one phase. Therefore \( u \) is also a solution of (1). We remark that this depends on the space in which we consider the local minima.

In dimension \( N > 1 \), a special assumption on (10) is needed. Let \( u_0 \in H^1 \),

\[
\int_G u_0 \, dx = 0 ; \text{then consider}
\]

(13)

\[
C_\varepsilon = \{ v \in H^1, \int_G v \, dx = 0 , \| v - u_0 \|_{H^1} < \varepsilon \}
\]

and suppose that the following relaxation result is valid:

(14)

\[
\inf \{ F(u), u \in C_\varepsilon \} = \min \{ F^{**}(u), u \in C_\varepsilon \}.
\]

If \( u_0 \) is a local minimum of (10), it is also a solution of the left-hand side of (14). From (14), \( F(u_0) = F^{**}(u_0) \). So \( Du_0(x) \in R^N - K \) a.e. in \( G \). From the Euler equation, we find that \( u_0 \) is also a solution of (10).

But (14) is not always true for all \( u_0 \) and \( \varepsilon \).

In general, if \( C \) is a convex closed subset of \( H^1 \) we have

\[
\inf \{ F(u) : u \in C \} = \inf \{ F_C(u), u \in C \}
\]

where \( F_C \) is the greatest functional lower semicontinuous in the weak topology of \( H^{1,2} \) less than \( F \) in \( C \). In particular

\[
F_C(v) = \inf \{ \liminf_{k} F(v_k), v_k \in C, v_k \rightharpoonup u, w-H^1 \}
\]
(see [7]) and $F_C(v) > F**(v)$.

Nevertheless, if $u_0$ and $\varepsilon$ are such that the problem in the right-hand role of (14) is valid. Note that the right-hand side of (14) is not strictly convex problem and then there are not uniqueness results. We state the following theorem [22] If $u_0$ is a local minima of (10) and the problem in the right-hand of (14), has at least two solutions, $u_0$ is also a solution of (10).

We proved existence results only on the cases in which $f**$ is linear on $k$. It is not known if this hypothesis is also necessary for the existence of minima.

Nevertheless, Marcellini in [24] showed that if $f**$ is not affine on $K$, a non convex problem lacks a solution. In particular, $f(p) = \phi(|p|)$, where $\phi$ is as in fig. 1. We have $f**(p) = \phi**(|p|)$ and $f**$ is not affine.

There is also a recent paper of Auber-Tahrabui [25]. In the framework of control theory, in which they use oru linearity assumption on $f**$ on $K$. For state the existence in some non convex problems.
REFERENCES

Stochastic Convergence Problems in the Calculus of Variations

Luciano Modica
University of Pisa

In our approach, nonlinear stochastic homogenization (N.S.H.) is a typical stochastic convergence problem in the Calculus of Variations.

Namely, let us consider the class $F$ of all integral functionals

$$F(u,A) = \int_A f(x,Du(x))dx$$

whose integrand $f(x,p)$ is convex in $p$ and fulfills the inequalities

$$c_1|p|^2 < f(x,p) < c_2(1+|p|^2)$$

where $c_1 > 0$, $c_2 > 0$, $\alpha > 1$ are fixed real constants. We construct a metric on $F$ so that $F$ is compact and the maps

$$F + m(F,\phi,u_0,A) = \min \{ F(u,A) + \int_A \phi dx : u = u_0 \text{ on } \partial A \}$$

are continuous on $F$ whenever $\phi$, $u_0$, $A$ are fixed. Note that the convergence of a sequence in $F$ is equivalent to $r$-convergence with respect to $L^\alpha$ topology.

Now, N.S.H. means for us to study the limit (almost everywhere, in probability, in law) as $\varepsilon \to 0^+$ of a family $(F_\varepsilon)$ of random functionals, i.e. measurable maps on a probabilistic space $\Omega$ with values in $F$, such that each $F_\varepsilon$ has the same law as the random functional $\rho F_\varepsilon$ given by

$$[\rho F_\varepsilon](\omega)(u,A) = \int_A f(\omega,\frac{x}{\varepsilon},Du(x))dx \quad (\omega \in \Omega)$$

where $F$ is a fixed random functional and $f(\omega,x,p)$ denotes its integrand.

