THE CORE OF A GAME WITH A CONTINUUM OF PLAYERS AND
FINITE COALITIONS: NONEMP'TINESS
WITH BOUNDED SIZES OF COALITIONS

By

Mamoru Kaneko

AND

Myrna Holtz Wooders

IMA Preprint Series # 126

January 1985

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
The Core of a Game With a Continuum of Players and
Finite Coalitions: Nonemptiness
With Bounded Sizes of Coalitions

by

Mamoru Kaneko* and Myrna Holtz Wooders**

* Institute of Socio-Economic Planning, University of Tsukuba, Ibaraki 305, Japan

** Department of Economics, University of Toronto, Toronto, Ontario, Canada
Acknowledgements

The authors are indebted to the Natural Sciences and Engineering Research Council of Canada for financial support. Part of this research was carried out while Wooders was at the Institute for Mathematics, University of Minnesota and the hospitality and support of the Institute is gratefully acknowledged.
Abstract

In this paper, we continue our study of the core, called the f-core, of a game with a continuum of players and finite coalitions. In previous work, we demonstrated nonemptiness of the f-core with the assumption that the players can be partitioned into a finite number of types. Here, we relax this assumption. To do this, we consider games where players are described by points in some space of attributes and the characteristic function is determined by the attributes of the players in each coalition. To formulate this structure, we take as given a pregame, a correspondence associating a set of "attainable payoffs" with every finite list of attributes. A game is then determined by the pregame and an attribute function, which ascribes attributes to each player. To obtain our theorem, we assume essential coalitions are bounded in size (i.e. numbers of players). Relaxing the type assumption of the previous paper makes it necessary for us to also assume a strong comprehensiveness property -- nevertheless, a weak condition. The theorem is applied to market games and other, immediate, applications are noted.
1. Introduction

In models of cooperative games with a continuum of players it has become standard to define the core of the game as the set of attainable payoffs that cannot be improved upon by any coalition of positive measure. As discussed in Kaneko and Wooders (1984), there are ambiguities in interpretation of these models. It is argued that the most reasonable interpretation is that each player is some subset of arbitrarily small but positive measure; otherwise, it is implausible that individual players are capable of facilitating cooperation. This interpretation, however, leaves the concept of the individual player imprecise. To give precision to this concept while maintaining plausibility of the idea that cooperation arises from the actions of individual players, in Kaneko and Wooders (1984) we introduced a new model of a game with a continuum of players. In our model, only finite coalitions (ones containing finite numbers of players) are allowed, and, consistent with this restriction, feasible outcomes must be attainable by partitions of the set of players into finite coalitions; the core, called the \textit{f-core}, is the set of attainable payoffs that cannot be improved upon by any finite coalition.

In the previous paper, Kaneko and Wooders (1984), we demonstrated nonemptiness of the f-core of a game satisfying the type assumption -- the assumption that the players can be partitioned into a finite number of types. The theorem was applied to a market game to obtain the non-emptiness of the f-core of the game with the type assumption but with otherwise considerably mild restrictions.
In this paper, we demonstrate the nonemptiness of the f-core without the type assumption. To relax this assumption we consider a class of games where the payoffs attainable by a coalition depend on the attributes of its members. To obtain nonemptiness of the f-core of games in this class, we assume boundedness of sizes of essential coalitions, i.e., for some integer \( n \), only coalitions containing no more than \( n \) members are essential. If we take coalition formation costs into account, this assumption can be justified.

In our framework, each player of a game is described by a point in some space of attributes. In a private goods exchange economy, for example, a point in the space of attributes would be a pair consisting of an initial endowment and a utility function. The characteristic function, describing the set of payoffs attainable by each coalition, then depends on the attributes of the members of the coalition. To formulate this situation, we take as given a pregame, i.e. a specification of a set of "attainable payoffs" for every finite list of attributes. A game is then determined by an attribute function which ascribes attributes to each player.

Our theorem is that, under some additional weak assumptions, any game with a continuum of players has a nonempty f-core. The conditions required are a continuity condition and strong comprehensiveness of payoff sets.

The direct application of our result to diverse game-theoretic and economic situations is immediate. We reference some examples: assignment games, introduced by Gale and Shapley (1962) and applied to market situation in Shapley and Shubik (1972), Crawford and Knoer (1981), Kaneko (1982), and Kelso and Crawford (1982); and partitioning games (a generalization of assignment games) in Kaneko and Wooders (1982).
2. Nonemptiness of the f-Core of a Continuum Game

2.1 General Framework

In this subsection, we repeat the general framework and description introduced in Kaneko and Wooders (1984).

Let \((N, B, \mu)\) be a measure space, where \(N\) is a Borel subset of a complete separable metric space; \(B\), the \(\sigma\)-algebra of all Borel subsets of \(N\); and \(\mu\), a nonatomic measure with \(0 < \mu(N) < +\infty\). Each element in \(N\) is called a player and \(N\) is the player set. The measure \(\mu\) represents the distribution of players. The \(\sigma\)-algebra \(B\) is necessary for measurability arguments but does not play any important game-theoretic role.

Let \(F\) be the set of all finite subsets of \(N\). Each element \(S\) in \(F\) is called a finite coalition or simply a coalition. As discussed briefly in Section 1, only finite subsets of players can form coalitions.

A function \(\psi\) from a set \(A\) in \(B\) to a set \(B\) in \(B\) is called a measure-preserving isomorphism from \(A\) to \(B\) iff (i) \(\psi\) is a measure-theoretic isomorphism, i.e., \(\psi\) is 1 to 1, onto, and measurable in both directions, and (ii) \(\mu(C) = \mu(\psi(C))\) for all \(C \in A\) with \(C \in B\). A partition \(p\) of \(N\) is feasible iff for any positive integer \(k\),

\[
N^p_k \equiv \bigcup_{S \in p} S \quad \text{is a measurable subset of} \quad N \quad \text{and;} \quad (2.1)
\]

\[
|S| = k
\]
each $N^P_k (k=1,2,...)$ has a partition $\{N^P_{kt}\}^k_{t=1}$, where each $N^P_{kt}$ is measurable, with the following property: there are measure-preserving isomorphisms $\psi^P_{k1}, \psi^P_{k2}, \ldots, \psi^P_{kk}$ from $N^P_{k1}$ to $N^P_{k1}, \ldots, N^P_{kk}$, respectively, such that $\{\psi^P_{k1}(i), \ldots, \psi^P_{kk}(i)\} \in p$ for all $i \in N^P_{k1}$.*

Note that (2.1) implies that for any $S \in p$ with $|S| = k$, we have $S = \{\psi^P_{k1}(i), \ldots, \psi^P_{kk}(i)\}$ for some $i \in N^P_{k1}$.

Let $\Pi$ be the set of feasible partitions.

**Example 2.1**

Consider a marriage (assignment) model where the girls are the points in $[0,1)$, and the boys, those in $[1,3)$, with Lebesgue measure. The partition $\{(i, i+2i): i \in [0,1)\}$ violates (2.1). An example of a feasible partition is $p = \{(i, 2+i): i \in [0,1)\} \cup \{(i): i \in [1,2)\}$.

For this partition $p$, we have $N^P_1 = \bigcup_{|S|=1} S \in p$, $N^P_2 = \bigcup_{|S|=2} S \in p$:

$$= [0,1) \cup [2,3), N^P_{21} = [0,1) \text{ and } N^P_{22} = [2,3);$$

then measure-preserving isomorphisms satisfying (2.1) are given by $\psi^P_{11}(i) = i$, $\psi^P_{21}(i) = i$ and $\psi^P_{22}(i) = 2+i$.

A **characteristic function game** $V$ without side payments is a correspondence on $F$ which assigns to each coalition $S \in F$ a subset $V(S)$ of $R^S$ with the following properties:

$$V(S) \text{ is a nonempty, closed subset of } R^S \text{ for all } S \in F; \quad (2.2)$$

$$V(S) \times V(T) \subseteq V(S \cup T) \text{ for any } S, T \in F \text{ with } S \cap T = \emptyset; \quad (2.3)$$

* $|S|$ denotes the number of players in the set $S$. 
\[ \inf \sup_{i \in \mathbb{N}} V(\{i\}) > -\infty ; \]  
\[ (2.4) \]

for any \( S \in F, x \in V(S) \) and \( y \in R^S \) with \( y \preceq x \) imply
\[ y \in V(S) ; \]  
\[ (2.5) \]

for any \( S \in F, V(S) - \bigcup_{i \in S} [\text{interior } V(\{i\})] \times R^{S-[i]} \]  
\[ (2.6) \]
is bounded.*

These assumptions are all innocuous and we note only that (2.4)
can always be obtained for a transformation of a game, and that conditions
(2.3) and (2.5) are called, respectively, super-additivity and comprehen-
siveness.

The game \( V \) will be required to be compatible with the measura-
bility structure of the player set. Let \( p \) be an arbitrary feasible
partition of \( N \). For any positive integer \( k \), we can regard
\[ V(\{\psi_{kl}^p(i), \psi_{k2}^p(i), \ldots, \psi_{kk}^p(i)\}) \] as a correspondence from \( N_{kl}^p \) to \( R^k \),
i.e.,
\[ V: N_{kl}^p \rightarrow 2^{R^k} \]
i \[ i \mapsto V(\{\psi_{kl}^p(i), \ldots, \psi_{kk}^p(i)\}) . \]

In this sense, we assume that
\[ V(\{\psi_{kl}^p(i), \ldots, \psi_{kk}^p(i)\}) \] has a measurable graph, i.e., the
\[ (2.7) \]
set \( \{(i,x) \in N_{kl}^p \times R^k : x \in V(\{\psi_{kl}^p(i), \ldots, \psi_{kk}^p(i)\}) \} \) is
measurable in \( N_{kl}^p \times R^k \).

* From (2.2) and (2.3) it is immediate that this set is nonempty.
Remark 2.1. In the previous paper, we assumed \( V(\{\psi_{kl}^i(i), \ldots, \psi_{kk}^i(i)\}) \) had an analytic graph; for the purposes of this paper, we simply assume the graph is measurable.

To define the set of possible outcomes of the game some preliminary definitions are required.

Given a feasible partition \( p \), define the set \( H(p) \) by

\[
H(p) = \{ h \in L(N,R) : (h(j))_{j \in p(i)} \in V(p(i)) \text{ for almost all } i \in N \},
\]

where \( L(N,R) \) is the set of measurable functions from \( N \) to \( R \) and \( p(i) \) is the element of the partition \( p \) containing player \( i \); \( H(p) \) is the outcome set relative to \( p \). Let

\[
H = \bigcup_{p \in \Pi} H(p). \tag{2.9}
\]

The following Lemma is proved in Kaneko and Wooders (1984).

Lemma 2.1. For any feasible partition \( p \) of \( N \), \( H(p) \neq \emptyset \).

