POSITIVE SOLUTIONS OF SHRÖDINGER EQUATIONS

BY

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POSITIVE SOLUTIONS OF SHRÖDINGER EQUATIONS

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1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator in R^n, where $V(x)$ is a real-valued function in $L_{p,\,loc}(R^n)$ with $p > n/2$. Then it is known (see [1], [5], [9]) that the equation $Hu = 0$ admits a positive solution in R^n if and only if $H \geq 0$, that is,

$$(1.1) \qquad \langle H\phi,\phi\rangle \equiv \int H\phi(x)\,\overline{\phi}(x)dx \geq 0 \quad \text{for all} \quad \phi \in C_0^\infty(R^n).$$

A natural question raised is how many and what kind of positive solutions exist.

The purpose of this paper is to show that for a Schrödinger operator with very short range potential there exists a positive solution having a prescribed behavior at infinity, and any positive solution is, except for one case, a constant multiple of it.

We assume the following conditions:

(VS) $V(x)$ is a real-valued function satisfying, for some $\rho > 2$,

$$\langle x\rangle^{\rho-n/p}\, V(x) \in L_p(R^n) \quad \text{for some} \quad p > n/2, \text{ if } \quad n > 1,$$
$$\langle x\rangle^{\rho-1}\, V(x) \in L_1(R^1) \quad \text{if} \quad n = 1,$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$.

(P) $H \geq 0$, i.e. (1.1) holds.

Note that by the condition (VS) every weak solution u of the equation $Hu = 0$ is locally Hölder continuous (see [10]). In what follows a solution u means a real-valued locally Hölder continuous function satisfying the equation $Hu = 0$ on R^n in the distribution sense.

Our main results are the following theorems.

Theorem 1.1. There exists a positive solution ψ or χ having the following asymptotic formula as $|x| \to \infty$:

$$(1.2)\qquad \psi(x) = \begin{cases} |x|^{2-n} + o\,(|x|^{2-n}) & (n \geq 2) \\ 1 + \mu x/|x| + o(1) \quad \text{for some constant } \mu & (n = 1)\,. \end{cases}$$

$$(1.3)\qquad \chi(x) = \begin{cases} 1 + o(1) & (n \geq 3) \\ \log |x| + O(1) & (n = 2) \\ |x| + o(|x|) & (n = 1). \end{cases}$$

Theorem 1.2. Both ψ and χ do not exist simultaneously.

Let us say that V is critical if ψ exists, and V is subcritical if χ exists.

Theorem 1.3. If $n \geq 2$ or V is critical, then any positive solution is a constant multiple of ψ or χ.

Theorem 1.4. If $n = 1$ and V is subcritical, then there exists another positive solution η such that

$$(1.4)\qquad \eta(x) = \begin{cases} ax + o(x) & \text{as } x \to \infty \\ b + o(1) & \text{as } x \to -\infty \end{cases}$$

for some positive constants a and b.

Remark 1.5. Theorems 1.1 and 1.2 are improvements of results in [6], where we investigated properties of positive solutions in connection to asymptotic behavior of the Schrödinger semigroup e^{-tH} as $t \to \infty$ under a condition stronger than (VS).

Remark 1.6. An interesting point of Theorem 1.3 is that this theorem asserts the "uniqueness" of positive solution even if V is subcritical. This type of uniqueness theorem does not hold, in general, for a potential decaying slower than $|x|^{-2}$ as $|x| \to \infty$. For example, the equation

$$(-\Delta + \langle x \rangle^{-\delta})u = 0 \quad \text{in} \quad R^n, \tag{1.5}$$

with $n \geq 2$ and $0 \leq \delta < 2$, admits infinitely many linearly independent positive solutions.

The rest of this paper is organized as follows. In section 2 we prove Theorems 1.1 and 1.4. In Section 3 we prove Theorems 1.2 and 1.3. In section 4 we show how to construct infinitely many positive solutions of the equation (1.5).

Extensive investigation of positive solutions of Schrödinger equations with potential decaying slower than or equal to $|x|^{-2}$ as $|x| \to \infty$ will be done elsewhere.

