EQUILIBRIA IN PRODUCTION ECONOMIES
WITH AN INFINITE-DIMENSIONAL COMMODITY SPACE

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ABSTRACT

This paper establishes the existence of competitive equilibria for economies with an infinite-dimensional space of commodities. The paper treats economies with or without production, and allows for commodity spaces whose positive cones have an empty interior. The crucial assumptions are on compactness of the set of feasible allocations, boundedness of the marginal rates of substitution in consumption, and boundedness of the marginal efficiency of production. Examples are presented to show that a competitive equilibrium may fail to exist if any of these assumptions fails to hold.
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1. INTRODUCTION

In the last dozen years, there has been an explosion of interest in economic models whose natural commodity spaces are infinite-dimensional. Examples* include:

(A) the use of $L_\infty$ to model infinite-horizon problems (Bewley (1972));

(B) the use of $L_\infty$ to model trading and production with uncertainty (Bewley (1972));

(C) the use of $\text{ca}(T)$ to model differentiated commodities (Mas-Colell (1975) and Jones (1984a));

(D) the use of $L_2$ to model financial instruments with uncertainty (Harrison and Kreps (1979), Duffie and Huang (1983) and Duffie (1984));

(E) the use of $\ell_1$ and $L_1$ to model perfect competition in large markets (Ostroy (1984)).**

Of particular interest in all these models is the existence of competitive equilibria. Although there are a number of existence results in the literature, none is entirely satisfactory. Bewley (1972) and Jones (1984a) treat only special spaces ($L_\infty$ or $L_\infty$ and $\text{ca}(T)$ respectively); Mas-Colell (1983) and Yannelis and Zame (1984) treat rather general spaces but restrict their attention to the exchange case; Duffie (1984) and Jones (1984c) require that the asymptotic production cone have a nonempty

* See Section 2 for definitions of these spaces.

** This list is not intended to be complete; see also the references herein and in Jones (1984b).
interior (this is a very unpleasant assumption in the infinite-dimensional case—see Section 5). *

This paper establishes the existence of equilibria in economies (with or without production) in rather general spaces and in rather general circumstances. These results allow for preferences which need not be transitive or complete ** or monotone. To describe the results a little more fully, it is necessary to make a few general remarks about equilibrium analysis in infinite-dimensional models.

There are two crucial issues which distinguish equilibrium analysis in the infinite-dimensional case from analysis in the finite-dimensional case. The first of these issues concerns compactness of the feasible consumption and production sets, without which optima—and hence equilibria—need not exist. In the finite-dimensional case, compactness of these sets is a consequence of the usual boundedness assumptions. In the infinite-dimensional case, boundedness will no longer suffice, and compactness assumptions must be made directly.

* Again, this list is not intended to be complete (see the references), but it presents a fair picture of the state of the art.

** Allowing for preferences which may not be transitive or complete seem particularly appropriate in infinite-horizon models, since the preferences of infinitely-lived agents may well arise from aggregation of the preferences of infinitely many finitely-lived agents. Since allowing for intransitive and incomplete preferences does not complicate any of the proofs given here, the principal drawback seems to be that Pareto optimality of equilibria cannot be tested, since it does not even make sense. However, Debreu (1954) has shown that for preferences which are transitive and complete, equilibria are Pareto optimal in very general circumstances, including the circumstances discussed here.
The other crucial issue concerns the existence of supporting hyperplanes. At equilibrium, the price functional defines hyperplanes which support the preferred set for each consumer and the production possibility set for each firm. In the finite-dimensional case, convexity alone allows us to find hyperplanes which support each of these sets individually; the difficulty lies in finding a family of supporting hyperplanes defined by a single price. In the infinite-dimensional case, however, convexity alone will not suffice even for the existence of individual supporting hyperplanes. In economic terms, the equilibrium price is a measure of the relative marginal rates of desirability of goods (to consumers) and the relative marginal rates of usefulness of goods (to firms). With infinitely many goods, these relative marginal rates may be unbounded in a way that precludes measurement by any price, so that equilibria need not exist.

Both of these issues present real difficulties. Section 3 presents a number of simple examples to illustrate how they may lead to nonexistence of equilibria.

In order to deal with the first of these issues, compactness assumptions will be imposed on the feasible consumption and production sets. Fortunately, these assumptions are quite weak, and are satisfied in most of the interesting models.*

---

* Aliprantis and Brown (1983) have given an equilibrium existence result (for exchange economies) which does not impose such compactness assumptions. However, Aliprantis and Brown take demand functions—rather than preferences—as the primitive notion. This is a different approach, and is not comparable to the preference-based approach taken here. If one were to begin with preferences as the primitive notion and attempt to establish the existence of equilibria via the methods of Aliprantis and Brown, the issue of compactness would arise.
In order to deal with the second of these issues, two different approaches will be taken. If the initial endowments lie in the interiors of the consumption sets, it will be shown (Theorem 1 of Section 5) that the difficulties with separating hyperplanes do not occur, and that the existence of equilibria can be established without any further assumptions. Without such interiority assumptions, additional assumptions will be imposed on preferences and on production possibility sets, leading to results on the existence of equilibria (Theorems 2 and 3 of Section 7). Roughly speaking, the assumption on the consumption side bounds the marginal rates of substitution for all bundles in terms of a fixed bundle; this assumption will be satisfied, for example, if preferences are given by monotone, concave utility functions defined on the entire commodity space. The assumption on the production side bounds the marginal efficiency of production processes.

I have not attempted to find the weakest possible assumptions which will guarantee the existence of equilibria. Rather, I have tried to make assumptions which seem to have natural economic interpretations and address directly the crucial issues; the reader will have to judge whether I have succeeded.

because, in the absence of compactness, demand need not be well-defined. For a discussion of demand in infinite-dimensional spaces, see Jones (1984b).

* This is an assumption which is frequently made in the classical finite-dimensional results, so it seems natural to call Theorem 1 a "neo-classical existence theorem." In the infinite-dimensional case, such an assumption is natural in some models and unnatural in others; see Section 5 for a more detailed discussion.
The remainder of the paper is organized in the following way. Section 2 presents background material and discusses the general framework, particularly the commodity spaces and prices which are considered. Examples of the non-existence of equilibria are presented in Section 3. Compactness of the feasible sets and continuity of preferences is discussed in Section 4, and Theorem 1 is presented in Section 5, together with some illustrative examples. The additional assumptions on consumption and production are discussed in Section 6, and Theorems 2 and 3 are presented in Section 7, together with some illustrative examples. The proof of Theorem 1 is given in Section 8, while the proofs of Theorems 2 and 3 are given in Section 9. Finally, Section 10 indicates some possible improvements and extensions and makes some concluding remarks.
2. ECONOMIES IN INFINITE-DIMENSIONAL SPACES

The commodity spaces we consider are normed spaces, or normed lattices. We begin with a brief review of these notions.

A **normed space** is a real vector space \( L \) equipped with a norm, i.e., a function \( \| \cdot \| : L \rightarrow [0, \infty) \) such that:

\[
\|x\| = 0 \text{ if and only if } x = 0;
\]
\[
\|\lambda x\| = |\lambda|\|x\| \text{ for each } x \in L, \lambda \in \mathbb{R};
\]
\[
\|x + y\| \leq \|x\| + \|y\| \text{ for each } x, y \in L.
\]

By the **dual space** of \( L \) we mean the space \( L' \) of continuous linear functionals \( \pi : L \rightarrow \mathbb{R} \). The dual space of \( L \) is also a normed space, when equipped with the norm

\[
\|\pi\| = \sup \{\|\pi x\| : x \in L, \|x\| \leq 1\}.
\]

A **normed lattice** is a normed space \( L \), together with a partial order \( \leq \) (i.e., a reflexive, antisymmetric transitive relation) satisfying:

(a) \( x \leq y \) implies \( x + z \leq y + z \), for \( x, y, z \in L \);

(b) \( x \leq y \) implies \( \lambda x \leq \lambda y \) for \( x, y \in L, \lambda \in \mathbb{R}^+ \);

(c) every pair of elements \( x, y \in L \) has a supremum (least upper bound) \( x \vee y \) and an infimum (greatest lower bound) \( x \wedge y \);

(d) \( |x| \leq |y| \) implies \( \|x\| \leq \|y\| \) for \( x, y \in L \).

Here we have written, as usual, \( |x| = x^+ + x^- \), where \( x^+ = x^0 \) (the **positive part** of \( x \)) and \( x^- = (-x)^0 \) (the **negative part** of \( x \)); we call \( |x| \) the **absolute value** of \( x \).
Recall that \( x = x^+ - x^- \) and that \( x^+ \land x^- = 0 \) (we say that \( x^+ \) and \( x^- \) are disjoint). Finally, we say that \( x \) is positive if \( x \geq 0 \); we write \( L^+ \) for the set of positive elements of \( L \) and refer to \( L^+ \) as the positive cone of \( L \).

The crucial property of normed lattices which we use several times is the Riesz Decomposition Property.

RIESZ DECOMPOSITION PROPERTY: If \( L \) is a normed lattice and \( x_1, \ldots, x_k, y \) are positive elements of \( L \) with \( y \leq \sum_{i=1}^{k} x_i \), then there are positive elements \( y_1, \ldots, y_k \) of \( L \) such that \( y = \sum_{i=1}^{k} y_i \) and \( y_i \leq x_i \) for each \( i \).

For more information about normed spaces and normed lattices, we refer to Schaefer (1971, 1974).

The following spaces, which have all been used in economic models, are all normed lattices (in fact, they are all Banach lattices, i.e., they are complete normed lattices):

\( l_1 \) — the space of summable real sequences;

\( l_2 \) — the space of square summable real sequences;

\( l_\infty \) — the space of bounded real sequences;

\( L_1(\Omega, \mathcal{F}, m) \) — the space of (equivalence classes of) integrable functions on a (sigma-finite\(^*\)) measure space;

\( L_2(\Omega, \mathcal{F}, m) \) — the space of (equivalence classes of) square integrable functions on a measure space;

\(^*\) The restriction to sigma-finite measure spaces is merely for convenience.
\( L_\infty (\Omega, \mathcal{G}, \mu) \) — the space of (equivalence classes of) bounded measurable functions on a measure space;

\( C(T) \) — the space of continuous, real-valued functions on a compact metric space;

\( ca(T) \) — the space of countably-additive, finite Borel measures on a compact metric space.

For \( L \) a normed space (or a normed lattice) an economy, with \( N \) consumers and \( M \) firms, with commodity space \( L \) (for short: an economy in \( L \)) is a family

\[
\mathcal{E} = \{ (X_i), (P_i), (\omega_i), (\theta_{ij}), (Y_j) \}
\]

where, as usual, \( X_i \) is the consumption set of the \( i \)-th consumer, \( P_i \) is the preference relation of the \( i \)-th consumer, \( \omega_i \) is the initial endowment of the \( i \)-th consumer, the numbers \( \theta_{ij} \) are firm shares, and \( Y_j \) is the production set of the \( j \)-th firm. By an allocation for the economy \( \mathcal{E} \) we mean an \( N+M \)-tuple \( ((x_i), (y_j)) \) where \( x_i \in X_i \) for each \( i \) and \( y_j \in Y_j \) for each \( j \). An allocation \( ((x_i), (y_j)) \) is feasible if

\[
\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j
\]

The projection into \( X_k \) of the set of feasible allocations is the feasible consumption set \( \hat{X}_k \) of the \( k \)-th consumer; the projection into \( Y_\ell \) of the set of feasible allocations is the feasible production set \( \hat{Y}_\ell \) of the \( \ell \)-th firm. Equivalently,

\[
\hat{X}_k = \left( \sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j - \sum_{i \neq k} x_i \right) \cap X_k
\]
\[ \hat{Y}_t = \left( \sum_{i=1}^{N} X_i - \sum_{i=1}^{N} \omega_i - \sum_{j \neq t} Y_j \right) \cap Y_t. \]

We shall always assume that the economy \( E \) enjoys the following properties, which we will refer to as the **Standard Assumptions:**

1. \( X_i \) is closed and convex and \( \omega_i \in X_i \) (for each \( i \));
2. \( P_i : X_i \to 2^X_i \) is a correspondence whose graph is a (relatively) open subset of \( X_i \times X_i \) (for each \( i \));
3. \( P_i(x) \) is convex and \( x \notin P_i(x) \) (for each \( i \) and each \( x \in X_i \));
4. for each \( x \) in the feasible consumption set \( \hat{X}_i \), \( x \) belongs to the boundary (relative to \( X_i \)) of \( P_i(x) \) (local nonsatiation);
5. \( 0 \leq \theta_{ij} \leq 1 \) (for each \( i, j \)) and \( \sum_{i=1}^{N} \theta_{ij} = 1 \) (for each \( j \));
6. \( Y_j \) is a closed convex set and \( 0 \in Y_j \) (for each \( j \)).

Note that we have not assumed any disposability at all; it is simply unnecessary. We have also made no boundedness assumptions; as indicated in the Introduction, boundedness assumptions are not adequate, and we shall make compactness assumptions instead. (See Theorems 1,2,3.)

An equilibrium for \( E \) is an \( N+M \)-tuple \((x_i),(y_j),\pi)\) where \( x_i \in X_i \) (for each \( i \)), \( y_j \in Y_j \) (for each \( j \)) and \( \pi \in L' \setminus \{0\} \) (i.e., \( \pi \) is a nonzero continuous linear functional on \( L \)) such that

1. \( \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j \); 
2. \( \pi(y_j) = \max \{ \pi(y_j^*) : y_j^* \in Y_j \} \) for each \( j \);
(3) \[ \pi(x_i) \leq \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j) \] for each \( i \);

(4) if \( x^*_i \in P_i(x_i) \) then \( \pi(x^*_i) > \pi(\omega_i) + \sum_{j=1}^{N} \theta_{ij} \pi(y_j) \).

A quasi-equilibrium is an \( N+M \)-tuple \((x_i, y_j, \pi)\) satisfying all of the above requirements except that instead of (4), we have

(4') if \( x^*_i \in P_i(x_i) \) then \( \pi(x^*_i) \geq \pi(\omega_i) + \sum_{j=1}^{N} \theta_{ij} \pi(y_j) \).

If \( L \) is a normed lattice, and \( \mathcal{E} \) is an economy in \( L \), we shall say that \( \mathcal{E} \) is irreducible if: whenever \( I \) and \( J \) are nonempty sets of consumers with \( I \cap J = \emptyset \) and \( I \cup J = \{1, 2, \ldots, N\} \), and \((x_i, y_j)\) is a feasible allocation for \( \mathcal{E} \), then there is a consumer \( m \in I \) and a consumer \( n \in J \) and a vector \( z \in x_n \) with \( z \leq \omega_n \) and \( x_m + z \in P_m(x_m) \). (See McKenzie (1981).)

We frequently refer to a vector \( x \in L \) as a commodity bundle or just a commodity. The reader should keep in mind, however, that in our abstract framework, there are no pure commodities.

A few comments on this framework seem in order. Normed lattices are natural infinite-dimensional generalizations of \( \mathbb{R}^n \); more general spaces could undoubtedly be used, at the cost of making the arguments more complicated. (Topological vector lattices—more specifically, Riesz spaces—are perhaps the "right" infinite-dimensional setting; see Aliprantis and Brown (1983).) The lattice structure of the commodity space plays an important role in Theorems 2 and 3 (but not in
Theorem 1, and Theorem 1 is formulated in normed spaces in order to emphasize this point). It might seem that ordered normed spaces—rather than normed lattices—would be the natural commodity spaces, but I think this is not so. The reason is that many economic notions lose their usual meanings if the lattice structure is missing. For example, given a production plan \( y \), we would like to write \( y = y^+ - y^- \), and interpret \( y^- \) as input and \( y^+ \) as output. But this does not make sense if the commodity space is not a lattice, since there will then be no canonical way to write \( y \) as the difference of positive elements (See also the discussion in Yannelis and Zame (1984).)