Define the outcome set \( H^* \) by

\[
H^* = \{ h \in L(N,R) : \text{for some sequence } \{h^v\} \text{ in } H, \{h^v\} \text{ converges in measure to } h \},
\]

(2.10)

where "\( (h^v) \) converges in measure to \( h \)" means that for any \( \varepsilon > 0 \),

\[
\mu(\{ i \in N : |h^v(i)-h(i)| > \varepsilon \}) \to 0 \text{ as } v \to \infty .
\]

Note that \( H(p) \subset H \subset H^* \) for all \( p \in \Pi \).
Remark 2.2. If \( \{h^\nu\} \) converges in measure to \( h \), then \( \{h^\nu\} \) has a subsequence which converges pointwise to \( h \) a.e. and, conversely, if \( \{h^\nu\} \) converges pointwise to \( h \) a.e., then \( \{h^\nu\} \) converges in measure. Therefore \( H^* \) can be defined by pointwise convergence and in our proofs we use whichever definition is convenient. Note that \( H^* \) is closed with respect to these convergences, i.e., if \( \{h^\nu\} \) in \( H^* \) converges either pointwise or in measure to \( h \), then \( h \in H^* \).

Remark 2.3. When we restrict essential coalitions to be in \( F_n \), it follows that if \( h \in H^* \), given \( \epsilon > 0 \) there exists a partition \( p \in \pi \), \( p \subset F_n \), and an outcome \( h' \in H(p) \) such that

\[
\mu(\{i \in N: |h'(i)-h(i)| > \epsilon\}) < \epsilon.\
\]

Let \( h \) be a function in \( L(N,R) \). We say that a coalition \( S \) in \( F \) can improve upon \( h \) iff for some \( y \in V(S) \), \( y_i > h(i) \) for all \( i \in S \). Now the \( f \)-core of the game \( V \) without side payments is defined to be the set \( C^f \):

\[
C^f = \{h \in H^*: \text{no coalition in } F \text{ can improve upon } h\}. \quad (2.11)
\]

An outcome \( h \) in the \( f \)-core \( C^f \) is stable in the sense that no coalition can improve upon \( h \) and is approximately feasible in the sense that \( h \in H^* \). That is, the outcome \( h \) can be approximated by a sequence \( \{h^\nu\} \) in \( H \) in the sense that for any \( \epsilon > 0 \), the measure of players whose payoffs under \( h^\nu \) are not within \( \epsilon \) of their payoffs under \( h \) can be made arbitrarily small by choice of \( \nu \) sufficiently large. In this sense, the \( f \)-core is the limit version of approximate cores, e.g., Shapley and Shubik (1966), Wooders (1983), Kaneko and Wooders (1982), Shubik and Wooders (1983) and Wooders and Zame (1984).

* A necessary and sufficient condition for \( \{h^\nu\} \) to converge in measure to \( h \) is that for any \( \epsilon > 0 \) there is a \( \nu_0 \) such that for all \( \nu \geq \nu_0 \),

\[
\mu(\{i \in N: |h^\nu(i)-h(i)| > \epsilon\}) < \epsilon.
\]
2.2 A Pregame $V^*$

We consider a class of games where the payoffs attainable by a coalition of players depend on the attributes of the members of the coalition and where the sizes (i.e. number of members) of essential coalitions are uniformly bounded by a given integer $n$. First, we define the space of attributes and a "pregame" which associates attainable payoffs to lists of (at most $n$) attributes. A pregame is not a game in the usual sense since it is not defined on coalitions of players but instead, on lists of attributes. Games will be derived from the pregame.

We assume that the set of attributes $E$ is given as a compact metric space with metric $d$. Let $E^* = \bigcup_{t=1}^{n} E^t = \bigcup_{t=1}^{n} E \times \ldots \times E$, where the integer $n$ is the bound on the sizes of essential coalitions. (It suffices to define the pregame only on lists of attributes of length at most $n$.)

A pregame $V^*$ is a function on $E^*$ which assigns to each $\alpha = (\alpha_1, \ldots, \alpha_t) \in E^t$ ($t=1, \ldots, n$) a subset $V^*(\alpha)$ of $R^t$ with the following properties:

\begin{equation}
V^*(\alpha) \text{ is a nonempty closed subset of } R^t; \tag{2.12}
\end{equation}

\begin{equation}
\text{if } \alpha = (\alpha^1, \alpha^2) = (\alpha_1^1, \ldots, \alpha_s^1, \alpha_1^2, \ldots, \alpha_t^2)(s + t \leq n), \text{ then} \tag{2.13}
V^*(\alpha) \supset V^*(\alpha^1) \times V^*(\alpha^2);
\end{equation}

\begin{equation}
\inf_{\alpha \in E} \sup_{\beta \in E} V^*(\alpha) > -\infty; \tag{2.14}
\end{equation}

\begin{equation}
x \in V^*(\alpha) \text{ and } y \in R^t \text{ with } y \preceq x \text{ imply } y \in V^*(\alpha); \tag{2.15}
\end{equation}
\[ V^*(\alpha_1', \ldots, \alpha_t') = \bigcup_{k=1}^{t} \left[ \text{interior } V^*(\alpha_k) \times R^{t-k} \right] \quad (2.16) \]

is a bounded subset of \( R^t \):

\[ \theta \cdot V^*(\alpha_1', \ldots, \alpha_t') = V^*(\alpha_{\theta(1)}', \ldots, \alpha_{\theta(t)}') \quad \text{for all } \theta \in \Theta_t, \quad (2.17) \]

where \( \Theta_t \) is the set of all permutations of \( \{1, \ldots, t\} \) and for a subset \( S \subseteq R^t \), \( \theta \cdot S = \{(x_{\theta(1)}, \ldots, x_{\theta(t)}): (x_1, \ldots, x_t) \in S \} \).

The pregame \( V^* \) assigns a set of attainable payoffs to each list of attributes. Conditions (2.12)-(2.16) correspond to (2.2)-(2.6).

Condition (2.17) means that the pregame \( V^* \) depends only upon attributes but not the labelling of attributes.

We introduce the sup metric \( d^* \) on each \( E^t \) \((t=1, \ldots, n)\):

for any \((\alpha_1', \ldots, \alpha_t'), (\beta_1', \ldots, \beta_t') \in E^t \),

\[ d^*((\alpha_1', \ldots, \alpha_t'), (\beta_1', \ldots, \beta_t')) = \max_{k} d(\alpha_k, \beta_k). \]

Also the \( t \)-dimensional Euclidean space \( R^t \) is endowed with the sup norm, i.e., \( ||x|| = \max |x_k| \). Let \( d^t_H \) be the Hausdorff metric for closed subsets of \( R^t \), so for closed subsets \( S, T \subseteq R^t \),

\[ d^t_H(S, T) = \max[\max_{x \in S} d^t(x, T), \max_{y \in T} d^t(S, y)] . \]

where \( d^t(x, y) = ||x-y|| \) and \( d^t(x, T) = \inf_{y \in T} d(x, y) \).

The pregame \( V^* \) is continuous iff for any \( t=1, \ldots, n \),

if \((\alpha_1^v, \ldots, \alpha_t^v) \rightarrow (\alpha_1', \ldots, \alpha_t')\), then \[ V^*(\alpha_1^v, \ldots, \alpha_t^v) \rightarrow V^*(\alpha_1', \ldots, \alpha_t')(v \rightarrow \infty) \] in the Hausdorff metric.
Remark 2.3. Since the attribute space $E$ is a compact metric space and since the set of all closed sets of $\mathbb{R}^t$ is a metric space, the continuity of $V^*$ implies uniform continuity.

The next assumption is strong comprehensiveness. The pregame $V^*$ is said to be strongly comprehensive iff there is a $b > 0$ such that for any $(\alpha_1, \ldots, \alpha_t) \in E^t$ ($t=1, \ldots, n$) and any $k(1 \leq k \leq t)$, if $x \in V^*(\alpha_1, \ldots, \alpha_t)$ and $x_k > \sup V^*(\alpha_k)$, then $y \in \mathbb{R}^t$ defined by its components

\[
Y_{\ell} = \begin{cases} 
  x_k - c & \text{if } \ell = k \\
  x_{\ell} + \frac{bc}{t-1} & \text{otherwise},
\end{cases}
\]

belongs to $V^*(\alpha_1, \ldots, \alpha_t)$ for any $c$ with $0 < c < x_k - \sup V^*(\alpha_k)$.

\[
V^*({1,2})
\]

(sup $V^*({1})$, sup $V^*({2})$)
This condition states that if a "player" (a personified point in attribute space) can get a larger payoff from joining a coalition with other players than he can achieve on his own, then he can transfer this increase in his payoff to the other players at a constant rate (which may be very small). In the figure above, from the point $x$, the point $y$ can be reached by player 1 transferring a part of his payoff to player 2 at the rate $b$. Similarly, $y'$ can be reached by transfers from player 2 to player 1. This is a weak condition and in a private goods economy suggests the presence of an infinitely divisible good for which the gradients of the utility functions are positive and uniformly bounded away from zero (see Section 3).

Lemma 2.2. The individually rational payoff vectors given by the pregame are uniformly bounded above, i.e., there is a constant $K$ such that

$$x \in V^\ast(a_1, \ldots, a_t), \quad 1 \leq t \leq n \quad \Rightarrow \quad x_k < K \quad \text{for all } k=1, \ldots, t.$$  

$$x_k > \max_{\gamma(k)} V^\ast(a_k) \quad \text{for all } k=1, \ldots, t.$$  

Proof. Appendix

From this lemma, we can assume without loss of generality that the pregame $V^\ast$ itself is uniformly bounded from above.

2.3 The Derived Game and the Main Theorem

An attribute function $\gamma$ is a function from $N$ to $E$ which is measurable in the sense that $\gamma^{-1}(B) \in B$ for any Borel subset $B$ of
E. The game $V$ derived from a pregame $V^*$ and an attribute function $\gamma$ is defined by

$$V(S) = V^*(\gamma(i_{i \in S})) = V^*(\gamma(i_1), \ldots, \gamma(i_s))$$

for all $S = \{i_1, \ldots, i_s\} \in F_n$; and

$$V(S) = \bigcup_{P(S) \in P(S)} \prod_{T \in P(S)} V(T) \text{ for all } S \in F - F_n,$$

where $F_n = \{S \in F: |S| \leq n\}$ and $P(S) = \{p_S: p_S \text{ is a partition of } S \text{ with } p_S \subset F_n\}$.

Condition (2.17) ensures that definition (2.21) does not depend upon the choice of an order $S = \{i_1, \ldots, i_s\}$. Formula (2.22) means that any coalition larger than $n$ can realize only those payoffs achievable by partitioning into coalitions in $F_n$; therefore "essential" coalitions are still in $F_n$. We note that this formulation is adopted to be consistent with the general framework of Subsection 2.1, but we can virtually ignore coalitions in $F - F_n$ without any change in the essence of our model at this point.