Of course, if the limit $F_0$ does exist, then the continuity of $m(\cdot,\phi,u_0,A)$ immediately yields the convergence (almost everywhere, in probability, in law) of the values of the Dirichlet minimization problems for $(F_\varepsilon)$ to the
WIENER'S POINTWISE POTENTIAL ESTIMATES

by

Umberto Mosco

In its simplest local form an obstacle problem is the minimization of an energy integral of Dirichlet type

$$\varepsilon(u) = \int_{\Omega} |Du|^2 \, dx$$

on some open domain \( \Omega \) of \( \mathbb{R}^n \), under a unilateral constraint

$$u > \psi \quad \text{in} \quad \Omega$$

on the solution \( u \). Here \( \psi \) is some given function on \( \mathbb{R}^n \), the obstacle of our problem.

In order to include the case of so-called thin obstacles, when the constraint is only prescribed on some lower dimensional subset \( E \) of \( \Omega \), the value \( -\infty \) is also allowed to \( \psi \) and, moreover, the inequality \( u > \psi \) is required to hold pointwise in \( \Omega \) except possibly a set of harmonic capacity zero in \( \mathbb{R}^n \).

The Euler inequality satisfied by any locally minimizing function \( u \) can be formally written as

\[ (*) \quad \min(u - \psi, Lu) = 0 \quad \text{in} \quad \Omega, \]

where \( L = -\Delta \) is the Laplace operator in \( \mathbb{R}^n \).

More generally, we take \( L \) to be a uniformly elliptic second order linear operator in divergence form, with bounded measurable coefficients in \( \mathbb{R}^n \).

Our goal in this paper is to describe the behavior of an arbitrary local weak solution \( u \) of \( (*) \) at a given point \( x_0 \) of the domain \( \Omega \), in terms of the behavior of \( \psi \) at that point. More precisely, we want also to estimate the modulus of continuity of \( u \).

Several results in this direction are available, both for regular (i.e., continuous) and irregular \( \psi \).

The material discussed appears as IMA Preprint #117

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Twinning in Crystals III

Gareth P. Parry

School of Mathematics
University of Bath
Bath, Avon
England BA2 7AY

Abstract

The modelling of twins in crystals using strain gradient theories provides interesting problems in thermodynamics and in the calculus of variations. Here, Dunn and Serrin's thermomechanical theory of interstitial working is used to set up a variational principle which governs the equilibria of materials with non-convex Helmholtz free energy. In some geometries, this principle reduces to a novel calculus of variations problem, and we provide an example where non-constant stable extrema connect symmetry-related equilibrium states.

References


THE USE OF NUMERICAL SPECTRAL METHODS IN FLUID DYNAMICS

Alfio Quarteroni

Spectral methods are a family of numerical methods to approximate the solutions of (initial) - boundary - value problems by discrete solutions which are for each time smooth functions over the domain of resolution. The discrete solution is typically a polynomial, either algebraic or trigonometric. Its representation is a finite expansion in terms of a given set of trial functions \( \{ \psi_k, k = 0, \ldots, N \} \), i.e.,

\[
(1) \quad u_N(x) = \sum_{k=0}^{N} a_k \psi_k(x)
\]

The \( \psi_k \)'s are independent, orthogonal functions under a given inner product \( (u,v) \) relative to a weight function \( w(x) \). Typically, they are the eigenfunctions of a singular Sturm-Liouville boundary-value problem. With this choice it can be proven that:

\[
\inf_{v_N \in S_N} \| u - v_N \|_{H^k_w} < C \left( \frac{1}{N} \right)^{s-k} \| u \|_{H^s_w}, \quad 0 < k < s
\]

where \( S_N = \text{span} \{ \psi_k : |k| < N \} \), and \( H^k_w \) denotes the Sobolev space of order \( k \) relative to the weight function \( w \).

Alternatively, \( u_N \) can be regarded as an interpolating function between the values at some suitable points \( \{ x_j, j = 0, \ldots, N \} \), i.e.,

\[
(2) \quad u_N(x) = \sum_{j=0}^{N} u_N(x_j) \varphi_j(x)
\]

The \( x_j \)'s are the nodes of a Gauss-type integration formula relative to the weight \( w(x) \), and \( \{ \varphi_j \} \) is the Lagrange basis. The set of coefficients \( \{ a_k \} \) is called the "frequency space" and \( (1) \) is the spectral representation of \( u_N \), while the set of grid-values \( \{ u_N(x_j) \} \) is called the "physical space", and \( (2) \) is the physical representation of \( u_N \).
The most popular representation of $u_N$ are either with complex trigonometric polynomials or with Chebyshev polynomials. The latter prove to be as good for non-periodic problems as the former are for periodic problems. They both allow the use of fast transform methods (namely the FFT) to efficiently implement the passage between the physical and the spectral space.