The restriction to equilibrium prices which are continuous linear functionals (rather than arbitrary linear functionals) also requires some comment. I have made this restriction because it seems mathematically and economically natural. In most circumstances, this is a mild restriction. For example, if the commodity space is a complete normed lattice (as are all of the examples given earlier), and equilibrium prices are positive (which will be the case if preferences are suitably monotone), the equilibrium prices will automatically be continuous (because positive linear functionals on complete normed lattices are automatically continuous—see Schaefer (1974)). If equilibrium prices are not positive, these remarks do not apply; however, the very existence of discontinuous linear functionals on complete normed lattices (or complete normed spaces) cannot be established without the Axiom of Choice (Wright, 1973); it is hard for me to imagine that such prices could have any very natural economic interpretation.
3. NONEXISTENCE OF EQUILIBRIA: EXAMPLES

This section presents several simple examples to illustrate how the difficulties described in the Introduction can lead to nonexistence of equilibria in the infinite-dimensional setting. The first example, which is due to Jones (1984b), illustrates the difficulty caused by the failure of the feasible consumption sets to be compact. For other examples of the same phenomenon, see Araujo (1974), Mas-Colell (1983) or Jones (1984b).

EXAMPLE 1: We consider an economy with commodity space $C[0,1]$ (the space of continuous, real-valued functions of the closed unit interval). There are two consumers, with consumption sets $X_1 = X_2 = C[0,1]^+$, and preferences given by the utility functions

$$u_1(x) = \int_0^1 tx(t) \, dt,$$

$$u_2(x) = \int_0^1 (1-t)x(t) \, dt.$$

Finally, the initial endowments of the two consumers are $\omega_1 = \omega_2 \equiv 1/2$. There is no reduction.

It is easy to see that the only optima for this economy allocate 0 to one or the other of the consumers; thus, no equilibrium exists. The difficulty here is that the feasible consumption sets,
\[ \hat{X}_i = \{ x : 0 \leq x(t) \leq 1 \text{ for each } t \} , \]

which are norm closed and bounded, are not compact (in any natural topology).

The remaining examples illustrate the difficulties that arise from the failure of the separating hyperplane theorem. The first example treats a one-consumer economy with no production; this example is taken from Yannelis and Zame (1984) and is a variant of an example in Mas-Colell (1983).

EXAMPLE 2: We consider an economy whose commodity space is \( \ell_1 \) (the space of summable real sequences). The consumption set of the single consumer is \( \ell^+_1 \), the positive cone. The consumer's preferences are given by the utility function

\[ u(x) = \sum_{n=1}^{\infty} \phi_n(x(n)) , \]

where

\[ \phi_n(t) = \begin{cases} 2^n \cdot t & \text{for } 0 \leq t \leq 2^{-2n} \\ 2^{-n} \cdot t + 2^{-n} - 2^{-3n} & \text{for } 2^{-2n} < t < \infty \end{cases} . \]

Finally, the consumer's initial endowment \( \omega \) is the sequence

\[ \omega(n) = 2^{-2n-1} . \]

To see that this economy has no equilibrium, note that the dual space of \( \ell_1 \) is \( \ell_\infty \) (the space of bounded sequences), so that a price is a bounded sequence; \( \pi = (\pi(1), \pi(2), \ldots) \). Since there is neither trade nor production, an equilibrium price \( \pi \) would need to have the property that \( \pi y > \pi \omega \) whenever \( u(y) > u(\omega) \). The
usual marginal utility analysis then yields that $\pi(1) > 0$, and that $\pi(n+1) \geq 2\pi(n)$ for each $n$, which contradicts the boundedness of the sequence.*

The next example shows how the failure of the separating hyperplane theorem can be manifested in the interaction between consumption and production.

EXAMPLE 3: We again consider a one-consumer economy with commodity space $\ell_1$. The consumer's consumption set is $\ell_1^+$ and his preferences are given by the utility function

$$u(x) = \sum_{n=1}^{\infty} x(n).$$

The consumer's initial endowment $\omega$ is the sequence

$$\omega(n) = 4^{-n}.$$

The economy has one firm; to describe its production possibility set, write $\delta_k$ for the sequence

$$\delta_k(n) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}. $$

The production possibility set $Y$ is the closed convex cone at $0$ generated by the set

$$\left\{-\delta_{2k-1} + k\delta_{2k} : k = 1, 2, \ldots \right\}. $$

It is easily checked that this economy satisfies all the standard assumptions and that, in fact, the feasible consumption and production sets are norm compact.

* It is very easy to show that a similar argument could be made without assuming continuity of $\pi$. See the comments in Section 2.
To see that this economy has no equilibrium, note that for an equilibrium price $\pi$, marginal utility analysis would yield

$$0 < \pi(2) = \pi(4) = \pi(6) \ldots$$

On the other hand, profit maximization by the firm yields

$$\pi(2k-1) \geq k\pi(2k) \text{ for each } k.$$

Combining these observations contradicts the fact that $\pi$ must be a bounded sequence.

The next example is a simple variant, with two firms, which illustrates how difficulties may arise from the interaction between firms.

**EXAMPLE 4:** Again, the commodity space is $l_1^+$, the consumption set of the single consumer is $l_1^+$, his preferences are given by the utility function

$$u(x) = \sum_{n=1}^{\infty} x(n),$$

and his initial endowment $\omega$ is the sequence

$$\omega(n) = 4^{-n}.$$

The economy has two firms. The production possibility set $Y_1$ of the first firm is the closed convex cone at 0 generated by the set

$$\left\{-\delta_{2k-1} + 2\delta_{2k} : k = 1, 2, \ldots \right\}.$$

The production possibility set $Y_2$ of the second firm is the closed convex cone at 0 generated by the set

$$\left\{-\delta_{2k} + \delta_{2k+1} : k \text{ not a power of 2} \right\}.$$
Again, it is easily checked that this economy satisfies all the standard assumptions and that in fact the feasible consumption and production sets are norm compact.*

As in Example 3, it may easily be seen that this economy has no equilibrium. For, if \( \pi \) were an equilibrium price, marginal utility analysis would yield:

\[
0 < \pi(2) = \pi(4) = \pi(8) = \pi(16) = \ldots
\]

On the other hand, profit-maximization by the first firm yields that

\[
\pi(2k - 1) \geq 2\pi(2k)
\]

for each \( k \), and profit-maximization by the second firm yields that

\[
\pi(2k) \geq \pi(2k + 1)
\]

provided that \( k \) is not a power of 2. Combining these inequalities contradicts boundedness of the sequence \( \pi \).

In the final example, the difficulty is all on the production side of the economy.

EXAMPLE 5: The commodity space is \( \ell_1 \). There is one consumer, whose consumption set is \( \ell_1^+ \), and whose initial endowment \( \omega \) is the sequence

\[
\omega(n) = 4^{-n}.
\]

We assume the consumer has monotone preferences (but the precise nature of the preferences is irrelevant). There is one firm, whose production possibility set \( Y \) is the closed convex hull (not a cone!) of the set

\[
Y_0 = \left\{ 2^{-k+1} \delta_k - 2^{-k} \delta_{k+1} : k = 1, 2, \ldots \right\}.
\]

* It is important here that \(-\delta_{2k} + \delta_{2k+1}\) does not belong to \( Y_2 \) if \( k \) is a power of 2.
Again, all the standard assumptions are easily checked, and the feasible consumption and production sets are norm compact; indeed, the entire production possibility set is norm compact.

To see that this economy has no equilibrium requires a little analysis. Suppose that \((x, y, \pi)\) were an equilibrium. Since \(y\) lies in the closed convex hull of a sequence converging to 0, we can write

\[
y = \sum_{k=1}^{\infty} \alpha_k (2^{-k+1} \delta_k - 2^{-k} \delta_{k+1})\]

where \(\alpha_1, \alpha_2, \ldots\) are real numbers, \(0 \leq \alpha_k \leq 1\) for each \(k\), and \(\sum_{k=1}^{\infty} \alpha_k \leq 1\).

Because markets clear, \(x = \omega + y\); in particular, \(y \geq -\omega\). Hence,

\[
y(k+1) = 2^{-k} \alpha_{k+1} - 2^{-k} \alpha_k \geq -\omega(k+1) = -4^{-(k+1)}
\]

for each \(k\), or equivalently,

\[
\alpha_k - \alpha_{k+1} \leq 2^{-k-2}
\]

for each \(k\). If we now sum these inequalities from \(n\) to infinity and note that the left-hand side telescopes, we obtain

\[
\alpha_n \leq \sum_{k=n}^{\infty} 2^{-k-2} = 2^{-n-1}
\]

for each \(n\). Hence

\[
\sum_{n=1}^{\infty} \alpha_n \leq \sum_{n=1}^{\infty} 2^{-n-1} = 1/2.
\]

Since \(Y\) is convex, this means that \(y + \frac{1}{2} z\) belongs to \(Y\) for each \(z\) in \(Y_0\). Since

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y is profit-maximizing, we conclude that $\pi z \leq 0$ for each $z$ in $Y_0$; hence
\[ \pi(n+1) \geq 2\pi(n) \]
for each $n$. On the other hand, monotonicity of preferences guarantees that $\pi(n) > 0$ for each $n$, and since $\pi \neq 0$, $\pi(m) > 0$ for some $m$. Once again, we conclude that $\pi$ cannot be bounded, a contradiction.

Notice what this analysis actually shows: no feasible production plan can be profit-maximizing (at any positive price).

It is useful to interpret the last four examples in economic terms. We view elements of $l$ as representing allocations of a single (non-renewable) resource over an infinite time horizon. In Example 2, the consumer's preference strongly emphasizes the future; an equilibrium price would have to keep pace with this emphasis, but it cannot do so (and still remain bounded*). In Example 3, the consumer's preferences are time-independent, but the firm's technology becomes unboundedly efficient in the future; once again, prices cannot keep pace.* (Note that the preferences serve to connect the periods.) Example 4 exhibits the same phenomenon, except that it is the aggregate technology that becomes unboundedly efficient, rather than the technology of the individual firms. Example 5 is a little more subtle; in effect, the nonexistence of an equilibrium can be traced to unboundedness of the marginal efficiency of production.

* Again, see the comments in Section 2 about the restriction to continuous prices.
4. COMPACTNESS AND CONTINUITY

As noted in the Introduction, the function of the usual boundedness assumptions in the finite-dimensional setting is to guarantee boundedness—and hence compactness—of the feasible consumption and production sets. Since boundedness will not suffice to guarantee compactness in the infinite-dimensional setting, we will make compactness assumptions directly.

Our compactness assumptions will be made in the following form. In Theorem 1, we will assume the existence of a Hausdorff vector space topology\(^*\) \(\mathcal{J}\) with respect to which the feasible and consumption sets are compact. (Requiring that the feasible consumption and production sets be \(\mathcal{J}\)-compact is exactly equivalent to requiring that the set of feasible allocations be compact in the product space \(L^{N+M}\), when we endow each factor with the topology \(\mathcal{J}\).) In Theorems 2 and 3 we also require (for technical reasons) that order intervals\(^**\) be \(\mathcal{J}\)-compact. In the Theorems, the topology \(\mathcal{J}\) will be left unspecified because the choice of \(\mathcal{J}\) will vary with the choice of commodity space. To illustrate this point, we consider several examples.

We first recall that if \(E\) is a normal space and \(E'\) is its dual space, then the weak-star topology on \(E'\) (sometimes written \(\sigma(E', E)\)) is the topology for which

\(^*\) That is, a Hausdorff topology in which the vector space operations are continuous.

\(^**\) An order interval in a normed lattice \(L\) is any set of the form \([a, b] = \{c \in L : a \preceq c \preceq b\}\).
convergence of the net \( \{\pi^\alpha\} \) to \( \pi \) means that the net of real numbers \( \{\pi^\alpha(x)\} \) converges to \( \pi(x) \) for each \( x \) in \( E \). The \textbf{weak topology} on \( E \) (sometimes written \( \sigma(E, E') \)) is the topology for which convergence of the net \( \{x^\alpha\} \) to \( x \) means that \( \{\pi(x^\alpha)\} \) converges to \( \pi(x) \) for each \( \pi \) in \( E' \). If \( E \) is a \textbf{reflexive space} (i.e., if \( E = (E')' \)) then the weak and weak–star topologies on \( E \) coincide.

(A) If the commodity space \( L \) is a dual space (such as \( \ell_\infty \) or \( L_\infty \) or \( \text{ca}(T) \)), it is natural to take for \( J \) the weak–star topology. A set is weak–star compact precisely when it is weak–star closed and norm bounded. It follows that the feasible consumption and production sets will be weak–star compact if the consumption and production sets are themselves weak–star closed* and the feasible consumption and production sets are norm bounded.** If \( L \) is a dual lattice (\( \ell_\infty \), \( L_\infty \) and \( \text{ca}(T) \) are all dual lattices) then order intervals will also be weak–star closed and norm bounded, hence weak–star compact.

(B) If the commodity space \( L \) is a reflexive space (such as \( \ell_2 \) or \( L_2 \)), we can take for \( J \) the weak topology, which (in reflexive spaces) coincides with the weak–star topology. This is a particularly pleasant choice, since the weak topology

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* Since the Mackey topology and the weak–star topology have the same closed convex sets, this is the same as saying that the consumption and production sets are Mackey closed.

** Note that these conditions are sufficient but not necessary: it is possible for the feasible consumption and production sets to be weak–star compact even if the consumption and production sets are not weak–star closed.
and the norm topology have the same closed convex sets, and sets are weakly compact exactly when they are weakly closed and norm bounded. Hence, the feasible consumption and production sets will be weakly compact whenever the consumption and production sets are norm closed (which is part of our Standard Assumptions) and the feasible consumption and production sets are norm bounded. Moreover, order intervals will always be weakly compact.

(C) If the commodity space is $L_1$, we may again take for $J$ the weak topology. Once again, the weak and norm topologies have the same closed convex sets. However, sets which are weakly closed and norm bounded need not be weakly compact. On the other hand, order intervals in $L_1$ are weakly compact. Thus, the feasible consumption and production sets will be weakly compact if they are order bounded; i.e., if they are contained in order intervals. (This would be the case, for example, if the consumption sets are order-bounded below and there is no production.)

(D) If the commodity space is $\ell_1$, it is natural to take for $J$ the norm topology itself, since in this case the weak topology and the norm topology have the same compact sets. The remarks of (C) apply in this case as well. Alternatively, we could take for $J$ the weak-star topology (viewing $\ell_1$ as the dual space of $c_0$, the space of sequences which converge to 0), in which case the remarks of (A) apply.

In view of the multitude of possible topologies, it seems natural to formulate results in a manner that includes all of them, rather than giving a separate formulation in each particular case. This is what we shall do.
It should be evident that compactness—in some topology—will be of no use unless preferences enjoy some continuity property in the same topology. We will require that each of the preference relations $P_i$ have a (relatively) open graph in $X_i 	imes X_i$, when we give the first factor the topology $\mathcal{J}$ and the second factor the norm topology. Equivalently, we will require that if $y \in P_i(x)$, then there is a (relatively) norm open subset $U$ of $X_i$, containing $y$, and a (relatively) $\mathcal{J}$-open subset $V$ of $X_i$, containing $x$, such that $y' \in P_i(x')$ whenever $y' \in U$ and $x' \in V$. If preferences are transitive and complete (i.e., if $P_i$ is the irreflexive part of a transitive, complete (weak) preference relation), this amounts to norm continuity together with $\mathcal{J}$-upper semicontinuity. If $\mathcal{J}$ is the weak topology, convexity of preferences implies that $\mathcal{J}$ upper semicontinuity follows automatically from norm continuity. For more on these points, see Yannelis and Zame (1984).