**Lemma 2.3.** The derived game $V$ satisfies condition (2.7).

**Proof.** Appendix

We are now in a position to state the main theorem.

**Theorem.** Assume that a pregame $V^*$ on $E^* = \bigcup_{t=1}^n E^t$ satisfies conditions (2.12)-(2.17), continuity (2.18), and strong comprehensiveness (2.19). Then given any attribute function $\gamma$, the game $V$ derived from $V^*$ and $\gamma$ has a nonempty $f$-core.
The proof of the Theorem will be given in Section 4. In the next section, 3, we apply the Theorem to market games; here, we give some immediate applications to related work.

An assignment game with a continuum of players is formulated as follows. Let \( n=2 \), and let \( \{E_1, E_2\} \) be a partition of \( E \). Let \( V^* \) be a pregame on \( E^* = E \cup (E \times E) \) such that

\[
V^*(a_1, a_2) = V^*(a_1) \times V^*(a_2) \quad \text{if} \quad a_1, a_2 \in E_1 \quad \text{or} \quad a_1, a_2 \in E_2.
\]

Here, \( E_1 \) and \( E_2 \) are the attributes of boys and girls respectively; our condition is that a coalition consisting of a pair of players of the same sex can do no better than each of the players singly. Thus, if there are any gains to coalition formation, the coalitions must contain boy-girl pairs -- these are the only possible essential coalitions. We now have an extension to a continuum of players of assignment games (c.f. Gale and Shapley (1962), Shapley and Shubik (1972), Crawford and Knoer (1981), Kelso and Crawford (1982), and Kaneko (1982)). A fortiori, an extension of partitioning games (a generalization of assignment games, considered in Kaneko and Wooders [1982]) is also covered by the Theorem.

Finally, as argued in Kaneko and Wooders [1984], a game \( v \) with side payments can be viewed as a special case of a game without side payments so our framework applies to games with side payments.
3. An Application to Market Games

In this section we show that our main theorem applies to a wide class of market games. We do, however, have to restrict the economies to ones with a bounded set of initial endowments, mainly because of the assumption of strong comprehensiveness (2.19).

Consider an exchange economy with \( m \) commodities. The consumption set \( \Omega \) is given as \( \Omega = I_+^c \times \mathbb{R}_+^{m-c} \) (with the sup norm \( \| \cdot \| \)) where \( c \) is an integer with \( 1 \leq c \leq m \) and \( I_+ \) is the set of all nonnegative integers. This means that the first \( c \) commodities are indivisible and the others are perfectly divisible.

The attribute space is \( E = U_\rho^M \times \tilde{\Omega} \) where \( U_\rho^M \) is a compact metric space of real-valued continuous functions \( u \) on \( \Omega \), (the utility functions), and \( \tilde{\Omega} \) is a compact subset of \( \Omega \). As will be apparent later, \( \rho \) and \( M \) are parameters on the space of utility functions.

Given \( E \), a pregame \( V^* \) for market games is defined in the standard way: for \( (a^1, \ldots, a_t) = ((u^1, \omega^1), \ldots, (u^t, \omega^t)) \in E^t = (U_\rho^M \times \tilde{\Omega})^t \) \( (t=1, \ldots, n) \),

\[
V^*(a_1, \ldots, a_t) = \{ x \in \mathbb{R}^t : \text{for some} \ (a^1, \ldots, a^t) \in \Omega^t \text{ with } \sum_{s=1}^t a^s s \leq \sum_{s=1}^t \omega^s s, \ x_s \leq u^s(a^s) \text{ for all } s=1, \ldots, t \} .
\]

It is clear that the pregame \( V^* \) satisfies conditions (2.12) - (2.17). Once the space \( U_\rho^M \) is completely specified, continuity (2.18)
and comprehensiveness (2.19) can be demonstrated.

The problem is to choose $U^M_\rho$ so that $E$ is compact and we have strong comprehensiveness (2.19). In this section we give a concrete example of an appropriate choice of $U^M_\rho$.

Given an attribute space $E = U^M_\rho \times \tilde{\Omega}$ and a pregame $V^*$ satisfying conditions (2.12) - (2.19), for any attribute function $\gamma$ the game $V$ derived from $V^*$ and $\gamma$ has a nonempty $f$-core $C_f$. This is a conclusion of our theorem. Then it follows from Kaneko and Wooders [1984, Theorem 1] that the market game specified by an attribute function has a nonempty $f$-core in allocation space (instead of payoff space).

(The $f$-core of a market game in allocation space is defined in Kaneko and Wooders [1984]).

We now specify the space $E = U^M_\rho \times \tilde{\Omega}$. Let $M$ be an arbitrary positive real number. Define $U^M$ by

$$U^M = \{ u : u \text{ is a real-valued function on } \tilde{\Omega} \text{ such that } u(0) = 0 \text{ and } |u(x) - u(y)| \leq M |x-y| \text{ for all } x, y \in \tilde{\Omega} \};$$

that is, $U^M$ is the class of Lipschitz continuous utility functions with normalization $u(0) = 0$.

**Remark 3.1.** If $u : \tilde{\Omega} \to R$ is Lipschitz continuous so that for some $M', |u(x) - u(y)| \leq M' |x-y|$ for all $x, y \in \tilde{\Omega}$, then $\frac{M}{M'} (u - u(0))$
belongs to the class $U^M$. Since we are ultimately concerned only with ordinal representations of utility functions, $u$ and $\frac{M}{M'} (u-u(0))$ are equivalent. That is, any preference relation that can be represented by a utility function satisfying Lipschitz continuity can be represented by a member of $U^M$.

We now define a metric $\sigma$ on the class $U^M$. Let $\{C^\nu\}_\nu=1^\infty$ be a partition of $R_+^m$ such that each $C^\nu$ can be represented as $\prod_{c=1}^m [k_c,k_c+1)$ for some nonnegative integers $k_1,k_2,\ldots,k_m$. Let $\{D^\nu\}_\nu=1^\infty$ be a partition of $\Omega$ such that each $D^\nu$ can be expressed as $D^\nu = \Omega \cap C^\nu$. Then define the metric $\sigma$ on $U^M$ by

$$\sigma(u^1, u^2) = \sum_{\nu=1}^\infty \frac{1}{\nu} \sup_{x \in D^\nu} \frac{|u^1(x) - u^2(x)|}{1 + |u^1(x) - u^2(x)|}$$  \hspace{1cm} (3.3)

for all $u^1, u^2 \in U^M$.

The following Lemma can be proved in the same way as Funaki and Kaneko [1983, Lemma 5].

**Lemma 3.1.** The class $U^M$ is a compact set with the metric $\sigma$.

Let $\tilde{\Omega}$ be a compact subset of the consumption set $\Omega$.

Initial endowments are restricted to be in $\tilde{\Omega}$.

It is necessary that we obtain strong comprehensiveness (2.19) only on the subset of $\tilde{\Omega}$ consisting of reallocations within coalitions

* There it is proved that a class of tax functions is compact with a similar metric.
containing no more than \( n \) members. Therefore we define the subset

\[
\tilde{\Omega} = \{ x \in \Omega : x \leq \sum_{t=1}^{n} y^t \text{ for some } y^1, \ldots, y^n \in \Omega \}.
\]

We take as given an arbitrary positive number \( \rho \), which will be a lower bound on the gradients of the utility functions on \( \tilde{\Omega} \) with respect to the \( m \)th commodity. We also need to assume a certain desirability of the \( m \)th commodity; we assume that any vector in \( \tilde{\Omega} \) has at least as high a utility as any vector \( x \), with \( x_m = 0 \), in \( \tilde{\Omega} \). Let \( e^m \) denote the \( m \)th unit vector of \( \mathbb{R}^m \). We define \( U^M_{\rho} \) as the set of utility functions \( u \in U^M \) satisfying:

\[
u(x + \epsilon e^m) - u(x) \geq \rho \epsilon \text{ for all } x \in \mathbb{R}^m \text{ and all } \epsilon \geq 0 \quad (3.4)
\]

such that \( x, x + \epsilon e^m \in \tilde{\Omega} \); and \( u(a) \geq u(x) \) for all \( a \in \tilde{\Omega} \) and all \( x \in \tilde{\Omega} \) with \( x_m = 0 \).

It is easy to see that \( U^M_{\rho} \) is a closed subset of \( U^M \), which implies that \( U^M_{\rho} \) is compact.

Define the metric \( d \) on \( U^M \times \tilde{\Omega} \) by

\[
d((u^1, x^1), (u^2, x^2)) = \max \{ \sigma(u^1, u^2), ||x^1 - x^2|| \} \quad (3.5)
\]

Now the attribute space \( E = U^M_{\rho} \times \tilde{\Omega} \) is a compact metric space with the metric \( d \).

The pregame \( V^* \) satisfies conditions (2.12)-(2.17), and
Lemma 3.2. The pregame $V^*$ satisfies the conditions of continuity
(2.18) and strong comprehensiveness (2.19).

Proof.

Continuity. Since the initial endowments are restricted to $\tilde{\Omega}$, the
condition that $(a^1, \ldots, a^t) \in \tilde{\Omega}^t (1 \leq t \leq n)$ with \[ \frac{t}{s} a^s < \frac{t}{s} \omega^s \quad \text{implies} \quad a^s \leq \omega^s \quad \text{for all} \quad s=1, \ldots, t. \]
When the domain of the functions in $U^M_{\rho}$ is re-
stricted to $\tilde{\Omega}$, the metric $\sigma$ can (and will) be viewed as a sup metric. That
is, it follows from (2.13) that for any $\varepsilon > 0$ and for any $u^1, u^2 \in U^M_{\rho}$
there is a $\delta_1 > 0$ such that

\[ \sigma(u^1, u^2) < \delta_1 \Rightarrow \sup_{a \in \tilde{\Omega}} |u^1(a) - u^2(a)| < \varepsilon/2. \]  

(3.6)

Now, to demonstrate continuity, let $\varepsilon > 0$ and $\delta_1$ as above be given.
Let $\delta_2$ be a real number with $0 < \delta_2 < \min(\delta_1, \frac{\varepsilon}{2M}, 1)$ and let us suppose
that $(\alpha_1, \ldots, \alpha_t) = ((u^1, \omega^1), \ldots, (u^t, \omega^t)) \in E^t$ and $(\beta_1, \ldots, \beta_t) =
((v^1, \pi^1), \ldots, (v^t, \pi^t)) \in E^t$ satisfy $\max_s d(\alpha_s, \beta_s) < \delta_2$. Since $0 < \delta_2 < 1$
and $\tilde{\Omega} = I^c_+ \times R^{m-c}_+$, we have, for all $s=1, \ldots, t$,

\[ a^s_k = \pi^s_k \quad \text{for all} \quad k=1, \ldots, c; \]  

and \[ \left| \pi^s_k - \pi^s_k \right| < \delta_2 \quad \text{for all} \quad k=c+1, \ldots, m. \]  