The different kind of equations satisfied by the discrete solution distinguishes between the three types of spectral methods, namely Galerkin, Tau or Collocation. Galerkin and Tau methods are variational methods. The two sets of trial and test functions coincide for the Galerkin method, while they are different for the Tau (or Lanczos) method, where the test functions are not supposed to satisfy the boundary conditions. Both Galerkin and Tau methods produce a set of equations in the frequency space. In Collocation methods, the differential equation is collocated at the points $\{x_j\}$, and the discrete problem is set in the frequency space. Boundary conditions can be satisfied either explicitly or implicitly.

Any type of spectral method may be regarded as a method of weighted residuals (or, equivalently, as a Petrov - Galerkin method). Indeed, it can be read as the projection of the differential equation upon a space $Y_N$ of test functions via an orthogonal projection operator $Q_N \cdot Y_N$ has the same dimension of the space $X_N$ of the trial functions to which the discrete solution $u_N$ belongs.

Due to their high accuracy, spectral methods allow the achievement of good results using only a few frequencies to approximate incompressible flows with smooth solutions.

Moreover, a proper use of filtering techniques applied to the frequencies $a_k$ guarantee accurate convergence even for incompressible flows with non-smooth solutions.
THE NASH-MOSER TECHNIQUE FOR AN INVERSE PROBLEM IN POTENTIAL THEORY RELATED TO GEODESY

C. Maderna, C. Pagani, and S. Salsa

1. We shall consider the following inverse problem of potential theory:
to determine the shape of a body from measurements of the Newtonian potential at
its surface, given some information about its mass distribution. More precisely,
we consider a class of bodies $G$ that can be parametrized as follows. Let $u$
be a smooth mapping of $S^2 \times \mathbb{R}$ with $|u| < \text{constant} < r_0$. (Here $S^2$
is the surface of the unit ball in $\mathbb{R}^3$). For $w \in S^2$, define

$$\phi_u(w) = w(r_0 + u(w)), \quad \Gamma_u = \phi_u(S^2).$$

Then $G \equiv G_u$ is the bounded domain whose boundary is $\Gamma_u$. (Thus $r_0$
is a reference sphere with center at the origin and radius $r_0$).

Let now $\delta: \mathbb{R}^3 \times \mathbb{R}$ be a given strictly positive function and $V_u$ the
potential created by a mass of density $\delta$ distributed over $G_u$; we can write

$$V_u(x) = \int_{S^2} dw \int_{0}^{\phi_u(w)} t^2 \delta(tw) |x-tw|^{-1} dt. \quad (1)$$

The measured datum (the potential on $\Gamma_u$) is then a function, $v$, defined on
$S^2$.

We want to find a function $u$ such that

$$A(u) \equiv V_u \circ \phi_u = v \quad \text{on} \quad S^2 \quad (2)$$

where $V_u \circ \phi_u$ denotes the composition of $V_u$ with $\phi_u$. We prove a local
result by linearizing near suitable surfaces and using a version of the inverse
function theorem of Nash-Moser.

The use of the ordinary Banach implicit function theorem is prevented by a
continuous loss of regularity in the iteration procedure involved in the solu-
tion of the equation. A point of interest is that the solvability of the
linearized equation depends on the mass distribution; namely, for particular densities, \( \delta \), including \( \delta = \text{constant} \), the presence of non trivial eigenspaces with dimension \( N \), depending on the density itself, forces the introduction of projections onto those eigenspaces. This combination of the Nash-Moser technique and projections on finite dimensional spaces leads to solve a modified equation of the form

\[
A(u) + w = v
\]

where \( w \) is an eigenvector of the linearized operator. We will describe the results in the particular case of small perturbations of an homogeneous sphere.

A problem similar to that described by equation (2) is the determination of the shape of the body where the potential is known on a surface surrounding (and far from) the body itself. A description of classes of stable solutions in both cases is proved in [P]. Two other problems showing some features in common with ours are described in the papers of [S] and [HÖ]; the first one arising in electrostatics and the second one in geodesy. In the latter, the problem is to find the shape of the earth from the knowledge of the potential and the gravity vector on its surface, while no assumptions are made on its internal structure. Both these problems have been dealt with by some version of the same inverse function theorem.