Notice that there is a tension between these compactness and continuity requirements. On the one hand, we would like the topology $\mathcal{J}$ to be very weak (so that it is easy for the feasible consumption and production sets to be $\mathcal{J}$-compact). On the other hand, we would also like the topology $\mathcal{J}$ to be very strong (so that it is easy for preferences to be $\mathcal{J}$-upper semicontinuous). Fortunately, there is usually a comfortable middle ground, as indicated above.
5. A NEO-CLASSICAL EXISTENCE THEOREM

As suggested in the Introduction, convexity of the preferred sets will not, of itself, suffice for the existence of supporting prices. What will suffice is convexity together with interior points. It should not seem surprising, therefore, that such assumptions, together with compactness, lead to an equilibrium existence theorem. Notice that this theorem looks very much like classical existence theorems, except that the usual boundedness assumptions are replaced by compactness assumptions. For a detailed discussion of the meaning of the compactness and continuity assumptions, see Section 4.

THEOREM 1: Let $L$ be a normed space and let $\mathcal{E}$ be an economy in $L$ which satisfies the Standard Assumptions. Let $\mathcal{J}$ be a Hausdorff vector space topology on $L$, which is weaker than the norm topology, such that:

1. the feasible consumption sets $\hat{X}_i$ are $\mathcal{J}$-compact (for each $i$);  
2. the feasible production sets $\hat{Y}_j$ are $\mathcal{J}$-compact (for each $j$);*  
3. the graph of the preference relation $P_i$ is a (relatively) $\mathcal{J} \times$ norm-open subset of $X_i \times X_i$ (for each $i$).

Then

(a) If the norm interior of the consumption set $X_i$ is not empty (for each $i$), then $\mathcal{E}$ has a quasi-equilibrium.

* As noted in Section 4, (1) and (2) together are equivalent to the requirement that the set of feasible allocations is compact in the product space $L^{N+M}$, where we give each factor the topology $\mathcal{J}$.
(b) If the initial endowment \( \omega_i \) belongs to the norm interior of \( X_i \) (for each \( i \)), then every quasi-equilibrium for \( \mathcal{E} \) is an equilibrium. *

We defer the proof of Theorem 1 to Section 8, but it may help to understand the statement if we consider a couple of examples.

**EXAMPLE 6.** Let the commodity space be \( L_\infty \) (of any sigma-finite measure space***) and take for \( \mathcal{J} \) the weak-star topology. Assume that the consumption and production sets are weak-star closed and that the feasible consumption and production sets are norm bounded. As noted in Section 4, this guarantees that the feasible consumption and production sets are weak-star compact. Hence, if the continuity assumptions (3) are satisfied and the consumption sets have nonempty norm interiors (for example, if each of the consumption sets is the positive cone), Theorem 1(a) assures us that a quasi-equilibrium exists. If the initial endowments lie in the interiors of the consumption sets, Theorem 1(b) assures us that an equilibrium exists. +

**EXAMPLE 7:** Let the commodity space be \( L_2 \) (of any measure space) and take for \( \mathcal{J} \) the weak topology (which coincides with the weak-star topology). Assume that the consumption set for each consumer is the set

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* See Remark 1 in Section 10 for ways in which this result may be extended.

** The restriction to sigma-finite measure spaces is made simply to ensure that \( L_\infty \) is the dual space of \( L_1 \).

+ This is closely related to Theorem 1 of Bewley (1972)—without transitive or monotone preferences.
\[ X = \left\{ x \in L_2^+ : \text{distance}(x, L_2^+) \leq 1 \right\} \]

and that the initial endowment for each consumer lies in \( L_2^+ \) (and hence in the interior of \( X \)). Assume also that the preferences of each consumer are given by norm-continuous, concave utility functions \( u_i : X \to \mathbb{R} \). Finally, assume that the production sets are norm closed and convex (as always) and that the feasible production sets are norm bounded. Then an equilibrium exists.

To see this, note that boundedness of the feasible production sets, together with the choice of consumption sets, implies that the feasible consumption sets are also norm bounded. As noted in Section 4, this implies weak compactness of the feasible consumption and production sets. In addition, as also noted in Section 4, norm continuity and concavity of the utility functions \( u_i \) implies their weak upper-semicontinuity. Hence Theorem 1 assures us that an equilibrium exists.

This example has a natural economic interpretation in terms of the stock-trading models of Harrison and Kreps (1979), Duffie and Huang (1983) and Duffie (1984). In these models, negative vectors in \( L_2 \) represent short sales of stock. Consumption sets such as \( X \) thus correspond naturally to the assumption that consumers are allowed limited short sales.

One of the virtues of Theorem 1 is its close resemblance to the usual finite-dimensional results. Its greatest drawback is the requirement that the consumption sets have nonempty interior. In many models, the natural consumption sets are the positive cone of the commodity space (or a subset of the positive cone); of the spaces we have discussed, only \( \ell_\infty \) and \( L_\infty \) have positive cones whose interiors are not empty. As Example 7 suggests, however, consumption sets not contained in the positive cone do arise in some models, so Theorem 1 is of value. We shall see in the proof of Theorem 2 that it is also a useful tool.
We note that Duffie (1984) and Jones (1984c) have established a result somewhat parallel to Theorem 1, in which the interiority assumptions are made on the asymptotic production cone, rather than on the consumption sets.* Such a result suffers from the same sort of drawback as Theorem 1: in many natural models, the asymptotic production cone does not have interior; indeed the asymptotic production cone may well consist only of the zero vector. There are circumstances, however, in which the Duffie and Jones results are useful tools.

Finally, it is worth noting the ways in which the assumptions of Theorem 1 prevent the kinds of misbehavior cited in the Introduction and Section 3.

1. The compactness and continuity assumptions guarantee the existence of nontrivial optima.

2. The interiority assumptions rule out preferences such as those in Example 2.

3. The interiority assumptions guarantee that the consumption sets are not too small; this also guarantees that the feasible production sets are not too small. This has two effects. With production sets such as those of Examples 3 and 4, the feasible production sets would not be compact if the consumption sets had interior; we have already ruled out this possibility. With production sets such as that of Example 5, larger feasible production sets means that it is easier to find feasible profit-maximizing plans; notice in particular that the misbehavior in

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* It seems likely that there is a general result which includes both Theorem 1 and the Duffie and Jones results, but formulating such a result seems cumbersome.
Example 5 depends critically on the fact that relatively few production plans are feasible.
6. EXTREMELY DESIRABLE COMMODITIES AND MARGINAL EFFICIENCY OF PRODUCTION

As we have noted, interiority assumptions on the consumption sets rule out the kinds of misbehavior illustrated in Examples 2 through 5 of Section 3. However, such interiority assumptions also rule out many of the interesting infinite-dimensional models. The purpose of this section is to introduce and discuss assumptions which allow us to obtain the existence of equilibria without interiority assumptions. As suggested by the Examples, we shall need assumptions on both the consumption and production sides of the economy.

Throughout this section, we will restrict our attention to the context of Theorems 2 and 3 of Section 7: the commodity space $L$ is a normed lattice, $\xi$ is an economy in $L$ which satisfies the Standard Assumptions, and the consumption sets coincide with the positive cone; i.e., $X_i = L^+$ for each $i$.\textsuperscript{*} We begin our discussion on the consumption side of the economy.

**Extremely Desirable Commodities**

We begin with a definition. Let $i$ be a consumer, let $v_i$ be a vector in the positive cone $L^+$ and let $W_i$ be a subset of $X_i = L^+$.

**DEFINITION:** We say that $v_i$ is extremely desirable for $P_i$ on $W_i$ if there is a real number $\mu_i > 0$ such that:

- if $x_i \in W_i$, if $\lambda$ is a real number with $0 < \lambda < 1$, and if $\sigma$ is a vector in $L$ with $\sigma \leq x_i + \lambda v_i$ and $\|\sigma\| < \lambda \mu_i$, then $x_i + \lambda v_i - \sigma \in P_i(x_i)$.

\textsuperscript{*} See Remark 3 in Section 10 for comments on how this assumption may be relaxed.
If any such number $\mu_i > 0$ exists, it is easy to see that there is a largest such number, which we call the marginal rate of desirability (of $v_i$ on $W_i$ with respect to $P_i$).

Informally, $v_i$ is extremely desirable if consumer $i$ would prefer to trade any commodity bundle $\sigma$ for an additional increment of the commodity bundle $v_i$, provided that such a trade is possible (notice that the restriction $\sigma \leq x_i + \lambda v_i$ guarantees that $x_i + \lambda v_i - \sigma$ is positive, and hence belongs to the consumption set $X_i = L^+$) and that the size of the bundle $\sigma$ (measured by $\|\sigma\|$) is sufficiently small compared to the size of the additional increment of the bundle $v_i$ (measured by $\lambda$). Evidently, extreme desirability is a kind of bound on the marginal rates of substitution, where we compare all bundles to the bundle $v_i$.

Extreme desirability is a condition introduced and discussed at some length in Yannelis and Zame (1984). We shall not repeat the discussion here, but a brief summary seems in order. The existence of an extremely desirable commodity for the preference relation $P_i$ is the nontransitive version of a condition introduced by Mas-Colell (1983), called "properness". Mas-Colell treats only complete, transitive preferences; in that setting, existence of an extremely desirable commodity is equivalent to properness. In infinite-horizon models, the existence of an extremely desirable commodity corresponds to the assumption that consumers do not emphasize the future. * Finally, if the positive cone $L^+$ has a nonempty interior, and preferences are monotone, the existence of extremely desirable commodities is automatic.

* Bewley's (1972) assumption that preferences are Mackey upper semicontinuous has a similar interpretation.
As an example of a preference relation which does not admit an extremely desirable commodity, we refer to the utility function $u$ described in Example 2, Section 4. The function $u$, which is defined on the positive cone $\ell_1^+$ of $\ell_1$, is continuous, concave and strictly monotone, but the preference relation it induces does not admit an extremely desirable commodity on any set containing the initial allocation $\omega$.

Rather than simply giving examples of preference relations which admit extremely desirable commodities, we prove a result which establishes the existence of extremely desirable commodities in many contexts. Looking ahead to Theorems 2 and 3, we will need to have a commodity which is extremely desirable on a relative $\mathcal{J}$-neighborhood of the feasible consumption set $\hat{X}_1$, where $\mathcal{J}$ is a topology for which $\hat{X}_1$ is compact and $P_i$ is upper semicontinuous. For illustrative purposes, we will confine ourselves to the case in which $\mathcal{J}$ is the weak topology on $L$; similar results could be proved in much greater generality.

PROPOSITION: Assume that there is a norm open set $U$ containing $L^+$ and a norm-continuous, concave, monotone function $u_i : U \to \mathbb{R}$ such that

$$P_i(x) = \{x' \in L^+ : u_i(x') > u_i(x)\}$$

for each $x \in L^+$. Assume also that $\hat{X}_1$ is weakly compact. Then there is a (relatively) weakly open subset $W_i$ of $L^+$ which contains $\hat{X}_1$ and a vector $v_i \in L^+$ which is extremely desirable for $P_i$ on $W_i$.

PROOF: Consider any point $x \in \hat{X}_1$. Local nonsatiation implies that there is a point $x^* \in L^+$ such that $x^* \in P_i(x)$; i.e., $u_i(x^*) > u_i(x)$. Write $\epsilon_x = \frac{1}{2} (u_i(x^*) - u_i(x))$, and set
\[ A(x) = \{ y \in U : u^*_1(y) > u^*_1(x) + \epsilon \} \]
\[ B(x) = \{ y \in U : u^*_1(y) < u^*_1(x) + \epsilon \} . \]

Evidently, \( A(x) \) and \( B(x) \) are norm open subsets of \( U \) (because \( u^*_1 \) is norm continuous), \( x \in B(x) \) and \( x^* \in A(x) \).

In fact, \( B(x) \cap L^+ \) is also a relatively open subset of \( L^+ \) in the weak topology.

To see this, set
\[ C(x) = \{ y \in L^+ : u^*_1(y) \geq u^*_1(x) + \epsilon \} . \]

Then \( C(x) \) is norm closed and convex (because \( u^*_1 \) is norm continuous and concave); since the norm topology and the weak topology have the same closed convex sets, \( C(x) \) is also weakly closed. Since \( B(x) \cap L^+ \) is the complement of \( C(x) \) in \( L^+ \), the set \( B(x) \cap L^+ \) is relatively weakly open.

The family \( \{ B(x) \cap L^+ : x \in \hat{X}_i \} \) is a covering of the weakly compact set \( \hat{X}_i \) by (relatively) weakly open sets, so it has a finite subcovering; write \( B(z_1), \ldots, B(z_T) \) for this finite subcovering, and \( z^*_1, \ldots, z^*_T \) for the corresponding preferred points.

As noted before, \( z^*_t \in A(z_t) \), and \( A(z_t) \) is a norm open subset of \( U \), so there is a real number \( \epsilon_t > 0 \) such that \( z \in A(z_t) \) whenever \( \| z - z^*_t \| < \epsilon_t \). Set:
\[ v_1 = \sup \{ z^*_1, \ldots, z^*_T \} , \]
\[ \mu_1 = \min \{ \epsilon_1, \ldots, \epsilon_T \} , \]
\[ W_i = \bigcup_{t=1}^{T} (B(z_t) \cap L^+) . \]

Evidently, \( v_1 \) belongs to \( L^+ \), \( \mu_1 \) is a positive real number, and \( W_i \) is a (relatively) weakly open subset of \( L^+ \) which contains \( \hat{X}_i \).
To see that \( v_i \) is extremely desirable for \( P_i \) on \( W_i \), we fix a vector \( x \in W_i \), a real number \( \lambda \) with \( 0 < \lambda < 1 \) and a vector \( \sigma \in L \) with \( \sigma \leq x + \lambda v_i \) and \( \| \sigma \| < \lambda \mu_i \); we are to show that \( x + \lambda v_i - \sigma \in P_i(x) \). By definition, \( x \in B(z_t^t) \) for some \( t \). Since \( \| \sigma \| < \lambda \mu_i \), it is also the case that \( \| \sigma / \lambda \| < \epsilon_t \), so that \( z_t^* - \sigma / \lambda \in A(z_t) \). Hence, \( u_1(z_t^* - \sigma / \lambda) > u_1(x) \), since our construction guarantees that \( u_1(a) > u_1(b) \) whenever \( a \in A(z_t) \) and \( b \in B(z_t) \). Concavity of \( u_1 \) implies that
\[
u_1((1 - \lambda)x + \lambda(z_t^* - \sigma / \lambda)) > u_1(x),
\]
and monotonicity of \( u_1 \) implies that
\[
u_1(x + \lambda v_i - \sigma) \geq u_1((1 - \lambda)x + \lambda(z_t^* - \sigma / \lambda)).
\]
Combining these, we obtain that
\[
u_1(x + \lambda v_i - \sigma) > u_1(x);
\]
in other words, \( x + \lambda v_i - \sigma \in P_i(x) \). Thus, \( v_i \) is extremely desirable for \( P_i \) on \( W_i \), as asserted. \( \square \)

One final comment about extremely desirable commodities seems in order.

In Theorems 2 and 3 we shall assume that for each \( i \), there is an extremely desirable commodity \( v_i \) for \( P_i \) on a set \( W_i \) which contains \( \hat{X}_i \) and is a relatively open subset of \( L^+ \) (in an appropriate topology). The meaning of these assumptions should be clear from the above discussions. However, we shall also assume that these commodities \( v_i \) may be chosen so that \( 0 \leq v_i \leq \omega \), where \( \omega = \sum_{i=1}^{\infty} \omega_i \) is the aggregate initial endowment. We make this additional assumption because we want to allow for the possibility that the aggregate initial endowment \( \omega \) is not strictly positive.*

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* Recall that a vector \( \omega \in L \) is \textit{strictly positive} if \( \pi \omega > 0 \) whenever \( \pi \) is a nonzero positive continuous linear functional on \( L \).
positivity of \( \omega \) is an assumption which is commonly made in finite dimensions. However, some of the interesting infinite-dimensional commodity spaces (such as the space \( \text{ca}(T) \) of countably additive Borel measures on a compact metric space \( T \) which has uncountably many points) contain no strictly positive elements whatsoever. It is therefore especially desirable to allow for the possibility that \( \omega \) might not be strictly positive. (See Mas-Colell (1983) and Yannelis and Zame (1984) for more discussion of this point.)