(3.7)

Let $x \in V^*(a_1, \ldots, a_t)$. Then, by (3.1) there is a $(a^1, \ldots, a^t) \in \tilde{\Omega}^t$ with
\[ \frac{t}{s} a^s < \frac{t}{s} \omega^s \]  

such that $x_s \leq u^s(a^s)$ for all $s=1, \ldots, t$. Define

$(b^1, \ldots, b^t)$ by
\[ b^s_k = \begin{cases} a^s_k & \text{if } k=1,\ldots,c \\ \max (a^s_k - t\delta_2, 0) & \text{otherwise} \end{cases} \tag{3.8} \]

for \( s=1,\ldots,t \). Then \( \sum_{s=1}^t b^s_k = \sum_{s=1}^t a^s_k - \sum_{s=1}^t w^s_k = \sum_{s=1}^t \pi^s_k \) for \( k=1,\ldots,c \)

by (3.7). For \( k=c+1,\ldots,m \), if \( b^1_k = \ldots = b^t_k = 0 \), then \( \sum_{s=1}^t b^s_k \leq \sum_{s=1}^t \pi^s_k \),

and if \( b^s_k = a^s_k - t\delta_2 > 0 \) for some \( 1 \leq s \leq t \), then \( \sum_{s=1}^t b^s_k \leq \sum_{s=1}^t a^s_k - t\delta_2 \leq \sum_{s=1}^t \omega^s_k - t\delta_2 \leq \sum_{s=1}^t \pi^s_k \) by (3.7). We have shown that

\[ \sum_{s=1}^t b^s_k \leq \sum_{s=1}^t \pi^s_k \]

which implies

\[ (v^1(b), \ldots, v^t(b)) \in V^*(\beta_1, \ldots, \beta_t). \tag{3.9} \]

Condition (3.6) and \( \max_{s} d(a^s, \beta^s) < \delta_2 \) imply

\[ |u^s(a^s) - v^s(b^s)| \leq |u^s(a^s) - u^s(b^s)| + |u^s(b^s) - v^s(b^s)| \tag{3.10} \]

\[ \leq M\delta_2 + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

for all \( s=1,\ldots,t \).

Define a \( y \in \mathbb{R}^t \) by its components

\[ y^s = v^s(b^s) - (u^s(a^s) - x^s) \]

for all \( s=1,\ldots,t \).

From (3.9) and comprehensiveness (2.15) we have \( y \in V^*(\beta_1, \ldots, \beta_t) \). By (3.10) we have \( ||x-y|| < \epsilon \). We have shown that for any \( x \in V^*(\alpha_1, \ldots, \alpha_t) \),
there is a \( y \in V^*(\beta_1, \ldots, \beta_t) \) such that \( \| x - y \| < \epsilon \). Conversely, we can prove that for any \( y \in V^*(\beta_1, \ldots, \beta_t) \), there is an \( x \in V^*(\alpha_1, \ldots, \alpha_t) \) with \( \| x - y \| < \epsilon \). This means that \( d_H(V^*(\alpha_1, \ldots, \alpha_t), V^*(\beta_1, \ldots, \beta_t)) < \epsilon \).

**Strong Comprehensiveness.** Let \( b = \frac{\rho}{M} \). Since \( M \) and \( \rho \) are fixed, \( b \) is also fixed. Suppose that \( x \in V^*(\alpha_1, \ldots, \alpha_t) = V^*((u_1^1, \omega_1^1), \ldots, (u_t^1, \omega_t^1)) \) and \( x_k > \sup V^*(\alpha_k^1) \) for some \( k \). We prove that for any \( c \) with \( 0 < c < x_k - \sup V^*(\alpha_k^1) \), the vector \( y \) defined by its components

\[
    y_s = \begin{cases} 
    x_k - c & \text{if } s = k \\
    x_s + \frac{bc}{t-1} & \text{otherwise}
    \end{cases}
\]

belongs to \( V^*(\alpha_1, \ldots, \alpha_t) \). It suffices to prove this in the case where

\[
x = (u_1^1(a_1^r), \ldots, u_t^1(a_t^r)) \quad \text{for some } (a_1^r, \ldots, a_t^r) \in \Omega^r \quad \text{with } \sum_{s=1}^{t} a_s \leq \sum_{s=1}^{t} \omega_s.
\]

Let \( \zeta^* = \sup \{ \zeta : u^k(a_k^r - \zeta e^m) > x_k - c \} \). Then \( u^k(a_k^r - \zeta e^m) = x_k - c \) and \( \zeta^* > 0 \) by (3.4). Define \( (b_1^r, \ldots, b_t^r) \in \Omega^t \) by

\[
    b_s = \begin{cases} 
    s - \zeta e^m & \text{if } s = k \\
    a_s + \frac{rs}{(t-1)} e^m & \text{otherwise}
    \end{cases}
\]

Then \( (b_1^r, \ldots, b_t^r) \) satisfies \( \sum_{s=1}^{t} b_s \leq \sum_{s=1}^{t} \omega_s \). From (3.2) we have

\[
    u^k(a^r) - u^k(b^r) \leq M\zeta^*; \quad (3.11)
\]
and from (3.4)

\[ u^s(b^s) - u^s(a^s) > \frac{\rho t^*}{t-1} \quad \text{for all } s \neq k \]  \hspace{1cm} (3.12)

It follows from (3.11) that

\[ c = u^k(a^k) - u^k(a^k - s^e m) \leq M^* \quad . \]  \hspace{1cm} (3.13)

Then we have

\[ y_k = x_k - c = u^k(b^k) \]

and for \( s \neq k \),

\[ y_s = x_s + \frac{bc}{t-1} = u^s(a^s) + \frac{\rho c}{M(t-1)} \]

\[ \leq u^s(a^s) + \frac{\rho M^*}{M(t-1)} \quad \text{(by (3.13))} \]

\[ = u^s(a^s) + \frac{\rho t^*}{t-1} \leq u^s(b^s) \quad \text{(by (3.12))} \quad . \]

These inequalities imply that \( y \in V^*(\alpha_1, \ldots, \alpha_t) \).

Q.E.D.
4. **Proof of the Theorem**

4.1 **Preliminaries**

Since the proof of the Theorem strongly depends upon Kaneko and Wooders [1984, Theorem 2], in this subsection we provide an explanation of the theorem and the necessary steps of its proof.

Let \((N,B,\mu)\) be as described in Section 2.1 and let \(V\) be a game without side payments. Players \(i\) and \(j\) are called substitutes iff for any \(S \in F\),

if \(i, j \notin S\), then \(x \in V(S \cup \{i\}) \notin x' \in V(S \setminus \{j\})\) \(x_i = x_j'\) for all \(\ell \in S\) and \(x_i = x_j'\); and

if \(x \in V(S)\) and \(i, j \in S\), then \(x' \in V(S)\), where \(x_i' = x_j\) for all \(\ell \in S - \{i, j\}\) and \(x_i' = x_i, x_j' = x_j\);

these conditions simply mean that the players \(i\) and \(j\) are identical with respect to the aspects described by the characteristic function \(V\). The game \(V\) has the \(r\)-property with respect to \(\{N_t\}_{t=1}^k\) iff

\[
\{N_t\}_{t=1}^k \text{ is a partition of } N \text{ with } \mu(N_t) > 0 \text{ for all } t=1, \ldots, k, \text{ and all players in each } N_t \text{ (}t=1, \ldots, k\text{) are substitutes;}
\]

a game with the \(r\)-property is simply one with a finite number of types and a positive measure of players of each type. Assume that \(V\) has the \(r\)-property with respect to \(\{N_t\}_{t=1}^k\). For any \(S \in F\), a payoff vector
$y \in V(S)$ has the equal-treatment property (the ETP) iff

$$y_i = y_j \text{ for all } i, j \in N_t \cap S \text{ and } t = 1, \ldots, k.$$  

(4.4)

The game $V$ is per-capita bounded with respect to $\{N_t\}_{t=1}^k$ iff

there is a positive real number $\delta$ ($0 < \delta < 1$) and a constant $K$ such that

$$S \in F \quad \Rightarrow \quad (1+\delta) \frac{\mu(N_t)}{\mu(N)} \geq \frac{|S \cap N_t|}{|S|} \geq (1-\delta) \frac{\mu(N_t)}{\mu(N)} \quad t = 1, \ldots, k \quad \Rightarrow \quad x_i < K$$

for all $i \in S$;

(4.5)

$x \in V(S)$ has the ETP

that is, there is a constant $K$ such that given any coalition $S$ with

approximately the same percentage of players of each type as $N$ and any

payoff $x$ with the ETP in $V(S)$, we have $x_i < K$ for all $i \in S$.

Theorem (Kaneko and Wooders [1984, Theorem 2]). Let $V$ be a game without

side payments. Assume $V$ is per-capita bounded and has the

r-property with respect to a partition $\{N_t\}_{t=1}^k$.

Then the $f$-core of the game is nonempty. Furthermore, there is an

outcome $h$ with the ETP in the $f$-core, i.e., $h(i) = h(j)$ if $i, j \in N_t$

($t = 1, \ldots, k$).

Two steps of the proof of this theorem are relevant. The first

one is as follows. It is proved that there is a sequence $\{(N^\lambda, V^\lambda)\}_{\lambda=1}^\infty$

such that

$$N^\lambda \subset N^{\lambda+1} \subset \ldots \subset N \text{ and } \mu(N^\lambda) \to \mu(N) \text{ (} \lambda \to \infty \text{)};$$

(4.6)
\( \mu(N^\lambda \cap N_t) \) is a positive rational number for \( t=1,\ldots,k \); \hspace{1cm} (4.7)

\( V^\lambda \) is the restriction of \( V \) to \( \{ S \in P : S \subseteq N^\lambda \} \) \hspace{1cm} (4.8)

for all \( \lambda=1,\ldots; \)

each \( (N^\lambda,V^\lambda) \) has an outcome \( h^\lambda \) with the ETP in its \hspace{1cm} (4.9)

\( f \)-core and \( h^\lambda \) converges uniformly to an outcome \( h \)
in the \( f \)-core of \( (N,V) \).