2. We consider now the linearized equation at a fixed \( u \). Standard computations give

\[
A'(u)\rho = V'_u \rho \circ \phi_u + \langle \text{grad}_x V_u \circ \phi_u , \phi'_u \rho \rangle
\]

where \( \langle , \rangle \) denotes the scalar product on \( \mathbb{R}^3 \). We have

\[
V'_u \rho(x) = \int_{S^2} \delta \circ \phi_u(w) \rho(w) \frac{|\phi'_u(w)|^2}{|x - \phi_u(w)|^3} \, dw
\]

and
\[ f_u(w) = -\int_{S^2} dw' \int_0^{|\phi_u(w')|} t^2 \delta(tw) \frac{<\phi_u(w') - tw', w>}{|\phi_u(w') - tw'|^3} \ dt. \]

Setting \( M_u \rho = V_u \rho \circ \phi_u \) and \( f_u(w) = <\nabla V_u \circ \phi_u, w> \), the linearized equation at \( u \) is as follows:

\[
(3) \quad M_u \rho(w) + f_u(w)\rho(w) = h(w) \quad (w \in S^2)
\]

a Fredholm integral equation of the second kind.

To see how this equation behaves we first consider the particular case in which \( u \) is a constant and \( \delta = \delta(|x|) \) is a radial function. In this case, putting \( r = r_0 + u \),

\[
f_u(w) = -\frac{4\pi}{r^2} \int_0^r t^2 \delta(t) dt = -\frac{m}{r^2}
\]

and equation (3) becomes

\[
(3') \quad r \delta(r) \int_{S^2} \rho(w') \ |w - w'|^{-1} dw' - mr^{-2} \rho(w) = h(w) \quad (w \in S^2).
\]

Observe now that the eigenvalues of the integral operator

\[
\rho(\cdot) + \int_{S^2} \rho(w') |\cdot - w'|^{-1} dw'
\]

are \( 4\pi/2n+1, \ n = 0,1,... \). The \( n \)-th eigenvalue has a \((2n+1)\)-fold degeneracy and the corresponding eigenfunctions are the \((2n+1)\) surface spherical harmonics of degree \( n \). Hence the discussion of (3) depends on \( \delta \), more precisely on the fact that the equation

\[
(4) \quad \int_0^r t^2 \delta(t) dt = r^3 \delta(r)/(2n+1)
\]

is satisfied or not for some \( n \). Therefore we distinguish two cases:

Case I. No integer satisfies equation (4). By expanding \( \rho \) and \( h \) in spherical harmonics,
\[ \rho = \sum_{0}^{\infty} \rho_n, \quad h = \sum_{0}^{\infty} h_n, \text{ it follows from (3')} \text{ that} \]

\[ \{4\pi/(2n+1)r\delta(r) - mn^{-2}\} \rho_n = h_n \quad (n = 0, 1, 2, \ldots) \]

and hence, for any \( h \) in the Sobolev space \( H^s(S^2) = H^s (s > 0) \) (3') has a unique solution \( \rho \) and the estimate

\[ \| \rho \|_S < c \| h \|_S \]

holds for every \( s > 0 \), with \( c \) independent of \( s \).

Case II. Equation (4) is satisfied for some \( \overline{n} \). In this case (3') admits a solution iff \( h_{\overline{n}} = 0 \), i.e., iff \( h \) belongs to a space of codimension \( 2\overline{n} + 1 \).

The map

\[ h + r^{2} \sum_{n \neq \overline{n}} \frac{2n+1}{2(\overline{n}-n)} h_n \]

solves (3') modulo \( \overline{n} \)-degree harmonics. A unique solution can be selected by imposing \( 2\overline{n} + 1 \) linearly independent conditions. Observe that when \( \delta \) is a constant, \( n = 1 \) satisfies (a) for every value of \( r \).

Our results on the linearized equation can be stated as follows:

**THEOREM 1.** Assume that for some radial function \( \overline{\delta} \) and for \( r = r_0 \) we are in case I. Then, for every \( h \in L^2 \), equation (3) has a unique solution \( \rho \in L^2 \), provided that \( |u|^1_{C^1(S^2)} \) and \( |\delta - \overline{\delta}|^1_{C^1(G_0)} \) are sufficiently small. Moreover, the estimate

\[ \| \rho \|_0 < c \| h \|_0 \]

holds.
As an example of the analogue of theorem 1 when $\delta u$ takes on values belonging to the spectrum of $M_u$, we consider a perturbation of an homogeneous sphere. Thus, assume $\delta$ close to a constant $\delta_0$ and consider equation (3) together with the conditions

$$B_{u, \delta \rho} = \int_{S^2} |\phi_u(w)|^3 \delta \phi_u(w) \rho(w) w_j \, dw = 0 \quad j = 1, 2, 3.$$  

The meaning of conditions (5) is that the center of mass of $G_u$ is the origin.