It should be noted, however, that requiring that \( v_i \) lie between 0 and \( \omega \) imposes no additional requirements whenever \( \omega \) is in fact strictly positive. To be more precise: if \( v_i \) is extremely desirable for \( P_i \) on \( W_i \) and if \( \omega \) is strictly positive, then there is a vector \( v_i^* \) which is also extremely desirable for \( P_i \) on \( W_i \), and enjoys the additional property that \( 0 \leq v_i^* \leq \omega \).

To see this, let \( \mu_i \) be the marginal rate of desirability for \( v_i \). Since \( \omega \) is strictly positive, there is a positive integer \( k \) such that \( \| k \omega \land v_i - v_i \| < \frac{1}{2} \mu_i \) (see Schaeffer (1974)). Set \( v_i^* = \frac{1}{k} (k \omega \land v_i) \) and \( \mu_i^* = \frac{1}{2k} \mu_i \). Now, for any \( x \in W_i \), any real \( \lambda \) with \( 0 < \lambda < 1 \) and any \( \sigma \in L \) such that \( \sigma \leq x + \lambda v_i^* \) and \( |\sigma| < \lambda \mu_i \), we can write

\[
x + \lambda v_i^* - \sigma = x + \frac{\lambda}{k} v_i - (\sigma + \frac{\lambda}{k} v_i - \lambda v_i^*)
\]

Clearly,

\[
\sigma + \frac{\lambda}{k} v_i - \lambda v_i^* \leq x + \frac{\lambda}{k} v_i
\]

Moreover,

\[
\| \sigma + \frac{\lambda}{k} v_i - \lambda v_i^* \| \leq \| \sigma \| + \frac{\lambda}{k} \| v_i k v_i^* \| < \frac{\lambda}{k} \mu_i
\]

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This latter inequality means that we can apply extreme desirability of \( v_i \) to the rewritten form of \( x + \lambda v_i^* - \sigma \) to conclude that \( x + \lambda v_i^* - \sigma \in \mathcal{P}_i(x) \). In other words, \( v_i^* \) is extremely desirable for \( \mathcal{P}_i \) on \( W_i \), and of course \( 0 \leq v_i^* \leq \omega \).

This completes our discussion of extremely desirable commodities. We now turn to the production side of the economy.

**Marginal Efficiency of Production**

As Examples 3 and 4 of Section 3 suggest, assumptions on the consumption side alone will not guarantee the existence of equilibria (in the absence of interiority assumptions). Indeed, Example 5 shows that production plans which are both feasible and profit-maximizing may fail to exist, quite independently of the nature of consumer preferences. As these examples suggest, the problem lies with the marginal efficiency of the production process.

A natural way to measure the efficiency of the production process is to compare the size (i.e., the norm) of the input with the size of the output. It seems natural, therefore, to measure the marginal efficiency of production by making a small change in the input, finding the corresponding change in the output, and comparing the size of these changes. (A little care must be exercised since production is described by a set of possibilities and not by a function.) We give two notions of boundedness of the marginal efficiency of production; we will use the stronger version in Theorem 2, while the weaker version will be adequate in Theorem 3 (where the production technology is assumed to display constant returns to scale).

In what follows, we write \( Y = \sum_{j=1}^{M} Y_j \) for the aggregate production set.
DEFINITION: We say the **marginal efficiency of production is weakly bounded** if there is a real number \( C > 0 \) such that:

if \( y \in Y \) and \( a \in L \) with \( 0 \leq a \leq y^- \), then there are a real number \( \rho > 0 \) and a vector \( b \) in \( L \) with \( 0 \leq b \leq y^+ \) such that

\[
(y^+ - b) - (y^- - \rho a) \in Y
\]

and

\[
\|b\| \leq C\|\rho a\|.
\]

DEFINITION: We say the **marginal efficiency of production is strongly bounded** if there is a real number \( C > 0 \) such that:

if \( y_j \in Y \) (for each \( j \)), \( y = \sum_{j=1}^{N} y_j \), and \( a \in L \) with \( 0 \leq a \leq y^- \), then there is a real number \( \rho > 0 \), vectors \( \hat{y}_j \in Y_j \) and a vector \( b \) in \( L \) with \( 0 \leq b \leq y^+ \) such that

\[
(y^+ - b) - (y^- - \rho a) = \sum_{j=1}^{M} \hat{y}_j
\]

\[
\|b\| \leq C\|\rho a\|;
\]

\[
\hat{y}_j \leq y_j^+ \quad \text{for each} \; j;
\]

\[
\hat{y}_j \leq y_j^- \quad \text{for each} \; j.
\]

These conditions have a simple interpretation. The vector \( a \) represents a direction, so \( \rho a \) represents a small change in the aggregate input, while the vector \( b \) represents a change in the aggregate output. To say that the marginal efficiency of production is weakly bounded is to say that the aggregate production technology can always accommodate a (sufficiently) small loss of input by a small loss of output; the constant \( C \) is a bound on the size of lost output compared with lost input.
To say that the marginal efficiency of production is strongly bounded is to say that this accommodation can be effected in such a way that the individual firms do not increase their inputs or outputs.

Before going any further, let us point out that both of these conditions are (vacuously) satisfied if there is no production at all. Thus the exchange case is definitely included in all our results.

Notice that these two conditions are equivalent if there is only one firm. If there are two (or more) firms, these conditions may not be equivalent. The reason is that the output of the first firm may include an intermediate good which is used as an input by the second firm. In such a case, it is possible that the aggregate technology can only accommodate a small loss of (aggregate) input—with a corresponding small loss of (aggregate) output—by a pair of production plans in which the first firm increases its output of this intermediate good and the second firm increases its input of the same intermediate good by exactly the same amount. (The intermediate good will then not appear in the aggregate production plan.)

Note that each of these conditions refers to the aggregate production set and not just to the production sets of the individual firms. There is a simple reason for this: Example 4 of Section 3 shows clearly that the aggregate marginal efficiency of production may be unbounded (and an equilibrium may not exist) even though the marginal efficiency of each individual firm is bounded. The reason here is that each firm produces intermediate goods which are used as inputs by the other firm.

Of course, Examples 3, 4 and 5 of Section 3 provide examples in which the marginal efficiency of production is not weakly bounded.
Here's a simple illustrative example in which the marginal efficiency of production is weakly bounded. Let $Z_1$ be a closed, convex, solid* subset of $L^+$, and let $Z_2$ be a closed, convex subset of $L^+$ which contains 0. (The sets $Z_1$ and $Z_2$ are to be interpreted as input and output sets, respectively.) Let $f_1 : Z_1 \to [0, \infty)$ and $f_2 : Z_2 \to [0, \infty)$ be continuous, monotone functions such that $f(0) = g(0) = 0$, and suppose that the aggregate production set $Y$ is of the form

$$Y = \{ y^+ \in Z_1, \ y^- \in Z_2, \ \text{and} \ f_2(y^+) \leq f_1(y^-) \}.$$

In this setting, the marginal efficiency of production will be weakly bounded if the (left) Gateaux derivatives of $f_1$ are bounded above and the (left) Gateaux derivatives of $f_2$ are bounded away from zero.

To see this, fix a production plan $y = y^+ - y^-$ in $Y$ and a vector $a$ with $0 \leq a \leq y^-$. Write $\hat{a} = a/\|a\|$ and $\hat{y}^+ = y^+/\|y^+\|$, so that $\hat{a}$ and $\hat{y}^+$ are unit vectors. The (left) Gateaux derivative of $f_1$ at $y^-$ in the direction of $\hat{a}$ is:

$$Df_1(y^-; \hat{a}) = \lim_{h \to 0^+} \frac{f_1(y^- - h\hat{a}) - f_1(y^-)}{h}.$$

(Note that $y^- - h\hat{a} \in Z_1$, for small $h$, because $Z_1$ is solid.) The (left) Gateaux derivative of $f_2$ at $y^+$ in the direction of $\hat{y}^+$ is:

$$Df_2(y^+; \hat{y}^+) = \lim_{h \to 0^+} \frac{f_2(y^+ - hy^+) - f_2(y^+)}{h}.$$

(Note that $y^+ - hy^+ \in Z_2$, for small $h$, because $Z_2$ is convex and contains 0.) By assumption, $Df_1(y^-; \hat{a}) \leq C_1$ and $Df_2(y^+; \hat{y}^+) \geq C_2 > 0$ for some constants $C_1$ and $C_2$.

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* A subset $Z_1$ of $L^+$ is **solid** if $z' \in Z_1$ whenever there is an element $z$ in $Z_1$ such that $0 \leq z' \leq z$. 

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C₂. Hence, if h₁ and h₂ are small enough (but strictly positive),

\[
\frac{f_1(y^-) - f_1(y^- - h_1 \hat{y})}{h_1} \leq 2C_1
\]

and

\[
\frac{f_2(y^+) - f_2(y^+ - h_2 \hat{y}^+)}{h_2} \geq \frac{1}{2}C_1
\]

Equivalently,

\[
f_1(y^- - h_1 \hat{y}) \geq -2C_1 h_1 + f_1(y^-),
\]

\[
f_2(y^+ - h_2 \hat{y}^+) \leq -\frac{1}{2}C_2 h_2 + f_2(y^+).
\]

Since y ∈ Y, we know that f₂(y⁺) ≤ f₁(y⁻). Hence,

\[
f_2(y^+ - h_2 \hat{y}^+) \leq f_1(y^- - h_1 \hat{y})
\]

provided that \(-\frac{1}{2}C_2 h_2 \leq -2C_1 h_1\), or equivalently, that \(h_2 \geq 4C_1 h_1 / C_2\). If we now set \(\rho = h_1 / \|a\|\), and \(b = (4C_1 h_1 / C_2) \hat{y}^+\), then \(y^- - h_1 a \in Z_1\) and \(y^+ - b \in Z_2\) (for \(h_1\) sufficiently small but strictly positive), and

\[
f_2(y^+ - b) \leq f_1(y^- - \rho a),
\]

so that \((y^+ - b) - (y^- - \rho a)\) belongs to Y. Since \(\|b\| = 4C_1 h_1 / C_2\) and \(\|\rho a\| = h_1\), this means that the marginal efficiency of production is weakly bounded (by \(4C_1 / C_2\)).

To illustrate the meaning of strong boundedness of the marginal efficiency of production, let us suppose that there are two firms, that the marginal efficiency of production of each firm* is bounded, and that none of the output of the first firm is used as an input by the second firm. Then the marginal efficiency of production is strongly bounded.

* That is, the marginal efficiency of production when only that firm is considered.
To see this, let $y_1, y_2$ be production plans for the two firms, let
\[ y = y_1 + y_2 = (y_1^+ - y_1^-) + (y_2^+ - y_2^-) \]
and let $a$ be a vector in $L$ with $0 \leq a \leq y^-$. Since
\[ 0 \leq y^- \leq y_1^- + y_2^- \], we can use the Riesz Decomposition Property to write
\[ a = a_1 + a_2, \]
where $0 \leq a_1 \leq y_1^-$ and $0 \leq a_2 \leq y_2^-$. (Notice that $y^-$ is not generally equal to $y_1^- + y_2^-,$
since the input of the first firm may include some of the output of the second firm;
such input will not be reflected in the aggregate input.) We can use boundedness of
the marginal efficiency of production of the second firm to find a $\rho_2$ and a $b_2$ such
that
\[ z_2 = (y_2^+ - b_2) - (y_2^- - \rho_2 a_2) \in Y_2 \text{ and } \|b_2\| \leq C_2 \|\rho_2 a_2\| \] (where $C_2$ is a bound on
the marginal efficiency of production of the second firm). Now, the vector $b_2$
represents a reduction in output of the second firm; part of $b_2$, however, might be
used as input by the first firm. To account for this, we notice that
\[ 0 \leq b_2 \leq y_2^+ \leq y_1^- + y^+, \]
so we may again use the Riesz Decomposition Property to
write $b_2 = b_2' + b_2''$ where $0 \leq b_2' \leq y_1^-$ and $0 \leq b_2'' \leq y^+$. We now make two applica-
tions of the boundedness of the marginal efficiency of production for the first firm
to find numbers $\rho_1, \rho_3$ and vectors $b_1, b_3$ such that
\[ z_1 = (y_1^+ - b_1) - (y_1^- - \rho_1 a_1) \in Y_1, \]
\[ z_3 = (y_1^+ - b_3) - (y_1^- - \rho_3 b_2') \in Y_1, \|b_1\| \leq C_1 \|\rho_1 a_1\|, \text{ and } \|b_3\| \leq C_1 \|\rho_3 b_2\| \] (where $C_1$
is a bound on the marginal efficiency of production of the first firm). By assumption,
outputs of the first firm are not used as inputs by the second firm, so that $b_1$ and
$b_3$ are disjoint from $y_2^-$. 

We now put these vectors together, using convexity of the production sets.

Choose a $t > 0$ so small that $t(\rho_2 \rho_3 + \rho_1 + \rho_1 \rho_3) < 1$, and set
\[ \hat{z}_1 = (1 - t \rho_2 \rho_3 - t \rho_1) y_1 + (t \rho_2 \rho_3) z_1 + (t \rho_1 z_3 \]
\[ = (y_1^+ - t \rho_2 \rho_3 b_1 - t \rho_1 b_3) - (y_1^- - t \rho_1 \rho_2 \rho_3 a_1 - t \rho_1 \rho_3 b_2') \]
and

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\[ \hat{z}_2 = (1 - \theta_1 \rho_3) y_2 + (\theta_1 \rho_3) z_2 \]
\[ = (y_2^+ - \theta_1 \rho_3 b_2' - \theta_1 \rho_3 b_2'') - (y_2^- - \theta_1 \rho_2 \rho_3 a_2) \cdot \]

Note that \( \hat{z}_1 \in Y_1 \) and \( \hat{z}_2 \in Y_2 \) (by convexity) and that \( \hat{z}_1^{+} \leq y_1^{+}, \hat{z}_1^{-} \leq y_1^{-}, \hat{z}_2^{+} \leq y_2^{+} \) and \( \hat{z}_2^{-} \leq y_2^{-} \) (by construction), so \( \hat{z}_1 \) and \( \hat{z}_2 \) represent production plans with decreased inputs and outputs. Moreover, because \( b_1 \) and \( b_3 \) are disjoint from \( y_2^{-} \) and because the terms involving \( b_2' \) cancel out, we obtain

\[ (\hat{z}_1 + \hat{z}_2)^+ = (y^+ - \theta_2 \rho_3 b_1 - \theta_1 \rho_3 b_2 - \theta_2 \rho_3 b_2'') \]

and

\[ (\hat{z}_1 + \hat{z}_2)^- = (y^- - \theta_1 \rho_2 \rho_3 a_1 - \theta_1 \rho_2 \rho_3 a_2) \]
\[ = (y^- - \theta_1 \rho_2 \rho_3 a) \cdot \]

Hence \( \hat{z}_1 + \hat{z}_2 \) represents an aggregate production plan with decreased aggregate input and output. To estimate the sizes of the loss of output and the loss of input, we need only put together the estimates for all the pieces and keep in mind the lattice property of the norm (if \( 0 \leq x \leq y \), then \( \|x\| \leq \|y\| \)); the end result is

\[ \|\theta_2 \rho_3 b_1 - \theta_1 \rho_3 b_1 - \theta_2 \rho_3 b_2''\| \leq (C_1 + C_1 C_2 + C_2) \|\theta_1 \rho_2 \rho_3 a\| \cdot \]

In other words, the marginal efficiency of production is strongly bounded (by \( C_1 + C_1 C_2 + C_2 \)), as asserted.