Pick \( (N^\lambda,V^\lambda) \) arbitrarily from the sequence \( \{(N^\lambda,V^\lambda)\} \). Then it \hspace{1cm} \hspace{1cm}

is proved that the game \( (N^\lambda,V^\lambda) \) can be approximated by a sequence

\( \{(A_r,V_r)\}_{r=1}^\infty \) of finite replica games in the following sense:

\[
A_r = \bigcup_{t=1}^k \{(t,1),\ldots,(t,m_{r,t})\} \text{ for all } r,
\] \hspace{1cm} (4.10)

\((m_1,\ldots,m_k)\) is a vector of positive integers with

\[
\frac{m_t}{\sum_{s=1}^k m_s} = \frac{\mu(N^\lambda \cap N_t)}{\mu(N^\lambda)} \text{ for all } t=1,\ldots,k;
\]

there is a 1 to 1 mapping \( g \) from \( \bigcup_{t=1}^k \{(t,1),(t,2)\ldots\} \)

\hspace{1cm} (4.11)

to \( N^\lambda \) with \( g(t,s) \in N^\lambda \cap N_t \) for \( t=1,\ldots,k \), and \hspace{1cm}

\( k=1,2,\ldots \), and \( V_r \) is defined by \( V_r(S) = V(g(S)) \) for \hspace{1cm}

all \( S \subseteq A_r \) and for all \( r \);
there is a sequence \( \{x(r)\}_{r=1}^{\infty} \) in \( \mathbb{R}^k \) such that \( \Pi_{t=1}^{r_k} x_t(r) = \Pi_{t=1}^{r_m} \left[ x_t(r) \times \ldots \times x_t(r) \right] \in \mathcal{V}_r(A_r) \) \( (i) \).

(ii) for any \( \epsilon > 0 \), there is an \( r_0 \) such that for any \( r \geq r_0 \) no coalition \( S \subseteq A_r \) has \( y \in \mathcal{V}_r(S) \) with \( y_i > x_t(r) + \epsilon \) for all \( i \in S \cap [t]_r \) and \( x_t(r) \to h^\lambda(i) (r \to \infty) \) if \( i \in N^\lambda_t \) and \( t = 1, \ldots, k \).

(iii) \( x_t(r) \to h^\lambda(i) (r \to \infty) \) if \( i \in N^\lambda_t \) and \( t = 1, \ldots, k \).

Here \( [t]_r = \{(t,1), \ldots, (t,r_m)\} \) for \( t = 1, \ldots, k \).

4.2 Approximation of the Game \( \mathcal{V} \) by Replica Games

In the following, let \((N,\mathcal{V})\) be a game satisfying the conditions of the theorem.

Let \( \{\delta^\nu\} \) be a decreasing sequence of positive numbers with \( \lim \delta^\nu = 0 \). Since the attribute space \( E \) is a compact metric space, for each \( \nu \) there is a finite covering \( \{\mathcal{U}^\nu(a_t^\nu)\}_{t=1}^{l^\nu} \) of \( E \) such that \( \mathcal{U}^\nu(a_t^\nu) \) is the \( \delta^\nu \)-ball centered at \( a_t^\nu \) for \( t = 1, \ldots, l^\nu \). Define \( \mathcal{U}^\nu(a_t^\nu) \) \( (t = 1, \ldots, l^\nu) \) inductively by

\[
\mathcal{U}^\nu(a_1^\nu) = \mathcal{U}^\nu(a_1^\nu) \quad \text{and}
\]

\[
\mathcal{U}^\nu(a_t^\nu) = \mathcal{U}^\nu(a_t^\nu) - \bigcup_{k=1}^{t-1} \mathcal{U}^\nu(a_k^\nu) \quad \text{for} \ t = 2, \ldots, l^\nu.
\]

Then \( \{\mathcal{U}^\nu(a_t^\nu)\}_{t=1}^{l^\nu} \) is a partition of \( E \) and also satisfies

\[
\mathcal{U}^\nu(a_t^\nu) \subset \mathcal{U}^\nu(a_t^\nu) \quad \text{for all} \ t = 1, \ldots, l^\nu.
\] (4.13)

For the purpose of the following arguments we can replace \( a_t^\nu \) by
any $\beta \in U^v(\alpha^v_t)$ if $\alpha^v_t \notin U^v(\alpha^v_t)$ so we can assume that $\alpha^v_t \in U^v(\alpha^v_t)$ for all $t = 1, \ldots, l_v$.

For each $v$, we define the $v$th player set $N^v$ by

$$N^v = \bigcup_{\mu(\gamma^{-1}(U^v(\alpha^v_t))) > 0} \gamma^{-1}(U^v(\alpha^v_t)).$$

Then it is clear that

$$\mu(N^v) = \mu(N) \text{ for all } v = 1, 2, \ldots. \quad (4.14)$$

Without loss of generality, we can assume that $\mu(\gamma^{-1}(U^v(\alpha^v_t))) > 0$ for all $t = 1, \ldots, l_v$.

Define the $v$th attribute function $\gamma^v: N^v \to E$ by

$$\gamma^v(i) = \alpha^v_t \text{ if } i \in U^v(\alpha^v_t). \quad (4.15)$$

We now have a sequence of games $\{(N^v, \gamma^v)\}_{v=1}^{\infty}$, where $V^v$ is the game derived from $V^*$ and $\gamma^v$ for all $v = 1, \ldots$. Each game $(N^v, \gamma^v)$ satisfies all the assumptions of the Theorem of Subsection 4.1. Therefore there exists a sequence $\{h^v\}$ such that each $h^v$ is an outcome with the ETP in the $f$-core of the game $(N^v, \gamma^v)$, i.e.,

$$\gamma^v(i) = \gamma^v(j) \Rightarrow h^v(i) = h^v(j). \quad (4.16)$$

Let $N = \bigcap_{v=1}^{\infty} N^v$. Then, of course, $\mu(N) = \mu(\bigcap_{v=1}^{\infty} N^v) = \mu(N)$ by (4.14).

We regard each outcome $h^v$ as a function in $L(N^\infty, R)$. We can now prove

**Lemma 4.1** The sequence $\{h^v\}$ satisfies

$$\forall \varepsilon > 0 \exists \delta > 0 \forall i, j \in N^\infty \forall v: d(\gamma^v(i), \gamma^v(j)) < \delta \Rightarrow |h^v(i) - h^v(j)| < \varepsilon. \quad (4.17)$$
Proof. On the contrary, suppose that the sequence \( \{h^v\} \) does not satisfy (4.17). Then we can assume without loss of generality that there are sequences \( \{i_v\} \) and \( \{j_v\} \) in \( \mathbb{N}^\infty \) such that

\[
d(v^v(i_v), v^v(j_v)) \to 0 \quad (v \to \infty) ; \quad \text{and} \quad \tag{4.18}
\]

for some \( c > 0 \), \( h^v(i_v) - h^v(j_v) > c \) for all \( v \). \quad \tag{4.19}

Let \( \varepsilon = \frac{bc}{15n} \) be fixed, where \( b \) is a positive number given by strong comprehensiveness (2.19). We can assume that \( 0 < b < 1 \) without loss of generality. Since the pregame \( V^\star \) is uniformly continuous by Remark 2.3, it follows from (4.18) that there is a number \( v_o \) such that for any \( (a_1, \ldots, a_{t-1}) \in \varepsilon^{t-1} \quad (t=2, \ldots, n) \),

\[
d_H(V^*(a_1, \ldots, a_{t-1}, v^v(i_v)), V^*(a_1, \ldots, a_{t-1}, v^v(j_v))) < \varepsilon \quad \tag{4.20}
\]

for all \( v \geq v_o \).

Let \( v(> v_o) \) be fixed in the following. As was mentioned in Subsection 4.1, the game \( (N^v, V^v) \) can be approximated by a sequence \( \{\{N^\lambda, V^\lambda\}\}_{\lambda=1}^\infty \) in the sense of (4.6)-(4.9). Let \( \{h^\lambda\}_{\lambda=1}^\infty \) be a sequence such that each \( h^\lambda \) is an outcome with the ETP in the f-core of \( (N^\lambda, V^\lambda) \) for \( \lambda=1,2,\ldots \) and \( h^\lambda \) converges uniformly to \( h^v \). Thus there is a \( \lambda_o \) such that for all \( \lambda \geq \lambda_o \)

\[
|h^\lambda(i) - h^v(i)| < \varepsilon \quad \text{for all} \quad i \in N^\lambda. \quad \tag{4.21}
\]

Let \( \lambda(> \lambda_o) \) be fixed. As was mentioned in Subsection 4.1, the game \( (N^\lambda, V^\lambda) \) can be approximated by a sequence of finite replica games
In the sense of (4.10)-(4.12). For \( h^V \), let \( \{ x(r) \}_{r=1}^\infty \) be a sequence in \( R^k \) as specified in (4.12) (recall that \( k \) is the number of \( \delta^V \)-balls). Since \( x_t(x) + h^V(i) (r \to \infty) \) for \( i \in N^V \cap N_t \) and \( t=1, \ldots, k \), there is an \( r_o \) such that for all \( r \geq r_o \),

\[
|h^V(i) - x_t(r)| < \varepsilon \quad \text{for} \quad i \in N^V \cap N_t \quad \text{and} \quad t=1, \ldots, k.
\]

(4.22)

Let \( r(\geq r_o) \) be fixed. Since the sizes of essential coalitions are bounded by \( n \), definition (2.22) and condition (4.12) imply that there is an \( S \subset A_r \) with \( |S| \leq n \) and an \( x_S = \{ x(t,q) \}_{(t,q) \in S} \in V_r(S) \) such that

\[
x(t,q) = x_t(r) \quad \text{for all} \quad (t,q) \in S; \quad \text{and}
\]

(4.23)

\[
\gamma^V(g(\bar{r},q)) = \gamma^V(i_v) \quad \text{for some} \quad (\bar{t},\bar{q}) \in S.
\]

(4.24)

Let \( (\bar{t},\bar{q}) \in A_r \) such that \( (\bar{t},\bar{q}) \notin S \) and \( \gamma^V(g(\bar{t},\bar{q})) = \gamma^V(j_v) \).

This choice of a \( (\bar{t},\bar{r}) \) is possible when \( r_o \) is large. Put

\[
T = (S - \{(\bar{t},\bar{q})\}) \cup \{(\bar{t},\bar{q})\}.
\]

Since \( V_r \) is derived from \( V^* \), \( \gamma^V \) and \( g \), it follows from (4.20) and (4.23) that there is a \( y \in V_r(T) \) such that

\[
|x(t,q) - y(t,q)| < \varepsilon \quad \text{for all} \quad (t,q) \in S \cap T
\]

(4.25)

\[
|x(\bar{t},\bar{q}) - y(\bar{t},\bar{q})| < \varepsilon.
\]

Define \( z_T = \{ z(t,q) \}_{(t,q) \in T} \) by
\[ z(t,q) = \begin{cases} 
  y(t,q) - \frac{c}{3} & \text{if } (t,q) = (\bar{t},\bar{q}) \\
  y(t,q) + \frac{bc}{3n} & \text{if } (t,q) \in S \cap T .
\end{cases} \tag{4.26} \]

Then it follows from strong comprehensiveness (2.19) that

\[ z_T \in V_T (T) . \tag{4.27} \]

We now compare \( z_T \) with \( x_S \), given by (4.23) and (4.24).