We have the following result:

THEOREM 2. Let $\delta_0$ be a constant. Then, for every $h \in L^2$, orthogonal to the first degree spherical harmonics, system (3), (5) has a unique solution, provided that $|u|_{C^1(S^2)}$ and $|\delta - \delta_0|_{C^1(G_0)}$ are sufficiently small. Moreover the estimate

$$\|\rho\|_0 \leq C \|h\|_0$$

holds.

3. We now treat the non linear equation.

Consider first the case in which the linearized equation is uniquely solvable. Thus, let $u_0$ and $\delta$ be functions verifying the conditions in Theorem 1 and set $A(u_0) = V_0$.

Our purpose is to show that if $V$ is sufficiently close to $V_0$ then there exists a solution $u$ close to $u_0$ of $A(u) = V$.

As we already said the use of the ordinary inverse function theorem is prevented by a loss of regularity phenomenon. In fact the derivative $A'(u)$ turns out to be discontinuous on Sobolev spaces. An estimate of the type

$$\|A'(u_1) - A'(u_2)\|_{C^0} \leq C \|\rho\|_{C^0}$$

is possible only with constants depending on a number of derivatives of $u_1 - u_2$ greater than $s$.

The appropriate tool to handle equation (2) is a Nash-Moser type technique.
The one we have used is presented in [H].

We briefly describe the procedure which gives the solution. We first introduce a class of smoothing operators $S_t, t > 0$, by taking a function $\psi \in C^\infty(\mathbb{R}), 0 < \psi(t) < 1, \psi = 0$ for $t < 0$, $\psi(t) = 1$ for $t > 1$ and defining for $u \in L^2(S^2)$, $u = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} c_{nj} Y_n$, $S_t u = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{nj} Y_n$

where $d_{nj} c_{nj} \cdot \psi(e^t - n - 1)$.

We can easily prove that $S_t u = 0$ if $t < 0$ and that

(i) $\|S_t u\|_s < e^{t(s-r)} \|u\|_r$ \hspace{1cm} $(r < s)$

(ii) $\|S_t u - u\|_s < e^{t(r-s)} \|u\|_r$ \hspace{1cm} $(r > s)$.

Consider now the iteration procedure given by the following differential equation for a path $u_t$:

(6) $u_t' = u \wedge (S_t u_t) S_t[v - A(u_t)]$ \hspace{1cm} $(u_t' = \frac{d}{dt} u_t)$

with $u_0$ as a starting point, where $u$ is a positive number and $\wedge$ denotes the inverse of $A$. We want to show that a solution exists for all times $t > 0$, and converges to a solution of $A(u) = v$ as $t \to +\infty$. Theorem 15.6.3 of [H] assures existence and uniqueness for small times. To go further we need uniform estimates in a neighborhood of $u_0$ for the operators $A, A', \wedge$ and for increments of the first derivative. We have the following result:

THEOREM 3. Let $u_0$ and $\delta$ verify the conditions of Theorem 1. Furthermore assume $\|u - u_0\|_4 < 1$. For $s > 0$ the following inequalities hold:
\[ \| A(u) - A(u_0) \|_S < c \| u - u_0 \|_{S+3} \]
\[ \| A'(u) \rho \|_S < c \cdot (\| \rho \|_2 \| u \|_{S+3} + \| \rho \|_{S+2}) \]
\[ \| A(u) h \|_S < c \cdot (\| h \|_2 \| u \|_{S+3} + \| h \|_{S+2}) \text{ (assuming } \| u-u_0 \|_3 \text{ small, depending on } s) \]
\[ \| A'(u+\sigma) \rho - A'(u) \rho \|_S < c(\| \rho \|_{S+2} \| \sigma \|_3 + \| \rho \|_2 \cdot \| \sigma \|_{S+3}) \]

where \( C = C(s) \).

We come now to prove the following theorem.

**THEOREM 4.** Let \( u_0 \) and \( \sigma \) verify the conditions of Theorem 1. Let \( u > q > g, s > 9, a > 2 \). If \( \| v - A(u_0) \|_q < \eta \), sufficiently small, then the path \( u_t \) defined by (6) exists for all times, converges in \( H_{S-a} \) norm as \( t \to \infty \) to a solution \( u_\infty \) of \( A(u) = v \) and the following estimates hold:

\[ \| u_t - u_\infty \|_{S-a} < c \cdot e^{-t} \| v - A(u_0) \|_{S+1} \]
\[ \| u_t - u_\infty \|_{S-a} < c \cdot \| v - A(u_0) \|_S \]

with \( c \) independent of \( t \).