If we wish to allow for the possibility that the second firm's production technology includes disposal activities, we should require only that the first firm's output is not an essential input for the second firm (i.e., even if \( y_1^+ \land y_2^- \neq 0 \), it should still be the case that \( y_2^+ - (y_2^- + y_1^+ \land y_2^-) \in Y_2 \), so that the same output can be
produced by the second firm without the use of the output of the first firm). This condition is easily generalized to an arbitrary number of firms: there should not be any chain of production plans (for individual firms) which circularly exchanges outputs and essential inputs. Combined with boundedness of the marginal efficiency of production of the individual firms, this condition implies that the (overall) marginal efficiency of production is strongly bounded. The argument is similar to the case of two firms, but more complicated (and less enlightening).
7. EXISTENCE OF EQUILIBRIA WITHOUT INTERIORITY

The assumptions discussed in the preceding section (existence of extremely desirable commodities and bounded marginal efficiency of production) together with compactness assumptions (which for technical reasons are slightly stronger than those of Theorem 1) are precisely the assumptions we need to establish the existence of equilibria without interiority assumptions on the consumption sets. We give two different existence theorems, which differ only in the combination of assumptions on the production side.

THEOREM 2: Let $L$ be a normed lattice and let $\mathcal{C}$ be an economy in $L$ which satisfies the Standard Assumptions. Let $\mathcal{J}$ be a Hausdorff vector space topology on $L$, which is weaker than the norm topology, such that:

1. the feasible consumption sets $\hat{X}_i$ are $\mathcal{J}$-compact (for each $i$);
2. the feasible production sets $\hat{Y}_j$ are $\mathcal{J}$-compact (for each $j$);
3. the graph of the preference relation $P_i$ is a (relatively) $\mathcal{J} \times$ norm-open subset of $X_i \times X_i$ (for each $i$);
4. every order interval* in $L$ is $\mathcal{J}$-compact;
5. $Y_j$ is $\mathcal{J}$-closed (for each $j$).

On the consumption side, assume that:

6. $X_i = L^+$ (for each $i$);
7. for each $i$, there is a vector $v_i$ with $0 \leq v_i \leq \sum_{k=1}^{N} \omega_k$, and a

* Recall that an order interval is any set of the form $[a, b] = \{ c \in L : a \leq c \leq b \}$.
relatively $\mathcal{J}$-open subset $W_i$ of $X_i$ which contains $\hat{X}_i$ such that $v_i$ is extremely desirable for $P_i$ on the set $W_i$.

On the production side, assume that:

(8) the marginal efficiency of production is strongly bounded.

Then $\mathcal{E}$ has a quasi-equilibrium $((\widetilde{x}_i), (\widetilde{y}_i), \widetilde{\pi})$ for which $\frac{\pi}{\pi} \left( \sum_{i=1}^{N} v_i \right) > 0$. If $\mathcal{E}$ is irreducible, then every such quasi-equilibrium is an equilibrium.

If production displays constant returns to scale (in the aggregate), we can make do with a weaker assumption on the marginal efficiency of production.

THEOREM 3: Let $L$ be a normed lattice and let $\mathcal{E}$ be an economy in $L$ which satisfies the Standard Assumptions. Let $\mathcal{J}$ be a Hausdorff vector space topology on $L$, which is weaker than the norm topology, such that:

(1) the feasible consumption sets $\hat{X}_i$ are $\mathcal{J}$-compact (for each $i$);

(2) the feasible production sets $\hat{Y}_j$ are $\mathcal{J}$-compact (for each $j$);

(3) the graph of the preference relation $P_i$ is a (relatively) $\mathcal{J}$ x norm-open subset of $X_i \times X_i$ (for each $i$);

(4) every order interval in $L$ is $\mathcal{J}$-compact;

(5) $Y_j$ is $\mathcal{J}$-closed (for each $j$);

(5') the aggregate production set $Y = \sum_{j=1}^{M} Y_j$ is $\mathcal{J}$-closed.

On the consumption side, assume that:

(6) $X_i = L^+$ (for each $i$);

(7) for each $i$, there is a vector $v_i$ with $0 \leq v_i \leq \sum_{k=1}^{N} \omega_k$, and a relatively $\mathcal{J}$-open subset $W_i$ of $X_i$ which contains $\hat{X}_i$ such that $v_i$ is extremely desirable for $P_i$ on the set $W_i$. 

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On the production side, assume that:

\begin{align*}
(8A) & \quad \text{the marginal efficiency of production is weakly bounded;} \\
(9) & \quad \text{the aggregate production set } Y = \sum_{j=1}^{M} Y_j \text{ is a cone at } 0.
\end{align*}

Then $\mathcal{E}$ has a quasi-equilibrium $((\tilde{x}_i), (\tilde{y}_i), \tilde{\pi})$ for which $\pi \left( \sum_{i=1}^{N} v_i \right) > 0$. If $\mathcal{E}$ is irreducible, then every such quasi-equilibrium is an equilibrium.

The lists of assumptions of these two theorems may seem rather daunting, but they should not. Notice that assumptions (1) - (5) of Theorem 2 and assumptions (1) - (5), (5') of Theorem 3 are purely topological assumptions, and in fact are very mild, and are satisfied in all the models discussed in the Introduction. (See the discussion and Section 4, and compare the topological assumptions of Theorems 2 and 3 with those of Theorem 1.) The assumptions in Theorem 2 with real teeth are (6), (7) and (8), which have been discussed in Section 6. In Theorem 3, the assumption of constant returns to scale allows for the substitution of weak boundedness of the marginal efficiency of production for the strong boundedness of the marginal efficiency of production used in Theorem 2. We refer to the Remarks in Section 10 for ways in which both these Theorems may be improved.

One further comment on these results seems in order. Notice that we have not assumed (in either Theorem 2 or Theorem 3) that the aggregate initial endowment $\omega = \sum_{i=1}^{N} \omega_i$ is strictly positive. (Assumption (7) does imply that $\omega$ is not zero.) We must therefore be careful to avoid the possibility of producing a trivial quasi-equilibrium. The requirement that $\tilde{\pi} \left( \sum_{i=1}^{N} v_i \right)$ be strictly positive, coupled with the assumptions that $0 \leq v_i \leq \omega$ for each $\omega$, rules out this possibility by guaranteeing that some consumer has income. As discussed in Section 6, we want to avoid
assuming that \( \omega \) is strictly positive, since this requirement cannot be met in some of the interesting models.

It may help to understand these results if we consider an example (which should be compared with Example 7 of Section 5).

EXAMPLE 8: Let the commodity space be \( L_2 \) (of some sigma-finite measure space), and take for \( J \) the weak topology. Assume that the consumption sets are the positive cone \( L_2^+ \) and that the preferences of each consumer are given by a norm-continuous, concave, strictly monotone utility function \( u_i : L_2 \rightarrow \mathbb{R} \). (It is important that the functions \( u_i \) be defined on the entire commodity space—or at least on a neighborhood of the positive cone.) Assume also that the aggregate initial endowments \( \omega_i \) are strictly positive (i.e., each \( \omega_i \) is a strictly positive function).

The discussions in Section 4, Example 7, and Section 6 show that assumptions (1) - (7) are all satisfied for this economy. Moreover, strict monotonicity of the utility functions (by which we mean that \( u_i(x + x') > u(x) \) whenever \( x' \) is strictly positive), together with positivity of the initial endowments, guarantees that this economy is irreducible. Hence Theorem 2 assures us that an equilibrium exists whenever the marginal efficiency of production is strongly bounded. In particular, an equilibrium exists if there is no production.

The proofs of these theorems are complicated and will be deferred to Section 9, but the central idea can be described by considering a very special setting. Let us suppose that the commodity space is \( L_1 \) (of some finite measure space), that the initial endowments \( \omega_i \) are (uniformly) bounded above and below (i.e., that \( 0 < C_1 < \omega(t) < C_2 \),

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for some constants \( C_1 \) and \( C_2 \) and that the production sets \( Y_j \) consist entirely of uniformly bounded functions (i.e., that there is a constant \( C_3 \) such that \( |y(t)| \leq C_3 \) for each \( y \in Y_j \)).

Consider a feasible allocation \( \langle (x_i), (y_j) \rangle \) for this economy. It is easily verified that

\[
0 \leq x_i(t) \leq C_2 + MC_3,
\]

and

\[
|y_j(t)| \leq C_3.
\]

In particular, the vectors \( x_i \) and \( y_j \) actually belong to \( L_\infty \). This suggests that we can find an equilibrium for \( \mathcal{E} \) by first looking at the restriction \( \mathcal{E}_\infty \) of \( \mathcal{E} \) to \( L_\infty \).

In fact, if we use the \( L_\infty \)-norm on \( L_\infty \) (instead of the norm \( L_\infty \) inherits as a subspace of \( L_1 \)), it can be verified that \( \mathcal{E}_\infty \) satisfies all the assumptions of Theorem 1(b) and hence has an equilibrium \( \langle (\tilde{x}_i), (\tilde{y}_j), \tilde{\pi} \rangle \).

Unfortunately, this does not solve our problem, since \( \tilde{\pi} \) is a (continuous) linear functional on \( L_\infty \) and not on \( L_1 \). However, we could extend \( \tilde{\pi} \) to a (continuous) linear functional \( \bar{\pi} \) on \( L_1 \) provided we knew that \( \bar{\pi} \) were continuous in the \( L_1 \) norm on \( L_\infty \).

If this were not so, there would be some commodity bundle in \( L_\infty \) which was small (in the \( L_1 \) norm) but very expensive. If such a commodity bundle were being consumed, the extreme desirability assumption would imply that some consumer would prefer to trade it for something cheaper. On the other hand, if such a commodity bundle were not being consumed it would have to be used in production, in which case the marginal efficiency assumption would imply that some firm could
increase profits by using less of this bundle in production. In either case, we would obtain a violation of the equilibrium conditions in $L_\infty$. Hence $\tilde{\pi}$ must be continuous in the $L_1$ norm, and has a continuous extension $\tilde{\pi}$ to $L_1$. Finally, it may be shown that $((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi})$ is an equilibrium for $\mathcal{E}$.

Although this is a very special setting, the reasoning involved is really quite general. Indeed the proof of Theorem 2 essentially combines this reasoning with a (rather unpleasant) limiting argument. Theorem 3 is easily derived from Theorem 2.
8. PROOF OF THEOREM 1

The proof of Theorem 1 is almost the same as the proof of Theorem 1 of Bewley (1972). We obtain the desired quasi-equilibrium as a limit of quasi-equilibria in finite-dimensional spaces. The crucial point is to be certain that the limiting price is nonzero. In Bewley's setting this follows from monotonicity; in our setting it must be proved using the interiority assumptions.

PROOF: We begin with some preliminary constructions. Fix an index \( i, \quad 1 \leq i \leq N \).

For each point \( x \in \hat{X}_i \), nonsatiation of preferences guarantees the existence of a point \( d_x \in X_i \) with \( d_x \in P_i(x) \). Since the graph of \( P_i \) is (relatively) \( \mathcal{J} \times \) norm-open in \( X_i \times X_i \), there is a \( \mathcal{J} \)-open set \( U_x \subset L \) with \( x \in U_x \), such that \( d_x \in P_i(x') \) whenever \( x' \in U_x \cap X_i \). If we do this for each \( x \in \hat{X}_i \), we obtain a family of pairs \( \{(d_x, U_x) : x \in \hat{X}_i \} \). Evidently, the family \( \{U_x\} \) is an open cover of \( \hat{X}_i \); compactness allows us to extract a finite subcover \( U_1, \ldots, U_r \). If we write \( d_1, \ldots, d_r \) for the corresponding points, our construction allows us to conclude that for each \( x \in \hat{X}_i \), there is a \( d_k \) \((1 \leq k \leq r)\) with \( d_k \in P_i(x) \). Since this construction may be repeated for each consumer \( i \), we may finally obtain a finite subset \( D \subset L \) such that for each \( i \) and each \( x \in \hat{X}_i \), there is a \( d \in D \) with \( d \in P_i(x) \).

We also choose, for each \( i \), a point \( z_i \) in the norm interior of \( X_i \).

We now let \( \mathcal{F} \) denote the family of all finite-dimensional subspaces \( F \) of \( L \) which contain \( D \cup \{x_i, z_i\} \). Notice that the family \( \mathcal{F} \) is directed by inclusion. For each \( F \in \mathcal{F} \), we define an economy \( \xi^F \) in \( F \) by setting:
\[ X_i^F = X_i \cap F , \]
\[ p_i^F(x) = p_i(x) \cap F , \]
\[ \omega_i^F = \omega_i , \]
\[ \theta_{ij}^F = \theta_{ij} , \]
\[ Y_j^F = Y_j \cap F . \]

Consider the economy \( \xi^F \). If we truncate the consumption and production sets, we obtain an economy to which the results of Shafer (1975) (Theorem 2 and various remarks) may be applied; hence the truncated economy has a quasi-equilibrium. The quasi-equilibrium allocation obtained is certainly feasible, so its components belong to the feasible consumption and production sets of the economy \( \xi \). Since \( \mathcal{J} \) is a Hausdorff vector space topology on \( L \), the restriction of \( \mathcal{J} \) to \( F \) coincides with the restriction of the norm topology (because all Hausdorff vector space topologies on a finite-dimensional vector space coincide). Thus, as we increase the size of the truncation, some subsequence of the sequence of quasi-equilibria of the truncated economies will converge to a quasi-equilibrium \( ((x_i^F), (y_j^F), p^F) \) for the economy \( \xi^F \). Of course, \( ((x_i^F), (y_j^F)) \) is a feasible allocation for the economy \( \xi \), and there is no loss of generality in assuming that \( \|p^F\| = 1 \).

We may use the Hahn-Banach Theorem to choose an extension of \( p^F \) to a continuous linear functional \( \pi^F \) on the entire space \( L \) with \( \|\pi^F\| = \|p^F\| = 1 \).

Since the family \( \mathcal{J} \) (of all finite-dimensional subspaces of \( L \) which contain \( D \cup \{\omega_i^F\} \cup \{z_j^F\} \)) is directed (by inclusion), we have constructed nets \( \{x_i^F\}, \{y_j^F\} \) and \( \{\pi^F\} \). Now, the feasible consumption and production sets are \( \mathcal{J} \)-compact (by
assumption) and the closed unit ball of the dual space of $L$ is weak-star compact (by Alaoglu's Theorem). Thus we may—passing to a subnet if necessary—find vectors $\bar{x}_i \in \hat{X}_i$, $\bar{y}_j \in \hat{Y}_j$ and a linear function $\bar{\pi}$ on $L$ such that $x^F_i$ converges in the topology $\mathcal{J}$ to $\bar{x}_i$ (for each $i$), $y^F_j$ converges in the topology $\mathcal{J}$ to $\bar{y}_j$ (for each $j$) and $\pi^F$ converges in the weak-star topology to $\bar{\pi}$. We are going to show that $((\bar{x}_i), (\bar{y}_j), \bar{\pi})$ is a quasi-equilibrium for the economy $\mathcal{E}$.

The first—and crucial—step is to show that $\bar{\pi}$ is not the zero functional. To this end, we first note that, since $X_1$ is a convex set with a (norm) interior point, it is actually the case that the interior of $X_1$ is (norm) dense in $X_1$. Since $P_1(\bar{x}_1)$ is a nonempty, relatively norm open subset of $X_1$, we can find a point $u$ which belongs both to $P_1(\bar{x}_1)$ and to the interior of $X_1$. Our continuity assumption implies that there is an open (norm) ball $B$, with center $u$ and radius some $\epsilon > 0$, and a $\mathcal{J}$-open set $U$ containing $\bar{x}_1$ such that $B \subset X_1$ and $b \in P_1(x)$ for each $b \in B$ and each $x \in X_1 \cap U$. Since $x^F_1$ converges to $\bar{x}_1$, this means in particular that there is a subspace $F_0 \in \mathcal{F}$ such that $b \in P_1(x^F_1)$ for each $b \in B$ and each $F \supseteq F_0$.