For an arbitrary \((t,q) \in S \cap T\),

\[ z(t,q) = y(t,q) + \frac{bc}{3n} \]

\[ > x(t,q) + \frac{bc}{3n} - \varepsilon \quad \text{(by (4.25))} \]

\[ > x(t,q) + \frac{bc}{3n} - \frac{bc}{15n} > x(t,q) + \varepsilon , \]

and for \((\bar{t},\bar{q})\),

\[ z(\bar{t},\bar{q}) = y(\bar{t},\bar{q}) - \frac{c}{3} \]

\[ > x(\bar{t},\bar{q}) - \frac{c}{3} - \varepsilon \quad \text{(by (4.25))} \]

\[ > h^\lambda (i_v) - \frac{c}{3} - 2\varepsilon \quad \text{(by (4.22))} \]

\[ > h^\nu (i_v) - \frac{c}{3} - 3\varepsilon \quad \text{(by (4.21))} \]

\[ > h^\nu (j_v) + \frac{2c}{3} - 3\varepsilon \quad \text{(by (4.19))} \]

\[ > h^\nu (j_v) + \frac{2c}{3} - 4\varepsilon \quad \text{(by (4.21))} \]
\[ x_{(t,q)} + \frac{2c}{3} - 5\varepsilon \text{ (by (4.22))} \]

\[ = x_{(t,q)} + \frac{2c}{3} - \frac{bc}{3n} \geq x_{(t,q)} + \frac{c}{3} \]

\[ \geq x_{(t,q)} + \varepsilon \text{ .} \]

This together with (4.27) means that the coalition \( T \) can \( \varepsilon \)-improve upon \( x_S \); that is,

\[ z(t,q) > x(t,q) + \varepsilon = x_{t}(r) + \varepsilon \text{ (by (4.23))} \]

for all \( (t,q) \in T \).

Since \( r(r_r) \) is arbitrary, this is a contradiction to (4.12). Therefore we have proved Lemma 4.1.

\[ \text{Q.E.D.} \]

4.3 The Limit of \( f \)-Core Outcomes in Replica Games

The next step is to obtain a limit of the functions \( \{ h^v \} \) defined above, using the preceding Lemma. For any \( \delta > 0 \), there is a \( v_o \) such that \( \delta_v < \delta/3 \) for all \( v \geq v_o \), which implies that for all \( v \geq v_o \),

\[ d(\gamma(i), a^v_t) < \delta/3 \text{ for all } \gamma(i) \in U^v(a^v_t) \text{ (t=1,...,l_v)} \text{ .} \]

Then it follows that for all \( v \geq v_o \),
\[ d(\gamma(i), \gamma(j)) < \delta/3 \Rightarrow d(\gamma^v(i), \gamma^v(j)) < \delta. \] (4.28)

Indeed, 
\[ d(\gamma^v(i), \gamma^v(j)) \leq d(\gamma^v(i), \gamma(i)) + d(\gamma(i), \gamma(j)) + d(\gamma(j), \gamma^v(j)) \leq 3 \times \delta/3 = \delta. \]

From (4.28) and Lemma 4.1, we have

for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) and a \( v_0 \) such that

\[ v \geq v_0 \text{ and } d(\gamma(i), \gamma(j)) < \delta \Rightarrow |h^v(i) - h^v(j)| < \varepsilon. \] (4.29)

Then the following Lemma holds.

**Lemma 4.2** Let \( \{h^v\} \) be a sequence of real-valued function on \( N^\infty \) such that \( \{h^v\} \) is uniformly bounded and satisfies condition (4.29) and for all \( v \), \( \gamma(i) = \gamma(j) \Rightarrow h^v(i) = h^v(j) \). Then there is a uniformly convergent subsequence \( \{h^{v^\lambda}\} \) of \( \{h^v\} \).

**Proof.** Appendix.

Without loss of generality, we can assume that the sequence \( \{h^v\} \) itself converges uniformly to the function \( h^* \) on \( N^\infty \). Let \( h^{**} \) be the function defined by

\[ h^{**}(i) = \begin{cases} h^*(i) & \text{if } i \in N^\infty \\ K & \text{otherwise} \end{cases} \] (4.30)

where \( K \) is the upper bound given by Lemma 2.2. Now we will prove that the function \( h^{**} \) belongs to the f-core of the original game \( (N, V) \).
Claim 1. No coalition $S \in P_n$ can improve upon $h^{**}$.

Proof. On the contrary, suppose that some $S \in P_n$ can improve upon $h^{**}$ with $y \in V(S)$. Then $S \subset N^\infty$ by (4.30). Since the radius $\delta^V$ of $U^V(a^V_t)$ tends to 0 uniformly by (4.1), $d(y(i), y^V(i)) \to 0$ for all $i \in S$. Therefore it follows from continuity (2.18) and Remark 2.3 that there is a $\nu_1$ such that for any $\nu \geq \nu_1$, $V^V(S)$ has a vector $\tilde{y}$ with the property:

$$\tilde{y}_i > \frac{y_i + h^*(i)}{2} \quad \text{for all } i \in S.$$

Since $h^V$ converges uniformly to $h^*$ on $N^\infty$, there is a $\nu_2$ such that for all $\nu \geq \nu_2$,

$$\frac{y_i + h^*(i)}{2} > h^V(i) \quad \text{for all } i \in S.$$

Therefore we have, for any $\nu \geq \max(\nu_1, \nu_2)$,

$$\tilde{y}_i > h^V(i) \quad \text{for all } i \in S.$$

This is a contradiction to the fact that $h^V$ is in the f-core of $(N^V, V^V)$.

Claim 2. The function $h^{**}$ is an outcome of the game $(N, V)$, i.e., $h^{**} \in H^*$.

Proof. Since $\mu(N) = \mu(N^\infty)$, it suffices to show that $h^*$ is an outcome of the game $(N^\infty, V)$. 
Since $h^\nu$ is in the $f$-core of $(N^\infty, h^\nu)$, there is a sequence 
\[ \{ h^\nu_{\lambda} \}_{\lambda=1}^\infty \] such that $h^\nu_{\lambda} \in H^\nu(\mathbb{P}^\nu)$ for all $\lambda = 1, \ldots$ and \{h^\nu_{\lambda}\} converges in measure to $h^\nu$. Hence for each $\nu$, there is a $\lambda_{\nu}$ such that for any $\lambda > \lambda_{\nu}$,
\[ \mu(\{ i \in N^\infty : |h^\nu_{\lambda}(i) - h^\nu(i)| > \frac{1}{2^\nu} \}) \leq \frac{1}{2^\nu} \] (4.31)

Since the number $\lambda_{\nu}$ is determined for each $\nu$, we have the "diagonal" sequence $\{g^\nu\} = \{h^\nu_{\lambda_{\nu}}\}$. Since $\{h^\nu\}$ converges uniformly to $h^*$ on $N^\infty$, it converges also in measure to $h^*$, i.e., for any $\varepsilon > 0$,
\[ \mu(\{ i \in N^\infty : |h^\nu(i) - h^*(i)| > \varepsilon \}) \to 0 \quad (\nu \to \infty). \] (4.32)

It follows from (4.31) and (4.32) that for any $\varepsilon > 0$ and any $\nu$ with $\frac{1}{2^\nu} < \varepsilon/2$,
\[ \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^*(i)| > \varepsilon \}) \]
\[ \leq \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^\nu(i)| + |h^\nu(i) - h^*(i)| > \varepsilon \}) \]
\[ \leq \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^\nu(i)| > \frac{\varepsilon}{2} \text{ or } |h^\nu(i) - h^*(i)| > \frac{\varepsilon}{2} \}) \]
\[ \leq \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^\nu(i)| > \frac{\varepsilon}{2} \}) \]
\[ + \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^*(i)| > \frac{\varepsilon}{2} \}) \]
\[ \leq \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^\nu(i)| > \frac{1}{2^\nu} \}) \]
\[ + \mu(\{ i \in N^\infty : |h^\nu_{\lambda_{\nu}}(i) - h^*(i)| > \frac{\varepsilon}{2} \}) \]
\[
\frac{1}{2^\nu} + \mu(\{ i \in \mathbb{N}^\infty : | h^\nu(i) - h^\nu(i) | > \frac{\epsilon}{2} \})
\]
\[
\to 0 \quad (\nu \to \infty).
\]

That is, the sequence \( \{ h^\nu \} \) converges in measure to \( h^* \). For
notational simplicity, we let the diagonal sequence \( \{ p^\nu \} \) be
represented as \( \{ p^\nu \} = \{ h^\nu \} \).

Finally we prove that there is a sequence \( \{ g^\nu_s \}_{s=1}^{\infty} \)
such that \( g^\nu_s \in H(p^\nu_s \nu_s) \) for all \( s = 1, 2, \ldots \) and \( g^\nu_s \)
converges in measure to \( h^{**} \). Let \( \{ \epsilon^s \} \) be a positive numbers with
\[
\lim_{s} \epsilon^s = 0.
\]
Pick an arbitrary \( \epsilon^s \). Define \( g^\nu_s \) by
\[
g^\nu_s(i) = p^\nu(i) - \epsilon^s \quad \text{for all} \quad i \in \mathbb{N}^\infty.
\]

Since \( \{ U^t(a^\nu_t) \}_{t=1}^{\infty} \) are \( \delta^\nu \)-balls and \( \delta^\nu \to 0 \), by the uniform continuity
of \( \nu^* \) (Remark 2.3), there is a \( \nu_s \) such that for all \( \nu \geq \nu_s \)
\[
x \in \nu^\nu(S) \quad \text{and} \quad s \in F_n \iff (x_i - \epsilon^s)_{i \in S} \in \nu^\nu(S).
\]

It follows from (4.33) that for all \( \nu \geq \nu_s \),
\[
\text{if} \quad p^\nu(i) \in F_n \quad \text{and if} \quad (q^\nu(j))_{j \in p^\nu(i)} \in \nu^\nu(p^\nu(i)) \quad \text{then} \quad (g^\nu_s(j))_{j \in p^\nu(i)} \in \nu^\nu(p^\nu(i)).
\]

If \( p^\nu(i) \notin F_n \), then \( \nu^\nu(p^\nu(i)) \) is the set of payoffs attained by
partitions of \( p^\nu(i) \) into members of \( F_n \), i.e. (2.21). This implies
that the above argument can be applied to \( p^\nu(i) \notin F_n \), and we have, for
all \( \nu \geq \nu_s \),
if \((g^v(j))\in_{p^v(i)}v^v(p^v(i))\), then \((g^v_s(j))\in_{p^v(i)}v(p^v(i))\) \hspace{1cm} (4.34')

Since \((g^v(j))\in_{p^v(i)}v^v(p^v(i))\) for almost all \(i\) in \(\mathbb{N}^\infty\),
it follows from (4.34') that for all \(v \geq v_s\)

\((g^v_s(j))\in_{p^v(i)}v(p^v(i))\) for almost all \(i\) in \(\mathbb{N}^\infty\), \hspace{1cm} (4.35)

i.e., \(g^v_s \in H(p^v)\).