We observe that, in general, \( \eta \) depends on \( s \). If the solution \( \rho \) of the linearized equation (3) at \( u = u_0 \) satisfies the inequality \( \| \rho \|_S < c \cdot \| h \|_S \) with \( c \) independent of \( s \) (this is true for instance, when \( u_0 \) = constant) then it is possible to choose \( \eta \) independent of \( s \).

4. We consider now the case of a perturbation of an homogeneous sphere (of radius \( r_0 \)) where \( f u \) takes on values in the spectrum of \( M_u \).

Let \( \delta \) be a positive constant and take \( u_0 \) and \( \delta \) as in Theorem 2. Our purpose is to show that, if \( v \) is sufficiently close to \( v_0 \) then there exists a \( u \) close to \( u_0 \) and small constants \( a_1, a_2, a_3 \) such that the equation

\[ A(u) - \sum_{j=1}^{3} a^j y_{lj} = v \]
is satisfied together with the conditions

\[ \int dw \int |\Phi U(w)| t^3 w_j \delta(tw) dt = 0 \quad (j=1,2,3). \]

Again the meaning of (8) is that the center of mass of $G_u$ is placed at 0.

The Nash-Moser iteration is to be set up for the unknown $V = (a^1, a^2, a^3, u)$ and the operator

\[ \tilde{A}(U) = A(u) - \frac{3}{2} \sum_{j=1}^3 \alpha_j y_{1j}. \]

The uniform estimates for $\tilde{A}$, $\tilde{A}^{-1}$, $\tilde{A}^{-1-1} \equiv \tilde{A}$, and the variation of $\tilde{A}'$ hold in the same form. In fact the linearized system is now

\[ A'(u) \rho = h + \sum_{j=1}^3 \alpha_j y_{1j} \] together with conditions (5).

Here $\alpha_j$ denotes the variation of $a_j$. The construction of the solution develops as before and the conclusion is the following result:

**THEOREM 4.** Let $\delta$ and $u_0$ verify the conditions of Theorem 2. Choose numbers $\mu > q > 9$, $s>9$, $a>2$. If $\|V - A(u_0)\|_q < \eta$, sufficiently small, the path $U_t = (a^1_t, a^2_t, a^3_t, u_t)$ defined by

\[ U_t = \mu \tilde{A}(S_t U_t) S_t [v - \tilde{A}(U_t)] \quad (S_t U = (a^1_t, a^2_t, a^3_t, S_t u)) \]

with initial condition $U_0 = (0,0,0,u_0)$, converges as $t \to \infty$ in $H^s_a$ norm to a solution $U_\infty$ of equation (7). Moreover the following estimates hold:

\[ \|u_t - u_\infty\|_{s-a} + \sum_{j=1}^3 |a_j^i| < c \cdot \|v - A(u_t)\|_s \]

\[ \|u_t - u_\infty\|_{s-a} + \sum_{j=1}^3 |a_j^i - a_j^i| < c \cdot e^{-t} \|v - A(u_0)\|_s \]

with $c$ independent of $t$. 

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5. We conclude with a uniqueness result for equation (2). An analogous result holds for equation (7).

THEOREM 5. Take $u_0$ and $\delta$ as in Theorem 1. If $\|u - u_0\|_4 < 1$, there exists a number $\rho_0$ such that, if $\|\rho\|_3 < \rho_0$, then $\|\rho\|_0 < c \|A(u+\rho) - A(u)\|_2$.

REFERENCES


AN INTRODUCTION TO INERTIAL MANIFOLDS

George R. Sell

School of Mathematics
and
Institute for Mathematics and its Applications
514 Vincent Hall
University of Minnesota

The partial differential equations

(1) \[ u_t - \nu \Delta u + B(u) = 0 \]

on \( \Omega = [0,2\pi]^2 \subseteq \mathbb{R}^2 \) and

(2) \[ u_t + \nu \Delta^2 u + \Delta u + \frac{1}{2} |\nabla u|^2 = 0, \]

on \( \Omega = [-\frac{L}{2}, \frac{L}{2}] \subseteq \mathbb{R}^1 \), with appropriate boundary conditions, generate nonlinear semi-flows on phase spaces \( H \), which are infinite dimensional Hilbert spaces. Equation (2) is the Kuramoto-Sivashinsky equation. This is a dissipative system [1]. We assume that the nonlinear term \( B(u) \) is fixed so that Eqn. (1) is also dissipative.