Now suppose that $\bar{\pi}$ is the zero functional. Since $\{\pi^F_1\}$ converges to $\bar{\pi}$ in the weak-star topology, this means that there is a subspace $G \in \mathcal{F}$ with $u \in G$ and $G \supseteq F_0$ such that $\pi^G(\omega_1) > -\varepsilon/3$ and $\pi^G(u) < \varepsilon/3$. By construction, $\pi^G$ and $p^G$ agree on $G$ and $\|p^G\| = 1$. Since $B \cap G$ is a ball in $G$ with center $u$ and radius $\epsilon$, we can find a vector $a \in B \cap G$ such that $p^G(a) < p^G(u) - 2\epsilon/3$. Combining these facts, we conclude that:
\[ p^G(a) < p^G(u) - \frac{2\epsilon}{3} \]
\[ = \pi^G(u) - \frac{2\epsilon}{3} \]
\[ < -\frac{\epsilon}{3} \]
\[ < \pi^G(\omega_1) \]
\[ = p^G(\omega_1) . \]

At a quasi-equilibrium, firms maximize profits; since each firm can always make a zero profit, it follows that

\[ p^G(a) < p^G(\omega_1) \leq p^G(\omega_1) + \sum_j \theta_{1j} p^G(y^G_j) . \]

In words, the vector \( a \) strictly satisfies the first consumer's budget constraint. On the other hand, \( a \) belongs to \( B \cap G \) and \( G \owns F_0 \), so \( a \in \mathcal{P}_1(x^G_1) \cap G = \mathcal{P}_1(x^G_1) \) (i.e., \( a \) is preferred to \( x^G_1 \)); this contradicts the fact that \( ((x^G_1), (y^G_j), p^G) \) is a quasi-equilibrium for the economy \( \xi^G \). We conclude therefore that the limit price \( \bar{\pi} \) is not the zero functional.

We next note that for each \( F \),

\[ \sum_i x^F_i = \sum \omega_i + \sum_j y^F_j . \]

Since the vector space operations are continuous in the topology \( \mathcal{J} \), it follows that

\[ \sum_i \bar{x}_i = \sum \omega_i + \sum_j \bar{y}_j . \]

In other words, \( ((\bar{x}_i), (\bar{y}_j)) \) is a feasible allocation for the economy \( \xi \).
Before verifying the remaining quasi-equilibrium conditions, it is convenient to prove the following fact.

\((\ast)\) If \(x, y_1, \ldots, y_M\) are vectors in \(L\) with \(x \in P_i(x_i)\) (for some \(i\)) and \(y_j \in Y_j\) (for each \(j\)), then

\[
\bar{\pi}(x) \geq \bar{\pi}(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \bar{\pi}(y_j) .
\]

For suppose this inequality were false. Then

\[
\bar{\pi}(x) < \bar{\pi}(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \bar{\pi}(y_j) ,
\]

so that (because \(\{\pi^F\}\) converges to \(\bar{\pi}\)) there is a subspace \(F_0 \in \mathcal{F}\) such that

\[
\pi^F(x) < \pi^F(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi^F(y_j) ,
\]

provided that \(F \in \mathcal{F}\) and \(F \supset F_0\). Our continuity assumptions also imply (just as before) that there is a subspace \(F_1 \in \mathcal{F}\) such that \(x \in P_i(x^F)\) whenever \(F \in \mathcal{F}\) and \(F \supset F_1\). Now choose any subspace \(G \in \mathcal{F}\) which contains the subspaces \(F_0\) and \(F_1\) and the vectors \(x, y_1, \ldots, y_M\). Since \(\pi^G\) and \(p^G\) agree on \(G\), profit-maximization by the firms and the fact that \(G \supset F_0\) imply that:

\[
p^G(x) < p^G(\omega_i) + \sum_{j=1}^{M} \theta_{ij} p^G(y_j)
\]

\[
\leq p^G(\omega_i) + \sum_{j=1}^{M} \theta_{ij} p^G(y_j) .
\]

Since \(x \in P^G_i(x_i^G)\) (because \(G \supset F_1\)), this contradicts the fact that \((x_i^G, y_j^G, p^G)\) is a quasi-equilibrium for \(\xi^G\). We conclude that the fact \((\ast)\) is indeed valid.
We can now verify all the budget constraints. Fix a consumer \( i \), and an \( \epsilon > 0 \). By local nonsatiation of preferences, we can find a vector \( x \) in \( P_i(x_i^-) \) with \( \|x - x_i^-\| < \epsilon \). Since \( \|\pi\| \leq 1 \), this implies that \( |\pi(x) - \pi(x_i^-)| < \epsilon \). Combining this with the fact (*) yields

\[
\pi(x_i^-) \geq \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j^-) - \epsilon .
\]

Since \( \epsilon > 0 \) may be made as small as we like, we conclude that

\[
\pi(x_i^-) \geq \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j^-)
\]

for each \( i \). If we sum over all \( i \), and rearrange terms, we obtain

\[
\pi\left(\sum_{i=1}^{N} x_i^-\right) = \sum_{i=1}^{N} \pi(x_i^-)
\]

\[
\geq \sum_{i=1}^{N} \left[ \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j^-) \right]
\]

\[
= \pi\left(\sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j^-\right) .
\]

Since \( (x_i^-), (y_j^-) \) is a feasible allocation, \( \sum x_i^- = \sum \omega_i + \sum y_j^- \). Thus, the inequality immediately above must in fact be an equality. Hence, for each \( i \),

\[
\pi(x_i^-) = \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi(y_j^-) ,
\]

which is the budget constraint we seek.
To see that each firm is maximizing profits, suppose that for some $k$ there is a vector $y \in Y_k$ with $\tilde{\pi}(y) > \tilde{\pi}(y_k)$. Since $\sum_{i=1}^{M} \theta_{ij} = 1$, there must be at least one consumer $i$ for which the quantity $\theta_{ik} \tilde{\pi}(y) - \theta_{ik} \tilde{\pi}(y_k)$ is strictly positive. Using local nonsatiation of preferences, as before, we can find a vector $x \in P_i(\tilde{x}_i)$ for which

$$\tilde{\pi}(x) < \tilde{\pi}(x_i) + \theta_{ik} \tilde{\pi}(y) - \theta_{ik} \tilde{\pi}(y_k).$$

Combining this with the budget constraint yields

$$\tilde{\pi}(x) < \tilde{\pi}(\omega_i) + \theta_{ik} \tilde{\pi}(y) + \sum_{j \neq k} \theta_{ij} \tilde{\pi}(y_j).$$

Of course, this again contradicts $()$; we conclude that each firm is indeed maximizing profits.

It remains to show that if $x \in P_i(\tilde{x}_i)$ then

$$\tilde{x}(x) \geq \tilde{\pi}(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \tilde{\pi}(y_j).$$

Of course, this is just the fact $(\sim)$ with $y_j = \tilde{y}_j$ for each $j$. Hence $((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi})$ is a quasi-equilibrium for $\xi$. This completes the proof of part (a).

The proof of part (b) of Theorem 1 is exactly the same as in the finite-dimensional case. If $((x_i^y), (y_j^y), \pi^y)$ is a quasi-equilibrium but not an equilibrium, then there is a consumer (whom we may as well assume to be the first consumer) and a vector $x_1$ with $x_1 \in P_1(x_1^y)$ and $\pi(x_1) = \pi(x_1^y)$. Since $\omega_1$ belongs to the interior of $X_1$ (by assumption), there is a vector $v \in X_1$ with $\pi(v) < \pi(\omega_1)$. Since $P_1(x_1^y)$ is relatively open, the vector $tx_1 + (1-t)v$ belongs to $P_1(x_1^y)$ if $t$ is sufficiently close to 1 (but less than 1). On the other hand, profit-maximization by
each firm ensures that \( \pi(\omega_1) \leq \pi(x_1^*) \). Combining all this information yields:

\[
\pi(t x_1 + (1 - t) v) = t \pi(x_1) + (1 - t) \pi(v) < t \pi(x_1^*) + (1 - t) \pi(\omega_1) \leq \pi(x_1^*) .
\]

Since \( x_1^* \) satisfies the consumer's budget constraint, this means that \( t x_1 + (1 - t) v \) strictly satisfies the consumer's budget constraint and is preferred to \( x_1^* \). This violates the quasi-equilibrium conditions, so we conclude that \( ((x_1^*), (y_j^*), \pi^*) \) is indeed an equilibrium. This completes the proof of Theorem 1. \( \Box \)
9. PROOFS OF THEOREM 2 AND THEOREM 3

Before beginning the proof of Theorem 2, we introduce some terminology. If $x$ is a positive element of $L$, then by the principal order ideal generated by $x$ we mean the set

$$L(x) = \{y \in L : -rx \leq y \leq rx \text{ for some } r \in \mathbb{R}^+\}.$$  

It is easily seen that $L(x)$ is a linear subspace of $L$ and a sublattice, and has the property that $z \in L(x)$ whenever there is an element $y$ of $L(x)$ for which $|z| \leq y$. Notice that if $x'$ is a positive element of $L$ and $x \in L(x)$ then $L(x') \subseteq L(x)$; if $x \leq x'$ and $x' \in L(x)$ then $L(x') = L(x)$.

The central idea of the proof of Theorem 2 has been described in Section 7, but a little more discussion may help to guide the reader. As in the proof of Theorem 1, we are going to obtain a quasi-equilibrium for the economy $\mathcal{E}$ as the limit of quasi-equilibria of subeconomies. By contrast with the proof of Theorem 1, however, these subeconomies will themselves be infinite dimensional. In fact, the commodity spaces for these subeconomies will be principal order ideals of $L$; the economies will also be truncated in a certain way. Obtaining equilibria in these subeconomies will be accomplished with the aid of Theorem 1. (This is the reason for our comment in Section 5 that Theorem 1 is a useful tool.) One of the key points in the argument will be that principal order ideals in a normed lattice bear much the same relationship to the entire lattice that the Lebesgue space $L_\infty$ (of a finite measure space) bears to the Lebesgue space $L_1$. 

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PROOF OF THEOREM 2: Let \( \omega = \sum_{i=1}^{N} \omega_i \), so that \( \omega \) is the aggregate initial endowment; as we noted following the statement of Theorem 2, assumption (7) implies that \( \omega \neq 0 \). Let \( \mathcal{X} \) be the family of all principal order ideals which contain \( \omega \). For each \( K \in \mathcal{X} \), choose and fix once and for all a positive vector \( e_K \) such that \( K = L(e_K) \) (i.e., \( K \) is generated by \( e_K \)). Since \( K \) contains \( \omega \), it follows from our comments about principal order ideals that \( K = L(e_K) = L(e_K + \omega) \). There is thus no loss of generality in assuming that \( e_K \geq \omega \).

We are going to construct economies in these principal order ideals. Simply restricting \( \mathcal{C} \) will not quite do; we will also need to truncate the production sets and alter the initial endowments. To this end, fix a principal order ideal \( K \in \mathcal{X} \) and positive integers \( s \) and \( t \). We define the economy \( \mathcal{C}^{(K,s,t)} \) by setting:

\[
X_i^{(K,s,t)} = X_i \cap K = L_i^+ \cap K = K_i^+ ;
\]

\[
p_i^{(K,s,t)}(x) = p_i(x) \cap K ;
\]

\[
\omega_i^{(K,s,t)} = \omega_i + \frac{1}{t} e_K ;
\]

\[
\theta_{ij}^{(K,s,t)} = \theta_{ij} ;
\]

\[
Y_j^{(K,s,t)} = Y_j \cap \left[ -se_{K'} + se_{K} \right] .
\]

(Recall that the order interval \( \left[ -se_{K'} + se_{K} \right] \) is the set of vectors \( z \in L \) such that \( -se_{K} \leq z \leq se_{K} \); of course, each such \( z \) belongs to \( K \).)

We want to use Theorem 1 to establish the existence of equilibria for these subeconomies. As matters stand, Theorem 1 does not apply, since the interiority
assumptions are not satisfied. To get around this problem, we equip $K$ with a new norm, different from the one it inherits as a subspace of $L$.

For $x \in K$, we define

$$
\|x\|_K = \inf \{ r \in \mathbb{R}^+: -re_K \leq x \leq +re_K \}.
$$

It is completely straightforward to verify that $\| \cdot \|_K$ is a norm on $K$, and that when equipped with this norm, $K$ is a normed lattice. Moreover, for each $x \in K$,

$$
-\|x\|_K e_K \leq x \leq +\|x\|_K e_K.
$$

Since $L$ itself is a normed lattice, this allows us to compare the original norm $\| \cdot \|$ and the new norm $\| \cdot \|_K$. For each $x \in K$, we obtain

$$
\|x\| < \|x\|_K e_K = [x\|_K e_K].
$$

In particular, this means that the topology on $K$ induced by the norm $\| \cdot \|_K$ is stronger than the topology on $K$ induced by the norm $\| \cdot \|$.

Let $\mathcal{Y}^K$ be the restriction to $K$ of the topology $\mathcal{Y}$; this is certainly a Hausdorff vector space topology and is certainly weaker than the $\| \cdot \|_K$-norm topology. We want to verify that, using the $\| \cdot \|_K$ norm and taking $\mathcal{Y}^K$ for our auxiliary topology, the economy $\zeta^{(K, s, t)}$ satisfies all the assumptions of Theorem 1, provided that the integer $t$ is sufficiently large. Since the $\| \cdot \|_K$-norm is stronger than the $\| \cdot \|$-norm topology, verification of all the Standard Assumptions is trivial, with the exception of local nonsatiation of preferences, which we defer for the moment. Verification of assumption (3) of Theorem 1 is also trivial.

To verify $\mathcal{Y}^K$-compactness of the feasible consumption and production sets, we will verify compactness of the set of feasible allocations. Suppose $((x_i), (y_j))$ is an allocation for the economy $\zeta^{(K, s, t)}$, so that $x_i \in K^+$ for each $i$ and

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\( y_j \in Y_j \cap \left[ -s e_K, +se_K \right] \). This allocation is feasible if and only if

\[
\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \omega_i + \sum_{j=1}^{M} y_j
\]

\[
= \omega + \frac{N}{t} e_K + \sum_{j=1}^{M} y_j.
\]

Since each \( x_i \) is positive, this implies that \( 0 \leq x_i \leq \omega + \frac{N}{t} e_K + M s e_K \). In other words, the allocation \( (x_i, y_j) \) is feasible if and only if all of the following conditions hold:

\[
x_i \in \left[ 0, \omega + \frac{M}{t} e_K + N s e_K \right] \text{ for each } i;
\]

\[
y_j \in Y_j \cap \left[ -s e_K, +se_K \right] \text{ for each } j;
\]

\[
\sum x_i = \omega + \frac{M}{t} e_K + \sum y_j.
\]

By our assumptions, all order intervals are \( J \)-compact, each of the sets \( Y_j \) is \( J \)-closed, and the vector space operations are \( J \)-continuous. The conditions above therefore express the set of feasible allocations for \( \xi^{(K, s, t)} \) as the intersection of subsets of \( L^{N+M} \), of which some are closed and some are compact (in the product topology on \( L^{N+M} \), endowing each of the factors with the topology \( J \)), so the set of feasible allocations is compact. As we noted in Sections 4 and 5, this means that the feasible consumption and production sets for \( \xi^{(K, s, t)} \) are \( J \)-compact; since these sets are contained in \( K \), they are of course \( J^K \)-compact.

Note that the initial endowments \( \omega^{(K, s, t)}_i \) belong to the \( \| \cdot \|_K \)-norm interior of \( K^+ \), since \( \omega^{(K, s, t)}_i = \omega_i + \frac{1}{t} e_K \), so that \( \omega^{(K, s, t)}_i - z \) is positive whenever
\[ \|z\|_K < 1/t \] (which is equivalent to saying that \( -\frac{1}{t} e_K \leq z \leq \frac{1}{t} e_K \)). Of course, this is the whole point of using the norm \( \cdot_K \), rather than the original norm.