Since we can assume that \(v_s \to \infty (s \to \infty)\), by definition the
sequence \(\{g^v_s\}\) converges in measure to \(h^*\). This, together with
(4.35), implies that \(h^*\) is an outcome of the game \((\mathbb{N}^\infty,v)\).

We now have shown that \(h^{**}\) is an outcome of the game \((\mathbb{N},v)\);
this completes the proof of the Theorem.

Q.E.D.
Appendix

Proof of Lemma 2.2. Let \( t \) and \( k \) be fixed with \( 1 \leq t \leq n \) and \( 1 \leq k \leq t \). Define a function \( X_k(a) = X_k(a_1, \ldots, a_t) \) on \( E^t \) by

\[
X_k(a) = \max(x_k; x \in V^*(a)) \quad \text{and} \quad x_k \geq \max V^*(a_k)
\]

for all \( k = 1, \ldots, t \) for all \( a \in E^t \).

From (2.12) and (2.16), this function \( X_k(a) \) is well defined.

Since \( E \) (and \( E^t \)) is a compact space, it suffices to prove that \( X_k(a) \) is a continuous function of \( a \). Let \( \{a^\nu\} \) be a convergent sequence in \( E^t \) with \( \lim_{\nu} a^\nu = a^o \). Let \( X_k(a^\nu) = x_k^\nu \) and \( x^\nu \in V^*(a^\nu) \) for all \( \nu \). Since \( V^* \) is continuous by (2.18), it follows that for any \( \varepsilon > 0 \), there is a \( \nu_0 \) such that for each \( \nu \geq \nu_0 \), there is a \( \nu_1 \) such that for each \( \nu \geq \nu_1 \), there is a \( x^\nu \in V^*(a^o) \) with \( ||x^\nu - y^\nu|| < \varepsilon \). This implies that \( \lim_{\nu} X_k(a^\nu) \leq X_k(a^o) \). Conversely, let \( X_k(a^o) = x_k^o \) and \( a^o \in V^*(a^o) \). Then for any \( \varepsilon > 0 \), there is a \( \nu_1 \) such that for each \( \nu > \nu_1 \), some \( z^\nu \in V^*(a^\nu) \) satisfies \( ||x^o - z^\nu|| < \varepsilon \) by (2.18). This implies that \( \lim_{\nu} X_k(a^\nu) \geq X_k(a^o) \). Therefore \( \lim_{\nu} X_k(a^\nu) = X_k(a^o) \).

Q.E.D.
Proof of Lemma 2.3. Let \( N_1, \ldots, N_t \) be disjoint subsets of \( N \) such that there are measure-preserving isomorphisms \( \psi_1, \ldots, \psi_t \) from \( N_1 \) to \( N_1, \ldots, N_t \) respectively. We will prove that \( V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \) has a measurable graph in \( N_1 \times R^t \).

First we show that the set \( \{ a \in E^t : x \in V^*(a) \} \) is a closed subset of \( R^t \) for any \( x \in R^t \). On the contrary, suppose that \( a^v \rightarrow a^0 \), \( x \in V^*(a^v) \) for all \( v \) and \( x \notin V^*(a^0) \). Then since \( V^*(a^0) \) is closed in \( R^t \) by (2.2), it holds that \( d^t(x, V^*(a^0)) > 0 \). However \( x \in V^*(a^v) \) implies \( d^t_H(V^*(a^v), V^*(a^0)) \geq d(x, V^*(a^0)) > 0 \). This is a contradiction to continuity (2.18) of \( V^* \).

Since \( \gamma \) is Borel-measurable, the set \( \{ i \in N_1 : (\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \in \{ a \in E^t : x \in V^*(a) \} \} \) is a measurable set for any \( x \in R^t \). This set can be represented as

\[
\{ i \in N_1 : x \in V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \}.
\]

Therefore the set \( \{ i \in N_1 : x \notin V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \} = N_1 - \{ i \in N_1 : x \in V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \} \) is also measurable.

Since the condition \( x \notin V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \) \((x \in R^t)\) is equivalent, by (2.12) and (2.15), to

\[
\text{for some vector } r \in R^t \text{ with rational components, } x \geq r
\]

and \( r \notin V^*(\gamma(\psi_1(i)), \ldots, \gamma(\psi_t(i))) \),

we have
\{(i, x) \in N_1 \times \mathbb{R}^t : x \notin V^*(\gamma(\psi(i)), \ldots, \gamma(\psi_t(i))) \}

= \bigcup_{r \in \mathbb{R}^t} \left[ \{i \in N_1 : x \notin V^*(\gamma(\psi(i)), \ldots, \gamma(\psi_t(i))) \} \times \{x \in \mathbb{R}^t : x \geq r\} \right].

This set is measurable in $N_1 \times \mathbb{R}^t$, which implies that the graph

\{(i, x) \in N_1 \times \mathbb{R}^t : x \notin V^*(\gamma(\psi(i)), \ldots, \gamma(\psi_t(i))) \}

is measurable in $N_1 \times \mathbb{R}^t$.

Let $p$ be a feasible partition of $N$, i.e., a partition satisfying (2.1). We will prove that for any $k \geq 1$, $V((\psi_{kl}^p(i)), \ldots, \psi_{kk}^p(i))$ has a measurable graph, where $\psi_{kl}^p, \ldots, \psi_{kk}^p$ are measure-preserving isomorphisms from $N_{kl}^p$ to $N_{kl}^p, \ldots, N_{kk}^p$. Let $K = \{1, \ldots, k\}$. For any $T = \{s_1, \ldots, s_t\} \subset K$ ($t \leq n$), we denote $\{\psi_{kl}^p(i), \ldots, \psi_{ks_t}^p(i)\}$ by $T(i)$. Then it follows from (2.22) that

$$V((\psi_{kl}^p(i)), \ldots, \psi_{kk}^p(i)) = \bigcup_{p \in P(K)} \bigcap_{T \in P_K} [V(T(i)) \times R^{K-T}],$$

where $P(K) = \{p_K : p_K$ is a partition of $K$ with $|T| \leq n$ for all $T \in p_K\}$. Since $V(T(i)) = V^*(\gamma(\psi_{kl}^p(i)), \ldots, \gamma(\psi_{ks_t}^p(i)))$ is shown to have a measurable graph in $N_{kl}^p \times \mathbb{R}^t$, the graph of $V((\psi_{kl}^p(i)), \ldots, \psi_{kk}^p(i))$, i.e.,
\[(i, x) \in N_{k,l}^p \times R^k : x \in \bigcup_{p \in P(K)} \bigcap_{T \in T_p} V(T(i)) \bigcap [V(T(i)) \times R^{k-T}] \bigcup \{i, x) \in N_{k,l}^p \times R^k : x \in \bigcup_{p \in P(K)} \bigcap_{T \in T_p} V(T(i)) \bigcap [V(T(i)) \times R^{k-T}] \}\]

is measurable in \( N_{k,l}^p \times R^k \).

Q.E.D.

Proof of Lemma 4.2. Put \( \Gamma = \gamma(N^\infty) \). Since \( E \) is a compact metric space and since \( \Gamma \) is subset of \( E \), \( \Gamma \) is a separable space. (See Royden [1963, p. 130, Proposition 6 and p. 163, Proposition 13]). That is, there is a countable subset \( \Gamma = \{\alpha_1, \alpha_2, \ldots\} \) of \( \Gamma \) such that the relative closure \( \bar{\Gamma} \) of \( \Gamma \) is \( \Gamma \) itself.

Since it holds that

\[ i, j \in \gamma^{-1}(a) \quad \text{and} \quad a \in \Gamma \implies h^\nu(i) = h^\nu(j) \]

we can define functions \( g^\nu \) on \( \Gamma \) by

\[ g^\nu(a) = h^\nu(i) \quad \text{for} \quad i \in \gamma^{-1}(a) \quad \text{and} \quad \nu = 1, \ldots. \quad (A.1) \]

Then \( g^\nu \) is a function on \( \Gamma \) to \( R \).

Step 1. Let us prove that there is a uniformly converging subsequence \( \{g^\nu \} \) of the sequence \( \{g^\nu \} \).

Consider the sequence \( \{g^\nu(a_1)\} \) of real numbers. Since for some \( K \), by uniform boundedness \( |g^\nu(a_1)| \leq K \) for all \( \nu \), \( \{g^\nu(a_1)\} \) has a convergent subsequence \( \{g^{\nu_v}(a_1)\} \). Similarly we can find a convergent
subsequence \( \{g^{2\nu}(a_2)\} \) of \( \{g^{1\nu}(a_2)\} \). Repeating this argument, we can construct a sequence of subsequences:

\[
\{g^{1\nu}(a_1)\} = \{g^{11}(a_1), g^{12}(a_1), \ldots\}
\]

\[
\{g^{2\nu}(a_2)\} = \{g^{21}(a_2), g^{22}(a_2), \ldots\}
\]

\[
\{g^{3\nu}(a_3)\} = \{g^{31}(a_3), g^{32}(a_3), \ldots\}
\]

..........................

..........................

These subsequences are chosen sequentially so that \( \{g^{k\nu}(a_k)\} \) is a convergent subsequence of \( \{g^{(k-1)\nu}(a_k)\} \) (\( k=2,3,\ldots \)).

Consider the "diagonal" subsequence \( \{f^{\nu}\} = \{g^{\nu\nu}\} \). Then it is clear by the construction of the sequences that at each point \( a_t \in \Gamma^0 \), the sequence \( \{f^{\nu}(a_t)\} \) is a convergent sequence of reals.

We now show that \( \{f^{\nu}\} \) is a convergent sequence of functions in the sup norm. Let \( \varepsilon > 0 \) be given. Condition (4.29) states that for some \( \delta > 0 \) and \( \nu_0 \),

\[
\nu \geq \nu_0 \quad \text{and} \quad d(\gamma(i), \gamma(j)) < \delta \Rightarrow |h^{\nu}(i) - h^{\nu}(j)| < \varepsilon/3 .
\]

This together with the definition of \( \{f^{\nu}\} \) implies that for some \( \nu_1 \),

\[
\nu \geq \nu_1 \quad \text{and} \quad d(a,a') < \delta \Rightarrow |f^{\nu}(a) - f^{\nu}(a')| < \varepsilon/3 . \tag{A.2}
\]

Consider the family \( \{U^\circ_\delta(a_t)\}_{t=1}^\infty \) of open balls in \( E \) with radius \( \delta \) centered on \( a_t \in \Gamma^0 \). Since \( \Gamma^0 \) is dense in \( \Gamma \), this family
\[ \{U_\delta^t(\alpha_t)\}_{t=1}^\infty \] is an open covering of \( \Gamma \). In fact, \( \{U_\delta^t(\alpha_t)\}_{t=1}^\infty \) forms an open covering of the closure \( \overline{\Gamma} \) of \( \Gamma \) relative to \( E \), because the radius \( \delta \) is uniform. Since \( \overline{\Gamma} \) is compact, the family \( \{U_\delta^t(\alpha_t)\}_{t=1}^\infty \) has a finite subcover, say \( \{U_\delta^t(\beta_{t})\}_{t=1}^k \), of \( \overline{\Gamma} \) (and, of course, of \( \Gamma \) itself).