These equations are examples of partial differential equations which have inertial manifolds. An inertial manifold \( M \) is a finite dimensional manifold in \( H \) with the following properties:

1) \( M \) is attracting, i.e., for every solution \( u(t) \) one has \( \text{dist}(u(t), M) \to 0 \) as \( t \to +\infty \).

2) \( M \) is (positively) invariant, i.e. if \( u(0) \in M \) then \( u(t) \in M \) for all \( t > 0 \).

If a partial differential equation has an inertial manifold, then the asymptotic behavior of the dynamics of the equation is completely described by an ordinary differential equation on a finite dimensional space.

In this lecture we show that the existence of inertial manifolds can be established by means of the contraction mapping principle. The lecture includes a
discussion of the dimension of the inertial manifold as a function of the physical parameters. For example in Eqn. (1) it is shown that

\[ \log (\text{dimension } M) < C \nu^{-1} \]

where \( C \) is a constant depending on \( B \).

The theory described here represents joint work with the author, C. Foias and R. Temam.

Reference

SYMMEY-BREAKING

by Joel Smoller and Arthur Wasserman

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109

We consider the bifurcation of monotone decreasing radially symmetric solutions of \( \Delta u + f(u) = 0 \), on an \( n \)-ball of radius \( R \), into asymmetric ones. For homogeneous Dirichlet boundary conditions, we show that a necessary condition for the symmetry to break is that \( u'(R) = 0 \), and as a consequence, the symmetry cannot break if \( f(0) > 0 \). We prove that the manifold of asymmetric bifurcating solutions is equal to \( n \), and we give examples of functions \( f(u) \) for which the symmetry breaks. For homogeneous Neumann boundary conditions, we show that if the symmetry breaks, than the manifold of asymmetric bifurcating solutions must equal the number

\[
k_N = \left( \binom{N+n-2}{2} \right) \frac{n+2N-2}{n+N-2},
\]

for some \( N > 1 \). Since \( k_N > n \) if \( N > 1 \), the symmetry breaks here in a more complex way. We construct an example of a function \( f(u) \), where \( f(u) > 0 \) only on an interval \( 0 < u < c \), with the following property. Namely, there is a sequence of monotone, radial solutions \( \{ u_N \} \) of the Neumann problem, for which there bifurcates out of each \( u_N \), a \( k_N \)-dimensional manifold of asymmetric solutions.
The theory of scalar nonlinear first order partial differential equations has been substantially developed with the introduction by M.G. Crandall and P.-L. Lions [6] of the correct class of generalized solutions. We continue by recalling their definition. Let $K$ be a subset of $\mathbb{R}^N$ and $F:K \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous (i.e., $F \in C(K \times \mathbb{R} \times \mathbb{R}^N)$). A function $u \in C(K)$ is called a viscosity solution of $F(y,u,Du) < 0$ on $K$ if for each real-valued function $\phi$ which is continuously differentiable in a neighborhood of $K$ and each local maximum $z \in K$ of $u-\phi$ relative to $K$ one has

\begin{equation}
F(z,u(z),D\phi(z)) < 0.
\end{equation}

Here $D\phi = (\phi_1, \ldots, \phi_N)$ is the gradient of $\phi$. We will use the notation $C^1(K)$ to mean the set of functions which are defined and continuously differentiable on a neighborhood of $K$. Similarly, a viscosity solution of $F(y,u,Du) > 0$ in $K$ is a $u \in C(K)$ such that for every $\phi \in C^1(K)$ and local minimum $z \in K$ of $u-\phi$ relative to $K$ one has

\begin{equation}
F(z,u(z),D\phi(z)) > 0.
\end{equation}

A viscosity solution of $F = 0$ on $K$ is a function which is a viscosity solution of both $F < 0$ and $F > 0$. We also call viscosity solutions of $F < 0$ ($F > 0$) viscosity subsolutions (respectively, supersolutions) of $F = 0$.

The notion of viscosity solutions is important in view of the interaction between equations of Hamilton-Jacobi type, control theory and differential games. The first uniqueness theorems using this notion are proved in M.G. Crandall and P.-L. Lions [6]. M.G. Crandall, L.C. Evans and P.-L. Lions [5] provides a simpler introduction to the subject while the book by P.-L.
Lions [20] and the review paper M.G. Crandall and P.E. Souganidis [12] provide a view of the scope of theory and the reference to much of the recent literature.

Here we want to provide a guide to more recent results. In view of the rate the literature is growing, this report will not be complete, but we hope it will be useful.