2. Existence

Lemma 2.1. (Corollary to Theorem C.8.1 in [9]) There exists a positive solution.

Lemma 2.2. Let u be a solution. If there exists $r > 0$ such that $u(x) > 0$ for x with $|x| \geq r$, then $u(x) > 0$ for all x in R^n.

Proof. The generalized maximum principle (see [8, p. 73, Theorem 10]) and Lemma 2.1 show that $u(x) > 0$ for $|x| < r$. Q.E.D.

For $1 \leq q \leq \infty$ and s in R^1, put

$$L_q^s = \{f \in L_{q,loc}(R^n) ; \langle x\rangle^s f(x) \in L_q(R^n)\}. \tag{2.1}$$

Let $K(x - y)$ be a fundamental solution of $-\Delta$: $K(x) = c_n|x|^{2-n}$ for a constant c_n when $n \neq 2$, and $K(x) = c_2 \log |x|$, $c_2 = -(2\pi)^{-1}$, when $n = 2$. Let G be an integral operator with kernel $K(x - y)$. Then elementary calculations yield:

Lemma 2.3. G is a bounded operator from L_p^s to X, where

$$s > 2 - n/p \text{ if } n \geq 2, \text{ and } s = 1 \text{ if } n = 1; \tag{2.2}$$

$$X = L_\infty^1 \text{ if } n = 1, \tag{2.3}$$

$$X = \{f \in L_{\infty,loc}; (\log\langle x\rangle + 1)^{-1}f(x) \in L_\infty\} \text{ if } n = 2,$$

$$X = L_\infty^r \text{ if } n \geq 3, \text{ with } r < s + n/p - 2 \text{ for } s \leq n/p'$$

$$\text{and } r = n - 2 \text{ for } s > n/p'.$$

Here p' is the conjugate of p: $1/p' + 1/p = 1$.

Lemma 2.4. There exists a positive continuous function u having the asymptotic formula as in (1.3) such that

$$Hu = f$$

for an L_p-function f with compact support.

Proof. For $R > 0$, define $V_R(x)$ by: $V_R(x) = V(x)$ for $|x| \geq R$ and $V_R(x) = 0$ for $|x| < R$. We can choose $R > 0$ so large that

$$\|GV_R\|_{X \to X} \leq 1/6,$$

where $X = L_\infty$ for $n \geq 3$ and X is the same Banach spaces as in (2.3) for $n \leq 2$. First, let $n \geq 3$. Solving the integral equation

$$u = 1 - GV_R u$$

by iteration method, we get a positive function u having the properies in the lemma. As for $n \leq 2$, we have only to solve the equation

$$u = G\phi - GV_R u,$$

where ϕ is a function in $C_0^\infty(R^n)$ such that $\phi \geq 0$ and $c_n \int \phi(y)dy = 1$.
Q.E.D.

Lemma 2.5. Suppose that there is no solution $u \neq 0$ in L_∞ such that $u(x) = o(1)$ as $|x| \to \infty$ if $n \geq 3$. Then there exists a bounded operator B from L_p^s to L_∞ with s satisfying (2.2), which has the following properties:

(2.4) $$H(Bf) = f;$$

(2.5) if $f \geq 0$, $f \neq 0$, and $f \in L_p^r$ for some $r > \max(n/p', 2 - n/p)$, then $Bf > 0$ and a constant multiple of Bf has the asymptotic formula as in (1.2).

Proof. When $n \geq 3$, we can construct B in the form $B = (-\Delta)^{-1}(1 + V(-\Delta)^{-1})^{-1}$ along the line given in the proof of Lemmas 3.3 and 3.5 in [7]. Let $n \leq 2$. Choose W in $C_0^\infty(R^n)$ so that $W \geq 0$ and $W \neq 0$. Put $L = -\Delta + W$. Then (5.47) and (5.55) in [7] show (see also Lemma 2.7 below) that there exists an operator C having the same properties as B in the lemma with H replaced by L. Thus B can be constructed in the form $B = C(1 + (V - W)C)^{-1}$. Q.E.D.

Lemma 2.6. Under the same assumption as in Lemma 2.5 there exists a positive solution χ having the asymptotic formula (1.3).