We still need to show that the preference relations \( P^{(K,s,t)} \) are locally nonsatiable (on the feasible consumption sets), provided that \( t \) is sufficiently large. To do this, notice that if \( x \) belongs to the (relatively) \( J \)-open set \( W_i \) prescribed in assumption (7), then \( x + \varepsilon v_i \) will belong to \( P_i(x) \) for every \( \varepsilon, 0 < \varepsilon < 1 \) (since \( v_i \) is extremely desirable on \( W_i \)). Of course, \( v_i \in K \) (since \( 0 \leq v_i \leq \omega \) and \( \omega \in K \)), and \( x + \varepsilon v_i \) will belong to \( K \) — and hence to \( P^{(K,s,t)} \) — if \( x \) belongs to \( K \). Thus we need only show that the feasible consumption sets \( \hat{X}_i^{(K,s,t)} \) for \( \hat{\varepsilon}^{(K,s,t)} \) are contained in the (corresponding) sets \( W_i \), provided \( t \) is sufficiently large.

Suppose this were not so. Then we could find arbitrarily large values of \( t \) and feasible allocations \( ((x^t_i), (y^t_j)) \) for \( \varepsilon^{(K,s,t)} \) such that \( x^t_i \) (say) does not belong to \( W_i \). Notice that for each \( t \), and each \( i \) and \( j \), the vector \( x^t_i \) belongs to the \( J \)-compact set \( \left\{ 0, \omega + N e_K + M s e_K \right\} \) and the vector \( y^t_j \) belongs to the \( J \)-compact set \( \left\{ -s e_K, + s e_K \right\} \). By passing to subnets if necessary, we may assume that there are vectors \( x^x_i \in \left[ 0, \omega + N e_K + M s e_K \right] \) and \( y^y_j \in \left[ -s e_K, + s e_K \right] \) such that \( \{x^t_i\} \) converges to \( x^x_i \) in the topology \( J \) and \( \{y^t_j\} \) converges to \( y^y_j \) in the topology \( J \) (for each \( i \) and \( j \)). Note that

\[
\sum x^t_i = \omega + \frac{N}{t} e_K + \sum y^t_j
\]

for each \( t \). Since the vector space operations are \( J \)-continuous, this implies that

\[
\sum x^*_i = \omega + \sum y^*_j.
\]
In other words, \((x_i^\ast, y_j^\ast)\) is a feasible allocation for the economy \(\xi\). In particular, \(x_1^\ast\) belongs to \(\hat{X}_1\). Since \(W_1\) is a (relatively) \(J\)-open neighborhood of \(\hat{X}_1\) in \(X_1 = L^+\), this implies that \(x_1^t\) also belongs to \(W_1\) for \(t\) sufficiently large. This is a contradiction, so we conclude that the feasible consumption sets \(\hat{X}_i^{(K,s,t)}\) are indeed contained in the (corresponding) sets \(W_i\), provided that \(t\) is sufficiently large.

As we have already noted, this implies that the preference relations \(P_i^{(K,s,t)}\) are locally nonsatiated on the feasible consumption sets. We have therefore verified that the economy \(\xi^{(K,s,t)}\) satisfies all the assumptions of Theorem 1(b), and hence has an equilibrium \(((x_i^{(K,s,t)}), (y_j^{(K,s,t)}), \pi^{(K,s,t)})\). We stress that—because we have used the norm \(\|\cdot\|_K\) on \(K\)—we are guaranteed only that \(\pi^{(K,s,t)}\) is a linear functional on \(K\) which is continuous in the norm \(\|\cdot\|_K\). We will, in fact, now prove that \(\pi^{(K,s,t)}\) is continuous in the norm \(\|\cdot\|_K\), but this is far from obvious.

Before embarking on this task, it is convenient to make a normalization. By assumption, \(v_i\) is extremely desirable for \(P_i\) on the set \(W_i\); let \(\mu_i\) be the marginal rate of desirability. Since \(x_1^{(K,s,t)} + \frac{1}{2} v_1 \in P_i(x_1^{(K,s,t)})\), it follows* that \(\pi^{(K,s,t)}(v_1) > 0\). Since the equilibrium nature of a price is unaffected by scaling, we are free to normalize \(\pi^{(K,s,t)}\) so that

\[
\pi^{(K,s,t)} \left(\sum_{i=1}^{N} \frac{v_i}{\mu_i} \right) = 1.
\]

* This of course uses the fact that we have constructed an equilibrium rather than just a quasi-equilibrium. That is one of the reasons we altered the original initial endowments; the other reason will become clear shortly.
We are now going to show that, as a functional on $K$ considered with the original norm $\| \cdot \|$, $\pi^{(K, s, t)}_{(K, s, t)}$ is continuous. In fact, its norm is at most $1 + C$, where $C$ is any bound on the marginal efficiency of production of the economy $\mathcal{E}$. We proceed in four steps.

For the first step, consider a vector $z \in K$ with $z \leq 0$. Choose a real number $\lambda$, with $0 < \lambda < 1$ and write $z^{\ast} = \lambda \frac{2}{\mu_1} z / \| z \|$, so that $\| z^{\ast} \| < \lambda \mu_1$. Of course, $z^{\ast} \leq x_1^{(K, s, t)}$, so extreme desirability implies that

$$x_1^{(K, s, t)} + \lambda v_1 - z^{\ast} \in P_1^{(K, s, t)}(x_1^{(K, s, t)}).$$

Hence

$$\pi^{(K, s, t)}_{(K, s, t)}(x_1^{(K, s, t)} + \lambda v_1 - z^{\ast}) > \pi^{(K, s, t)}_{(K, s, t)}(x_1^{(K, s, t)}).$$

Thus,

$$\pi^{(K, s, t)}_{(K, s, t)}(z^{\ast}) \leq \lambda \pi^{(K, s, t)}_{(v_1)}.$$

Equivalently,

$$\pi^{(K, s, t)}_{(K, s, t)}(z) \leq \frac{\pi^{(K, s, t)}_{(v_1)}}{\lambda \mu_1} \| z \|.$$

Our normalization implies that $\pi^{(K, s, t)}_{(v_1)} / \mu_1 \leq 1$; taking $\lambda$ arbitrarily close to 1 implies that $\pi^{(K, s, t)}_{(K, s, t)}(z) \leq \| z \|$ (whenever $z \in K$ and $z \leq 0$).

For the second step, consider a vector $z \in K$ with $0 \leq z \leq \sum_{i=1}^{N} x_i^{(K, s, t)}$. By the Riesz Decomposition Property, we can find elements $z_1, \ldots, z_N$ in $K$ such that $0 \leq z_i \leq x_i^{(K, s, t)}$ for each $i$ and $\sum_{i=1}^{N} z_i = z$. As before, choose a real number $\lambda$ with $0 < \lambda < 1$; write $z_i^{\ast} = \lambda \frac{2}{\mu_1} z_i / \| z_i \|$, so that $\| z_i^{\ast} \| < \lambda \mu_1$. Since $z_i^{\ast} \leq x_i^{(K, s, t)}$, extreme desirability implies that
\[ x_i^{(K, s, t)} + \lambda v_i - z_i^* \in F_i(x_i^{(K, s, t)}) , \]

so that, by the equilibrium conditions,

\[ \pi^{(K, s, t)}(x_i^{(K, s, t)}) + \lambda v_i - z_i^* > \pi^{(K, s, t)}(x_i^{(K, s, t)}) . \]

If we unwind this as before, we conclude that

\[ \pi^{(K, s, t)}(z) = \pi^{(K, s, t)}\left( \sum_{i=1}^{N} z_i \right) \]

\[ \leq \sum_{i=1}^{N} \frac{\pi^{(K, s, t)}(v_i)}{\lambda \mu_i} \|z_i\| . \]

Since \( 0 \leq z_i \leq z \), we also have that \( \|z_i\| \leq \|z\| \); since \( \lambda \) may be chosen as close to 1 as we like, we conclude that (by our normalization,

\[ \pi^{(K, s, t)}(z) \leq \sum_{i=1}^{N} \frac{\pi^{(K, s, t)}(v_i)}{\mu_i} \|z\| = \|z\| , \]

whenever \( z \in K \) and \( 0 \leq z \leq \sum x_i^{(K, s, t)} \). Combining this estimate with that of the previous paragraph, we conclude that

\[ (9.1) \quad \pi^{(K, s, t)}(z) \leq \|z\| \text{ for every } z \in K \text{ such that } z \leq \sum x_i^{(K, s, t)} . \]

For the third step, we let \( z \) be an arbitrary element of \( K \); we want to show that \( \pi^{(K, s, t)}(z) \leq (1 + C)\|z\| \). There is no loss of generality in assuming that

\[ \|z\| \leq 1/t , \text{ so that } z \leq \frac{1}{t} e_K . \]

Write \( y = \sum_{j=1}^{M} y_j^{(K, s, t)} \) and \( y = y^+ - y^- \). Since \( ((x_i^{(K, s, t)}), (y_j^{(K, s, t)})) \) is a feasible allocation for the economy \( \xi^{(K, s, t)} \), we have that

\[ \sum x_i^{(K, s, t)} = \omega + \frac{1}{t} e_K + y^+ - y^- . \]
Equivalently,

\[
\sum x_i^{(K, s, t)} + y^- = \omega + \frac{1}{t} e_K + y^+.
\]

Since \( z \leq \frac{1}{t} e_K \), it is also the case that \( z \leq \sum x_i^{(K, s, t)} + y^- \). Write \( z = z^+ - z^- \) and apply the Riesz Decomposition Property to \( z^+ \) to find vectors \( z_1, z_2 \) in \( K \) with \( 0 \leq z_1 \leq \sum x_i^{(K, s, t)} \), \( 0 \leq z_2 \leq y^- \), and \( z^+ = z_1 + z_2 \). It is certainly the case then that \( z_1 - z^- \leq \sum x_i^{(K, s, t)} \) and that \( \| z_1 - z^- \| \leq \| z \| \). Combining this with (9.1) yields

\[
\pi^{(K, s, t)}(z_1 - z^-) \leq \| z \|.
\]

It remains to estimate \( \pi^{(K, s, t)}(z_2) \). To do this, we use the assumption that the marginal efficiency of production (for the economy \( \xi \)) is strongly bounded to find vectors \( y_j \) in \( Y_j \) (for each \( j \)), a vector \( z_3 \in L \) with \( 0 \leq z_3 \leq y_+ \), and a real number \( \rho \) with \( 0 < \rho < 1 \) such that

\[
0 \leq y_j^+ \leq y_j^{(K, s, t)}^+ \quad \text{(for each \( j \)},
\]

\[
0 \leq y_j^- \leq y_j^{(K, s, t)}^- \quad \text{(for each \( j \)},
\]

\[
\sum_{j=1}^{M} y_j = (y^+ - z_3) - (y^- - \rho z_2),
\]

\[
\| z_3 \| \leq C \| \rho z_2 \|.
\]

Since \( K \) is a principal order ideal, the first two inequalities imply that the vectors \( y_j \) belong to \( K \), and hence to \( Y_j^{(K, s, t)} \) (for each \( j \)). At equilibrium, each firm maximizes profits, so the aggregate profit is also maximized. Hence
\[
\pi^{(K,s,t)}_{(y)} \geq \pi^{(K,s,t)}_{(y^+ - z_3^+)} - (y^- - \rho z_2^-)
\]

Thus,
\[
\pi^{(K,s,t)}_{(-z_3^+ + \rho z_2^-)} \leq 0,
\]

and
\[(9.4) \quad \pi^{(K,s,t)}_{(z_3^+)} \geq \rho \pi^{(K,s,t)}_{(z_2^-)}.
\]

Now, since \(0 \leq z_3 \leq y^+\), (9.2) implies that \(z_3 \leq \sum x^1_{(K,s,t)} + y^-\). On the other hand, \(y^+\) and \(y^-\) are disjoint, so in fact \(z_3 \leq \sum x^1_{(K,s,t)}\). We can therefore use (9.1) to conclude that
\[
\pi^{(K,s,t)}_{(z_3^+)} \leq \|z_3\| \leq C\|\rho z_2\|.
\]

Combining this with (9.4) yields that
\[
\rho \pi^{(K,s,t)}_{(z_2^-)} \leq C\|\rho z_2\|.
\]

Dividing by \(\rho\) yields
\[
\pi^{(K,s,t)}_{(z_2^-)} \leq C\|z_2\| \leq C\|z\|.
\]

Since \(z = z^+ - z^- = x^1_2 + z^-\), we may combine this inequality with our previous estimate for \(\pi^{(K,s,t)}_{(z_1^- - z^-)}\) to obtain:

\[(9.5) \quad \pi^{(K,s,t)}_{(z)} \leq (1 + C)\|z\| \text{ for every } z \text{ in } K.
\]

The fourth and final step is trivial. Since (9.5) is valid for every \(z\) in \(K\), replacing \(z\) by \(-z\) gives
\[
\left| \pi^{(K,s,t)}_{(z)} \right| \leq (1 + C)\|z\|
\]

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for every $z$ in $K$. In other words, the norm of $\pi^{(K, s, t)}$, computed with respect to the original norm $\|\cdot\|$, is at most $1 + C$. This is the estimate we desire. 

We now use the Hahn-Banach Theorem to extend $\pi^{(K, s, t)}$ to a linear functional on $L$ whose norm is at most $1 + C$; we will denote this extension by $\pi^{(K, s, t)}$, since no confusion should result.

The next step is the construction of a quasi-equilibrium for $\xi$ via a three-state limiting process. (The reason for the three stages will become clear shortly.) For the first stage, we fix a principal order ideal $K$ and a positive integer $s$. For $t$ sufficiently large, we have an equilibrium $\langle (x_i^{(K, s, t)}, (y_j^{(K, s, t)}), \pi^{(K, s, t)} \rangle$. As we have already noted, the consumers' allocations and the firms' production plans lie in $J$-compact subsets of $L$, and $\pi^{(K, s, t)}$ lies in the ball with center $0$ and radius $1 + C$ in the dual space of $L$; Alaoglu's Theorem guarantees that such balls are weak-star compact. Hence, passing to subnets if necessary, we may find vectors $x_i^{(K, s)}$ and $y_j^{(K, s)}$ in $L$ and a linear functional $\pi^{(K, s)}$ on $L$ (of norm at most $1 + C$) such that $\{x_i^{(K, s, t)}\}_i$ converges to $x_i^{(K, s)}$ in the topology $J$ (for each $i$), $\{y_j^{(K, s, t)}\}_j$ converges to $y_j^{(K, s)}$ in the topology $J$ (for each $j$) and $\{\pi^{(K, s, t)}\}_j$ converges to $\pi^{(K, s)}$ in the weak-star topology. As we showed earlier, $\langle (x_i^{(K, s)}, (y_j^{(K, s)}), \pi^{(K, s)} \rangle$ is a feasible allocation for the economy $\xi$. (It is true that $\langle (x_i^{(K, s)}, (y_j^{(K, s)}), \pi^{(K, s)} \rangle$ is a quasi-equilibrium for a certain economy, but we shall have no use for this fact.) This completes the first stage.

Note that this calculation depends strongly on the fact that the aggregate initial endowment in the economy $\xi^{(K, s, t)}$ is in the interior (relative to $\|\cdot\|_K$) of $K^+$. This is of course the other reason for altering the original initial endowments.
For the second stage, we fix a principal order ideal $K$ in $\mathcal{X}$. Just as before—passing to subnets if necessary, and recalling that the set of feasible allocations for $\xi$ is compact (when we give each factor of $L^{N+M}$ the topology $J$)—we may find vectors $x_i^K, y_j^K$ in $L$, and a linear functional $\pi^K$ on $L$, of norm at most $1+C$, such that $\{x_i^{(K,s)}\}$ converges to $x_i^K$ in the topology $J$ (for each $i$), $\{y_j^{(K,s)}\}$ converges to $y_j^K$ in the topology $J$ (for each $j$), and $\{\pi^{(K,s)}\}$ converges to $\pi^K$ in the weak-star topology. (Notice that there is no reason to suppose that $x_i^K$ or $y_j^K$ belong to $K$. However, $\{(x_i^K), (y_j^K)\}$ is a feasible allocation for the economy $\xi$.) This completes the second stage.