REFERENCES


Typical computations in rheology deal with flow problems for high-polymer solutions or melts. One difficulty is that available equations are not known at all precisely, so one cannot trust predictions which are peculiar to a particular set of equations. Another is that, for some common types of problems, it is not easy to find numerical techniques which are reliable. The lecture described personal experiences in dealing with these difficulties.
LIQUID CRYSTAL BISTABLE DISPLAYS*

R.N. Thurston
Bell Communications Research
Holmdel, NJ 07733

ABSTRACT

Bistability, by which is intended the presence of two stable equilibrium states, implies storage of information, which means memory. The practical significance of bistability is that it eliminates the need to refresh, which is a major obstacle to matrix-addressed displays that have a large number of elements.

We review displays based on bistable configurations of the director orientation of the liquid crystal. Two types of such displays are distinguished by whether the bistable configurations are topologically distinct, requiring disclination motion in switching, or not. The disclination-free type can operate at low voltages of about 5V, whereas a high voltage of 100V or more is required for rapid switching by disclination motion.

An example of the high voltage type is an electrically written, thermally erased display based on the topologically distinct vertical and horizontal states. A 20 row by 40 column model has been demonstrated.

Examples of the low voltage, disclination-free type are a cholesteric twist cell and a display based on bistable boundary layer configurations. The boundary layer display bistability is insensitive to variations in holding voltage, cell parameters, material properties, and temperature. Switching is simple: a short, low voltage dc pulse selects one or the other of the bistable states, depending on the polarity of the pulse.

---

GREEN'S FORMULAE FOR LINEARIZED PROBLEMS WITH LIVE LOADS

Giorgio Vergara-Caffarelli
Dipartimento di Matematica
Universita di Pisa

For a pair of linear operators \((L,M)\), which arise for example as the linearizations of a boundary value problem of elastostatics at some state, we look for a Green's formula generalizing Betti's reciprocity theorem. Finding such a formula is a crucial point in order to apply the theory of linear boundary value problems for elliptic operators. If \(\Omega\) is smooth or piecewise smooth, normality of the boundary operator \(M\) can be used to construct an appropriate formula. If \(M\) is not normal, ambiguities may arise.

This work is more fully described in IMA preprint 121.
INTERIOR REGULARITY FOR SOLUTIONS OF
OBSTACLE PROBLEMS

by

William Ziemer

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $K$ be the subset of the Sobolev space $W^{1,\alpha}(\Omega)$ consisting of all $v$ such that $v$ agrees with a prescribed boundary function $\theta$ on $\partial\Omega$ in the sense of trace theory and $v(x) > \psi(x)$, for almost all $x \in \Omega$, where $\psi$ is a function (the "obstacle") defined on $\Omega$. We consider the variational inequality

$$
\int_{\Omega} A(x,u,vu) \cdot \nabla \psi + B(x,u,vu) \psi \, dx > 0
$$

for all $\psi \in W^{1,\alpha}(\Omega)$ with $\psi(x) > \psi(x) - u(x)$ for almost all $x \in \Omega$. The functions $A$ and $B$ are Borel measurable and satisfy the following structure conditions:

$$
|A(x,z,p)| < |p|^{\alpha-1} + \mu |z|^{\alpha-1} + \nu
$$

$$
p \cdot A(x,z,p) > |p|^{\alpha-1} - \mu |z|^{\alpha-1} - \nu
$$

$$
|B(x,z,p)| < \mu |p|^{\alpha-1} + \nu |z|^{\alpha-1} + \nu
$$

for $x \in \Omega$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, where $\mu$, $\nu$ are non-negative constants. The obstacle $\psi$ is assumed only to be a bounded measurable function that is defined everywhere except perhaps for a set of $\alpha$-capacity 0. We discuss the results of [MZ] in which the continuity of the solution is investigated. Drawing upon an idea in [CK], it is shown how weak-type Harnack inequalities can be used to establish continuity of the solution at points where the obstacle satisfies a very weak form of upper-semicontinuity. In case $\psi$ is the characteristic function of a compact set, the condition imposed on the obstacle is identical to the Wiener condition as employed in [GZ]. In the context of linear equations with bounded, measurable coefficients, these results were obtained by Frehse and
Mosco, [FM]. In particular, our results show that the solution is continuous if
the obstacle has a Lebesgue point everywhere and is upper-semicontinuous.
Moreover, if the obstacle is Holder continuous, then \( u \) is also Holder con-
tinuous, thus recovering the result in [DV].

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