Proof. Let u be a function constructed in Lemma 2.4. By Lemma 2.5,

$$H(u - Bf) = 0$$

and $u - Bf$ has the same asymptotics as in (1.3). Thus Lemma 2.2 shows that $u \quad Bf > 0$. Putting $\chi = u - Bf$, we complete the proof of Lemma 2.6. Q.E.D.

It remains to show that there exists a positive solution ψ if there is a solution u in L_∞ such that $u(x) = o(1)$ as $|x| \to \infty$ if $n \geq 3$.

Lemma 2.7. Let W be a function such that $W \geq 0$, $W \neq 0$, and W satisfies the condition (VS) with V replaced by W. Then $H + W$ satisfies the hypothesis in Lemma 2.5.

Proof. Let u be a bounded solution of the equation $(H + W)u = 0$ such that $u(x) \to 0$ as $|x| \to \infty$ if $n \geq 3$. Choose a C_0^∞-function ϕ such that $\phi \geq 0$, $\phi(x) = 1$ for $|x| \leq 1$, and $\phi(x) = 0$ for $|x| \leq 2$. For $R > 0$, put $\phi_R(x) = \phi(x/R)$. First, let $n = 1$. Then we have that for some constant λ

$$u(x) = \lambda - \int c_1 |x - y|(V + W)u(y)dy, \tag{2.6}$$

with $\int(V + W)u(y)dy = 0$. This implies that $u'(x) = O(|x|^{1-\rho})$ as $|x| \to \infty$.

Thus we get

(2.7) $$\langle Hu,u\rangle = \lim_{R\to\infty} \langle H(\phi_R u),\phi_R u\rangle.$$

Hence $\langle Wu,u\rangle \leqslant \langle (H + W)u,u\rangle = 0$, which implies $Wu = 0$, and so $u = 0$. Next, let $n \geqslant 3$. With the same notation as in lemma 2.3 we have that

(2.8) $$u(x) = -c_n \int |x - y|^{2-n}(V + W)u(y)dy.$$

This together with Lemma 2.3 shows that $u \in L_\infty^{n-2}$ or $u \in L_\infty^{\varepsilon}$ for some ε such that $0 < \varepsilon < n - 2$ and $(n - 2)/\varepsilon$ is not an integer. If $u \in L_\infty^{\varepsilon}$, Lemma 2.3 and (2.8) imply that $u \in L_\infty^{r}$ with $r = \min(2\varepsilon, n - 2)$. Finite repetitions of this argument show that $u \in L_\infty^{n-2}$. This implies that $\nabla u \in L_2$. Hence we get (2.7), from which it follows that $u = 0$. Finally, let $n = 2$. We have that for some λ

(2.9) $$u(x) = \lambda - \int c_2 \log|x - y|(V + W)u(y)dy,$$

with $\int(V + W)u(y)dy = 0$. In order to show that $u = 0$ we have only to prove that

(2.10) $$\langle (H + W/2)u,u\rangle \geqslant 0.$$

First, assume that the equation $(H + W/2)f = 0$ has a bounded solution $f \neq 0$. Then (2.9) holds with u and W replaced by f and $W/2$. If $\lambda = 0$, then we obtain that $\langle Hf,f\rangle \geqslant 0$, which implies that $f = 0$. This is a contradiction. Thus $\lambda \neq 0$. In view of Lemma 2.2 we may assume that $f > 0$. Then Lemma 2.1 in [6] shows that (2.10). Next, assume that there exists no non-trivial bounded solution f of $(H + W/2)f = 0$. Then Lemma 2.6 asserts that there exists a positive solution χ of the equation $(H + W/2)\chi = 0$ having the asymptotic formula (1.3). Then Lemma 2.1 in [6] implies (2.10). Q.E.D.

The following lemma completes the proof of Theorem 1.1.

Lemma 2.8. Assume that there exists a bounded solution $g \neq 0$ such that $g(x) \to 0$ as $|x| \to \infty$ if $n \geq 3$. Then there exists a positive solution ψ having the asymptotic formula as in (1.2).