Since \( \{f^\nu(\beta_{t})\} \) converges for \( t=1, \ldots, k \), there is a \( \nu_2 \) such that

\[ \nu, \lambda \geq \nu_2 \Rightarrow |f^\nu(\beta_{t}) - f^\lambda(\beta_{t})| < \frac{\varepsilon}{3} \text{ for all } t=1, \ldots, k. \] (A.3)

Let \( \nu_3 = \max(\nu_1, \nu_2) \). For each \( \alpha \in \overline{\Gamma} \), we can find a \( \beta_{t} \in \{\beta_1, \ldots, \beta_k\} \) with \( d(\alpha, \beta_{t}) < \delta \). Then it follows from (A.2) and (A.3) that for any \( \alpha \in \overline{\Gamma} \),

\[ \nu, \lambda \geq \nu_3 \Rightarrow |f^\nu(\alpha) - f^\lambda(\alpha)| \leq |f^\nu(\alpha) - f^\nu(\beta_{t})| \]

\[ + |f^\nu(\beta_{t}) - f^\lambda(\beta_{t})| \]

\[ + |f^\lambda(\beta_{t}) - f^\lambda(\alpha)| \leq 3 \times \frac{\varepsilon}{3} = \varepsilon. \] (A.4)

Therefore the sequence \( \{f^\nu\} \) is a Cauchy sequence in the sup norm.

For each point \( \alpha \in \overline{\Gamma} \), \( \{f^\nu(\alpha)\} \) is also a Cauchy sequence of real numbers, which implies that \( \{f^\nu(\alpha)\} \) converges to some real number \( f^*(\alpha) \). Having \( \lambda \to \infty \) in (A.4), we have,

there is a \( \nu \) such that for any \( \alpha \in \overline{\Gamma} \),

\[ \nu \geq \nu \Rightarrow |f^\nu(\alpha) - f^*(\alpha)| \leq \varepsilon. \] (A.5)

This means that the sequence \( \{f^\nu\} \) converges uniformly to \( f^* \).
Since the sequence \( \{ f^v \} \) is a subsequence of \( \{ g^v \} \), we can denote \( \{ f^v \} \) by \( \{ g^v\lambda \} \), and will consider the corresponding subsequence \( \{ h^v\lambda \} \) of \( \{ h^v \} \). Then this sequence \( \{ h^v\lambda \} \) converges uniformly to \( h^* = f^* \circ \gamma \) (i.e., \( h^*(i) = f^*(\gamma(i)) \) for all \( i \in \mathbb{N}^\infty \)). Indeed, it follows from (A.1) and (A.5) that for any \( \epsilon > 0 \), there is a \( \lambda \) such that for any \( i \in \mathbb{N}^\infty \) with \( \gamma(i) = a \),

\[
\lambda \geq \lambda \Rightarrow |h^v\lambda(i) - h^*(i)| = |g^v\lambda(a) - \lim_{\lambda} g^v\lambda(a)|
\]

\[
= |g^v\lambda(a) - f^*(a)| \leq \epsilon .
\]

Q.E.D.
References


Funaki, Y. and M. Kaneko, [1983], Economies with labor indivisibilities Part I - Optimal tax schedules, ISEP DP No. 200, University of Tsukuba.


Kaneko, M. and M.H. Wooders, [1984], The core of a game with a continuum of players and finite coalitions: The model and some results, IMA Preprint No. 111, Institute for Mathematics and its Applications, University of Minnesota.


<table>
<thead>
<tr>
<th>#</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>78</td>
<td>Abstracts for the Workshop on Bayesian Analysis in Economics and Game Theory</td>
<td></td>
</tr>
<tr>
<td>79</td>
<td>G. Chichilnisky, G.M. Heal</td>
<td>Existence of a Competitive Equilibrium in L and Sobolev Spaces</td>
</tr>
<tr>
<td>80</td>
<td>Thomas Selden</td>
<td>Time-dependent Solutions of a Nonlinear System in Semiconductory Theory, II: Boundedness and Periodicity</td>
</tr>
<tr>
<td>81</td>
<td>Yakar Kannan</td>
<td>Engaging in R&amp;D and the Emergence of Expected Non-convex Technologies</td>
</tr>
<tr>
<td>82</td>
<td>Herve Moulin</td>
<td>Choice Functions over a Finite Set: A Summary</td>
</tr>
<tr>
<td>83</td>
<td>Herve Moulin</td>
<td>Choosing from a Tournament</td>
</tr>
<tr>
<td>84</td>
<td>David Schmeidler</td>
<td>Subjective Probability and Expected Utility Without Additivity</td>
</tr>
<tr>
<td>85</td>
<td>I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan</td>
<td>The Numerical Computation of Invariant Circles of Maps</td>
</tr>
<tr>
<td>86</td>
<td>F. William Lawvere</td>
<td>State Categories, Closed Categories, and the Existence of Semi-Continuous Entropy Functions</td>
</tr>
<tr>
<td>87</td>
<td>F. William Lawvere</td>
<td>Functional Remarks on the General Concept of Chaos</td>
</tr>
<tr>
<td>88</td>
<td>Steven R. Williams</td>
<td>Necessary and Sufficient Conditions for the Existence of a Locally Stable Measure Process</td>
</tr>
<tr>
<td>89</td>
<td>Steven R. Williams</td>
<td>Implementing a Generic Smooth Function</td>
</tr>
<tr>
<td>90</td>
<td>Dhill Abreu</td>
<td>Infinitely Repeated Games with Discounting: A General Theory</td>
</tr>
<tr>
<td>91</td>
<td>J.S. Jordan</td>
<td>Instability in the Implementation of Walrasian Allocations</td>
</tr>
<tr>
<td>92</td>
<td>Myrna Holtz Wooders, William R. Zamo</td>
<td>Large Games: Fair and Stable Outcomes</td>
</tr>
<tr>
<td>93</td>
<td>J.L. Noakes</td>
<td>Critical Sets and Negative Bundles</td>
</tr>
<tr>
<td>94</td>
<td>Graciela Chichilnisky, Von Neumann-Morgenstern</td>
<td>Utilities and Cardinal Preferences</td>
</tr>
<tr>
<td>95</td>
<td>J.L. Ericksen</td>
<td>Twinning of Crystals</td>
</tr>
<tr>
<td>96</td>
<td>Anna Nagurney</td>
<td>On Some Market Equilibrium Theory Paradoxes</td>
</tr>
<tr>
<td>97</td>
<td>Anna Nagurney</td>
<td>Sensitivity Analysis for Market Equilibrium</td>
</tr>
<tr>
<td>98</td>
<td>Abstracts for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations</td>
<td></td>
</tr>
<tr>
<td>99</td>
<td>Millard Beatty</td>
<td>A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability</td>
</tr>
<tr>
<td>100</td>
<td>Filomena Pacella</td>
<td>Central Configurations of the N-Body Problem via the Equivariant Morse Theory</td>
</tr>
<tr>
<td>101</td>
<td>D. Carlson and A. Hoger</td>
<td>The Derivative of a Tensor-valued Function of a Tensor</td>
</tr>
<tr>
<td>102</td>
<td>Kenneth Mount</td>
<td>Privacy Preserving Correspondence</td>
</tr>
<tr>
<td>103</td>
<td>Millard Beatty</td>
<td>Finite Amplitude Vibrations of a Neo-hookean Oscillator</td>
</tr>
<tr>
<td>104</td>
<td>D. Emmons and N. Yannelis</td>
<td>On Perfectly Competitive Economies: Loeb Economies</td>
</tr>
<tr>
<td>105</td>
<td>E. Mascolo and R. Schianchi</td>
<td>Existence Theorems in the Calculus of Variations</td>
</tr>
<tr>
<td>106</td>
<td>D. Kinderlehrer</td>
<td>Twinning of Crystals (II)</td>
</tr>
<tr>
<td>107</td>
<td>R. Chen</td>
<td>Solutions of Minimax Problems Using Equivalent Differentiable Equations</td>
</tr>
<tr>
<td>108</td>
<td>D. Abreu, D. Pearce, and E. Stacchetti</td>
<td>Optimal Cartel Equilibria with Imperfect Monitoring</td>
</tr>
<tr>
<td>109</td>
<td>R. Lauterbach</td>
<td>Hopf Bifurcation from a Turning Point</td>
</tr>
<tr>
<td>110</td>
<td>C. Kahn</td>
<td>An Equilibrium Model of Quits under Optimal Contracting</td>
</tr>
<tr>
<td>111</td>
<td>M. Kaneko and Myrna Holtz Wooders</td>
<td>The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results</td>
</tr>
<tr>
<td>112</td>
<td>Halm Brezis</td>
<td>Remarks on Sublinear Equations</td>
</tr>
<tr>
<td>113</td>
<td>D. Carlson and A. Hoger</td>
<td>On the Derivatives of the Principal Invariants of a Second-order Tensor</td>
</tr>
<tr>
<td>114</td>
<td>Raymond Beneckere and Steve Pelican</td>
<td>Competitive Chaos</td>
</tr>
<tr>
<td>115</td>
<td>Abstracts for the Workshop on Homogenization and Effective Moduli of Materials and Media</td>
<td></td>
</tr>
<tr>
<td>116</td>
<td>Abstracts for the Workshop on the Classifying Spaces of Groups</td>
<td></td>
</tr>
<tr>
<td>117</td>
<td>Umberto Mosco</td>
<td>Pointwise Potential Estimates for Elliptic Obstacle Problems</td>
</tr>
<tr>
<td>118</td>
<td>Jose-Francisco Rodrigues</td>
<td>An Evolutionary Continuous Casting Problem of Stefan Type</td>
</tr>
<tr>
<td>119</td>
<td>C. Mueller and F. Weissler</td>
<td>Single Point Blow-up for a General Semilinear Heat</td>
</tr>
<tr>
<td>120</td>
<td>D.R.J. Chillingworth</td>
<td>Three Introductory Lectures on Differential Topology and its Applications</td>
</tr>
</tbody>
</table>