For the third stage, we recall that $\mathcal{X}$ is directed by inclusion. Once again—passing to subnets if necessary—we may find vectors $\tilde{x}_i, \tilde{y}_j$ in $L$ and a linear functional $\tilde{\pi}$ on $L$, of norm at most $1+C$, so that $\{x_i^K\}$ converges to $\tilde{x}_i$ in the topology $J$ (for each $i$), $\{y_j^K\}$ converges to $\tilde{y}_j$ in the topology $J$ (for each $j$), and $\{\pi^K\}$ converges to $\tilde{\pi}$ in the weak-star topology. This completes the third stage of the construction.

We want to show that $\{(\tilde{x}_i), (\tilde{y}_j), \tilde{\pi}\}$ is a quasi-equilibrium for $\xi$. Note first that $\{(\tilde{x}_i), (\tilde{y}_j)\}$ is a feasible allocation (since it is the limit of feasible allocations). Moreover, $\tilde{\pi}\left(\sum_{\mu_i}^{V_i}\right) = 1$ (since $\pi^{(K,s)}\left(\sum_{\mu_i}^{V_i}\right) = 1$, so that $\tilde{\pi}\left(\sum_{\mu_i}^{V_i}\right) = 1$, and thus $\pi^{K}\left(\sum_{\mu_i}^{V_i}\right) = 1$, so that $\tilde{\pi}\left(\sum_{\mu_i}^{V_i}\right) = 1$). In particular, $\tilde{\pi}$ is not the zero functional.

The remainder of the argument is very similar to the end of the proof of Theorem 1(a). We first verify the fact:
If \( x, y_1, \ldots, y_M \) are vectors in \( L \) with \( x \in P_i(\bar{x}_i) \) (for some \( i \)) and \( y_j \in Y_j \) (for each \( j \)), then
\[
\bar{\pi}(x) \geq \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \bar{\pi}(y_j).
\]

For suppose this were not so. Then
\[
\bar{\pi}(x) < \pi(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \bar{\pi}(y_j),
\]
so that, because \( \{\pi^K_j\} \) converges to \( \bar{\pi} \),
\[
\pi^K(x) < \pi^K(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi^K(y_j)
\]
for all \( K \) containing some element \( K_0 \) of \( \mathcal{X} \). Moreover, continuity of preferences together with the fact that \( \{x^K_i\} \) converges to \( \bar{x}_i \) implies that \( x \in P_i(x^K_i) \) for all \( K \) containing some element \( K_1 \) of \( \mathcal{X} \). Let \( K_2 \) be the principal order ideal generated by \( \omega + x + \sum \bar{x}_1 + \sum |\bar{y}_1| + \sum |y_j| \), so that \( K_2 \in \mathcal{X} \). Since \( \mathcal{X} \) is directed by inclusion, we can choose and fix an element \( K^* \) of \( \mathcal{X} \) which contains \( K_0, K_1 \) and \( K_2 \); in particular, \( K^* \) contains all the vectors \( \omega_i, x, \bar{x}_i, \bar{y}_j, y_j \). Moreover,
\[
\pi^K^*(x) < \pi^K^*(\omega_i) + \sum_{j=1}^{M} \theta_{ij} \pi^K^*(y_j)
\]
and \( x \in P_i(x^K_i) \).

We may now use convergence of \( \{\pi^{(K^*, s)}_i \} \) to \( \pi^{K^*}_i \) and of \( \{x^{(K^*, s)}_i \} \) to \( x^{(K, s)}_i \) to choose an \( s^* \) so large that
\[ \pi(K^*, s^*)(x) < \pi(K^*, s^*)(\omega) + \sum_{j=1}^{M} \theta_{ij} \pi(K^*, s^*)(y_j) \]

and \( x \in P_{\omega} \pi(K^*, s^*) \). Moreover, we may also choose \( s^* \) so large that

\[ y_j \in Y_j \cap [-s_{K^*}, +s_{K^*}] \]

for each \( j \).

Finally, we use convergence of \( \{\pi(K^*, s^*, t^*)\} \) to \( \pi(K^*, s^*) \) and of \( \{x_i(K^*, s^*, t^*)\} \) to \( x_i(K^*, s^*) \) to choose a \( t^* \) so large that

\[ \pi(K^*, s^*, t^*)(x) < \pi(K^*, s^*, t^*)(\omega) + \sum_{j=1}^{M} \theta_{ij} \pi(K^*, s^*, t^*)(y_j) \]

and \( x \in P_{\omega} \pi(K^*, s^*, t^*) \). Since \( \omega_i(K^*, s^*, t^*) = \omega_i + \frac{1}{t^*} e_k \in K^* \), we may also choose \( t^* \) large enough so that

\[ \pi(K^*, s^*, t^*)(x) < \pi(K^*, s^*, t^*)(\omega) + \sum_{j=1}^{M} \theta_{ij} \pi(K^*, s^*, t^*)(y_j) \].

On the other hand, the vectors \( y_j \) belong to the production sets \( Y_j^{(K^*, s^*, t^*)} \), so profit-maximization by the firms guarantees that

\[ \pi(K^*, s^*, t^*)(x) < \pi(K^*, s^*, t^*)(\omega) + \sum_{j=1}^{M} \theta_{ij} \pi(K^*, s^*, t^*)(y_j) \].

Since \( x \) belongs to the consumption set \( X_i^{(K^*, s^*, t^*)} \), this means that \( x \) satisfies consumer \( i \)'s budget constraint in the economy \( \xi^{(K^*, s^*, t^*)} \), which violates the equilibrium conditions. This contradiction implies that fact (*) is indeed true.

This may be a good place to point out why our three-stage construction—although cumbersome—is necessary. The difficulty is that, even when \( K \subset K' \), the
vectors \( e_K \) and \( e_{K'} \) may be entirely unrelated. Hence, the production sets \( Y_j^{(K,s,t)} \) and \( Y_j^{(K',s',t')} \) may also be unrelated. The approach we have taken avoids this difficulty, at the cost of going through several stages.

The remainder of the verification that \( ((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi}) \) is a quasi-equilibrium for \( \xi \) follows exactly the end of the proof of Theorem 1(a), with just the same modifications as above.

We conclude that \( ((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi}) \) is a quasi-equilibrium for \( \xi \). Since

\[ \tilde{\pi}\left(\sum \frac{v_i}{\mu_i}\right) = 1 \] and \( \tilde{\pi}(v_i) \geq 0 \) for each \( i \) (because \( v_i \) is extremely desirable), it follows that \( \tilde{\pi}\left(\sum v_i\right) > 0 \). This completes the proof of the first part of Theorem 2.

For the second part, we wish to show that if \( \xi \) is irreducible and \( ((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi}) \) is any quasi-equilibrium for \( \xi \) with \( \tilde{\pi}\left(\sum v_i\right) > 0 \), then \( ((\tilde{x}_i), (\tilde{y}_j), \tilde{\pi}) \) is actually an equilibrium. This follows exactly as in the finite-dimensional case.

We let \( I \) denote the set of consumers \( i \) such that there is a vector \( z_i \) with \( 0 \leq z_i \leq \omega_i \) and \( \tilde{\pi}(z_i) \neq 0 \) (so consumers in the set \( I \) have income). Notice that \( I \) is not empty. For, since \( 0 \leq v_i \leq \omega \) for each \( i \), we may use the Riesz Decomposition Property to find vectors \( u_i \) in \( L \) with \( 0 \leq u_i \leq \omega_i \) for each \( i \) and

\[ \frac{1}{N} \sum_{i=1}^{N} v_i = \sum_{i=1}^{N} u_i. \] Since \( \tilde{\pi}\left(\sum v_i\right) > 0 \), it must be the case that \( \tilde{\pi}(u_i) > 0 \) for at least one consumer \( i \), and this consumer belongs to \( I \).

We next show that the equilibrium conditions are satisfied for each consumer in \( I \). Fix \( i \) in \( I \) and a vector \( z_i \) with \( 0 \leq z_i \leq \omega_i \) and \( \tilde{\pi}(z_i) \neq 0 \). Let \( x \in P_{\tilde{x}_i} \); we know that
(9.6) \[ \tilde{\pi}(x) \geq \tilde{\pi}(\omega_i) + \sum_{j=1}^{N} \theta_{i,j} \tilde{m}(y_j) \]

and we want to show that the inequality is strict. If it is not, we distinguish two cases. CASE 1: \( \tilde{\pi}(\omega_i) > 0 \). Then \( \tilde{\pi}(x) > 0 \) (since \( \sum_{j=1}^{\infty} \theta_{i,j} \tilde{m}(y_j) > 0 \)) so \( \tilde{\pi}(tx) < \tilde{\pi}(x) \) if \( 0 < t < 1 \). On the other hand, \( tx \in P_i(\tilde{x}) \) if \( t \) is sufficiently close to 1 (by continuity) so \( tx \) strictly satisfies consumer \( i \)'s budget constraint and is strictly preferred to \( \tilde{x}_i \), a contradiction. CASE 2: \( \tilde{\pi}(\omega_i) \leq 0 \). We consider the vector \( x' = x + \lambda \omega_i + \lambda' z_i \), where \( \lambda > 0 \), \( |\lambda'| < \lambda \) and the sign of \( \lambda' \) is the opposite of the sign of \( \tilde{\pi}(z_i) \). Since \( |\lambda'| < \lambda \) and \( 0 \leq z_i \leq \omega_i \), the vector \( x' \) belongs to \( L^+ = X_i^+ \); since \( \pi(\omega_i) \leq 0 \) and \( \lambda' \) has the opposite sign from \( \tilde{\pi}(z_i) \), \( \tilde{\pi}(x') < \tilde{\pi}(x) \). But then, if \( \lambda, \lambda' \) are sufficiently small, the vector \( x' \) will strictly satisfy consumer \( i \)'s budget constraint and be strictly preferred to \( \tilde{x}_i \), a contradiction. We conclude that the equilibrium conditions are satisfied for each consumer in \( I \).

It remains to show that every consumer belongs to \( I \). If not, irreducibility implies that there is a consumer \( i \) in \( I \), a consumer \( k \) not in \( I \) and a vector \( z \) in \( L \) such that \( 0 \leq z \leq \omega_k \) and \( \tilde{x}_i + z \in P_i(\tilde{x}_i) \). Since consumer \( k \) has no income, \( \tilde{\pi}(z) = 0 \); but this contradicts the equilibrium condition just established for consumers in \( I \).

We conclude that every consumer belongs to \( I \), and hence that the quasi-equilibrium \( (\tilde{x}_i, \tilde{y}_j, \tilde{\pi}) \) is indeed an equilibrium. This completes the proof of Theorem 2.

PROOF OF THEOREM 3: We construct a new economy \( \mathcal{E}' \), apply Theorem 2 to obtain a quasi-equilibrium, and then show that the quasi-equilibrium for \( \mathcal{E}' \) yields a quasi-equilibrium for \( \mathcal{E} \).
The economy $\mathcal{E}'$ has the same consumers, consumption sets, preferences and initial endowments as $\mathcal{E}$. There is only one firm in $\mathcal{E}'$; its production possibility set is $Y = \sum_{j=1}^{M} Y_j$. Firm shares in $\mathcal{E}'$ may be chosen arbitrarily. It is evident that $\mathcal{E}'$ satisfies the Standard Assumptions and also assumptions (1) - (8) of Theorem 2. (Note that $Y$ is $J$-closed, by assumption, and hence also norm closed.) Moreover, the marginal efficiency of production for $\mathcal{E}'$ is certainly weakly bounded, since this is the case for $\mathcal{E}$ and this is purely a property of the aggregate production set. But, since $\mathcal{E}'$ has only one firm, the marginal efficiency of production for $\mathcal{E}'$ is also strongly bounded. Hence we may apply Theorem 2 to the economy $\mathcal{E}'$ to obtain a quasi-equilibrium $((\bar{x}_1, \bar{y}, \pi))$ for $\mathcal{E}'$ with $\pi(\sum \bar{y}_1) > 0$.

Since $Y = \sum Y_j$, we can find vectors $\bar{y} \in Y_j$ such that $\bar{y} = \sum \bar{y}_j$. We claim that $((\bar{x}_1), (\bar{y}_j), \pi)$ is a quasi-equilibrium for $\mathcal{E}$. To see this, note first of all that it is a feasible allocation. To see that each firm is maximizing, suppose that $y^*_k \in Y_k$ and $\pi(y^*_k) > \pi(\bar{y}_k)$. Write $y^* = y^*_k + \sum_{j \neq k} \bar{y}_j$; then $y^* \in Y$ and $\pi(y^*) > \pi(\bar{y})$, which contradicts profit-maximization in $\mathcal{E}'$; hence each firm is indeed maximizing. Moreover, since each of the production possibility sets $Y_j$ is a cone at 0, profit-maximization implies that each firm is actually making zero profit. Hence the consumers' budget constraints in $\mathcal{E}$ are the same as in $\mathcal{E}'$. It follows immediately that $((\bar{x}_1), (\bar{y}_j), \pi)$ is a quasi-equilibrium for $\mathcal{E}$, and $\pi(\sum \bar{y}_1) > 0$.

The second part of Theorem 3 is proved in exactly the same way as the second part of Theorem 2. This completes the proof. \[\square\]
10. REMARKS

We have noted in a number of places that the results of this paper could be improved in various ways. This section seems an appropriate place in which to indicate some of the improvements.

(1) We have required that the preferred sets be convex and that the preferences of each consumer be independent of the consumption of others. However, each of these requirements could be relaxed. All of the results would remain valid if we assumed that the preferences of each consumer were given by a relation $P_i(x_1, \ldots, x_N)$ which depends on the consumption of all consumers, and satisfies the requirement that $x_i$ does not belong to the convex hull of $P_i(x_1, \ldots, x_N)$; only trivial modifications in the proofs would be necessary.

(2) The notion of an extremely desirable commodity could be weakened a bit without altering the validity of Theorems 2 and 3. It would suffice, for example, merely to require that $x + \lambda v_i - \sigma \in P_i(x)$ provided that $\lambda$ is a sufficiently small positive number, $\sigma \leq x$ and $\|\sigma\| < \lambda \mu_i$ (rather than the requirement we have made, that $x + \lambda v_i - \sigma \in P_i(x)$ provided that $0 < \lambda < 1$, $\sigma \leq x + \lambda v_i$ and $\|\sigma\| < \lambda \mu_i$). Again, only trivial modifications in the proofs would be necessary.

In fact, this requirement could be weakened even more, by allowing the vector $v_i$ (but not the constant $\mu_i$) to depend on $x$ and on $\sigma$. However, this change would entail substantial complications in the proof. Moreover, in the presence of monotone preferences, the weaker requirements would coincide with the original requirements. In all, such an improvement does not seem terribly worthwhile.
(3) Perhaps the most troublesome assumption in Theorems 2 and 3 is that consumption sets must be the positive cone of the commodity space. However, a close look at the proofs shows that this assumption is only used in two ways. The first is to guarantee that the consumption sets in the subeconomies $\xi^{(K, s, t)}$ have nonempty interior (in the norm $\| \cdot \|_K$). The second is to guarantee that certain vectors of the form $x + \lambda v_i - \sigma$ actually belong to the consumption sets. Other assumptions could be made on the consumption sets, however, which would have the same effect. For example, it would suffice to assume that for each $i$ the consumption set $X_i$ is contained in the positive cone $L^+$ and, in addition, that for each $i$ there is a relatively $\mathcal{J}$-open subset $U_i$ of $L^+$ such that $\hat{X}_i \subset U_i \subset X_i$. The case of completely general consumption sets seems to me to be very hard.
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