Proof. We may assume that $g > 0$ in a nonempty open set. Choose $W \in C_0^\infty$ such that $W \geq 0$, $Wg \geq 0$, and $Wg \neq 0$. By Lemmas 2.5 and 2.7, there exists a positive function f such that $(H + W)f = Wg$ and λf has the same asymptotics as in (1.2) for some λ. Since

$$(H + W)(g - f) = 0,$$

Lemma 2.7 shows that $g = f$. Thus we have only to put $\psi = \lambda g$. Q.E.D.

Proof of Theorem 1.4. If χw is a solution, then w satisfies the equation

$$w'' + (2\chi'/\chi)w' = 0. \tag{2.11}$$

Noting that $(\chi(0)/\chi(x))^2$ is integrable on R^1, we put

$$w(x) = \int_{-\infty}^{x} (\chi(0)/\chi(y))^2 dy.$$

This is a solution of (2.11) satisfying

$$\lim_{x \to \infty} w(x) = \int_{-\infty}^{\infty} (\chi(0)/\chi(y))^2 dy \equiv a,$$

$$\lim_{x \to -\infty} w(x)/|x| = \int_{-\infty}^{-1} (\chi(0)/y)^2 dy \equiv b.$$

Hence $\eta = \chi w$ is a positive solution satisfying (1.4). Q.E.D.

3. Uniqueness

Proof of Theorem 1.2. First, let $n \neq 2$. With the same notation as in Lemma 2.7 we see that

$$\lim_{R \to \infty} \int |\nabla \phi_R|^2 \psi^2 + |\nabla \phi_R| \; |\nabla \psi| \; \phi_R \psi \, dx = 0$$

and $|\nabla \psi| \in L_2$. Thus (2.7) holds with u replaced by ψ. On the other hand, we have (see [5, p. 224]) that

$$\langle H(\phi_R \psi), \phi_R \psi \rangle = \int |\nabla(\phi_R \psi / f)|^2 f^2 \, dx$$

for any positive solution f. Thus,

$$\langle H\psi, \psi \rangle = \int |\nabla(\psi/f)|^2 f^2 \, dx. \tag{3.1}$$

This implies that f is a constant multiple of ψ, which proves the theorem for $n \neq 2$. Next, let $n = 2$. If both ψ and χ exist, the same argument as in the proof of Lemma 2.1 of [6] shows that (3.1) holds with f replaced by χ. This is a contradiction. Q.E.D.

Actually, we have shown:

Lemma 3.1. Let $n \neq 2$, and assume that there exists a positive solution ψ having the asymptotic formula (1.2). Then any positive solution is a constant multiple of ψ.

It remains to prove Theorem 1.3 in the case that $n = 2$, or $n \geq 3$ and V is subcritical. In what follows we assume that $n \geq 2$.

Lemma 3.2. Let u be a positive solution and $v = \log u$. Then there is a positive constant C depending only on n such that

$$\int_{Q(x_0,r)} |\nabla v|^2 \, dx \leq C(1 + \|V\|_{n/2}) r^{n-2} \tag{3.2}$$

for any $r > 0$ and $x_0 \in R^n$, where $Q(x_0,r)$ is a cube with center x_0 and edge length r.

Proof. Choose a C^∞-function α such that $\alpha \equiv 1$ on $Q(x_0,r)$, $\alpha \equiv 0$ outside of $Q(x_0,2r)$, and $|\nabla\alpha| \leq 2/r$. Since $-\Delta v + V - |\nabla v|^2 = 0$ on R^n, we have

$$\begin{aligned}\int \alpha^2|\nabla v|^2 dx &= 2 \int \alpha\nabla v \cdot \nabla\alpha\, dx + \int \alpha^2 V\, dx \\ &\leq 2^{-1} \int \alpha^2|\nabla v|^2\, dx + 2 \int |\nabla\alpha|^2\, dx \\ &\quad + (\int \alpha^{2n/(n-2)} dx)^{1-2/n}\, \|V\|_{n/2}.\end{aligned}$$

This implies (3.2). Q.E.D.

Lemma 3.3. There exist a positive constant K depending only on n and

(3.3) $$\sup\, \{\|V\|_{L_p(Q(x_0,1))}\ ;\ x_0 \in R^n\}$$

and a positive integer k depending only on n and $\|V\|_{n/2}$ such that

(3.4) $$\max_{|x| \leq r} u \leq K\, u(0) r^k, \qquad r > 1,$$

for any positive solution u.

Proof. We can deduce from Lemma 3.2 along the line in the proof of Lemmas 8.2 and 8.3 of [10] the inequality

(3.5) $$\int_{Q(0,s)} u^\alpha dx \int_{Q(0,s)} u^{-\alpha} dx \leq \beta^2 s^{2n}, \qquad s > 0,$$

where α and β are positive constants depending only on n and $\|V\|_{n/2}$. On the other hand, Harnack's inequality ([10, Theorem 8.1]) asserts that for any $x_0 \in R^n$

(3.6) $$\max_{Q(x_0,1)} u \leq C \min_{Q(x_0,1)} u,$$

where C is a constant depending only on (3.3) and n. Put $s = 2r + 4$ and

$u(x_r) = \max\{u(x); |x| \leq r\}$. Then we have by (3.5) and (3.6) that

$$(u(x_r)/C)^{\alpha}(Cu(o))^{-\alpha} \leq \beta^2(2r + 4)^{2n}.$$

This implies (3.4). Q.E.D.

Lemma 3.4. ([4, Theorem 1.1]) Let q and s be real numbers such that $1 < q < \infty$, $s < n/q'$, and $s + n/q$ is not an integer. Then for any $f \in L_q^s$ there exists a solution g of the equation

$$\Delta g = f \quad \text{on} \quad R^n$$

such that $D^{\alpha} g \in L_q^{s - 2 + |\alpha|}$ for $|\alpha| \leq 2$.

Lemma 3.5. Let u be a positive solution. Then $u \in L_{\infty}$ if $n \geq 3$, and $u(x)(\log\langle x\rangle + 1)^{-1} \in L_{\infty}$ if $n = 2$.

Proof. We prove the lemma only for the case $n \geq 3$, since the proof for the other case is similar. Let $n \geq 3$. Choose q, ε, and δ so that $n/2 < q < \infty$, $\varepsilon > 0$, $2 - n/q < \delta < n/q'$, $j\varepsilon + \delta + nk/q$ $(j = 1,2,\ldots)$ are not integers, and

$$V \in L_q^{\varepsilon+\delta}.$$

By Lemma 3.3, $u \in L_{\infty}^{-k}$. If $-k + \varepsilon \geq 0$, then $Vu \in L_q^{\delta}$ with $\delta > 2 - n/q$. By Lemma 2.3, there exists a function $g \in L_{\infty}$ such that $\Delta g = Vu$. Since $\Delta u = Vu$, we obtain that $\Delta(u - g) = 0$, which implies that for a harmonic polynomial P

$$u = g + P. \tag{3.7}$$

Let $m = \deg P$. If $m > 0$, then there exists a nonempty open cone K with vertex zero such that

$$\lim_{R \to \infty} R^{-m} \int_{K_R} P(x)dx < 0, \quad K_R = \{x \in K;\ R < |x| < R + 1\}. \tag{3.8}$$

This together with (3.7) implies that $m = 0$, since $u > 0$ and $g \in L_\infty$. Hence $u \in L_\infty$, which proves the lemma in the case $-k + \varepsilon \geq 0$. Let $-k + \varepsilon < 0$. Since $Vu \in L_q^{-k+\varepsilon+\delta}$ with $-k + \varepsilon + \delta < n/q'$, Lemma 3.4 shows that there exists a solution g of $\Delta g = Vu$ such that $D^\alpha g \in L_q^{-k+\varepsilon+\delta-2+|\alpha|}$ for $|\alpha| \leq 2$. This implies that $\Delta(\langle x\rangle^{-k+\varepsilon}g(x)) \in L_q^\delta$ with $\delta > 2 - n/q$. Note that $\langle x\rangle^{-k+\varepsilon}g(x) \in L_q^{\delta-2}$ with $\delta - 2 > -n/q$, and apply the same argument as above to g instead of u. Then we obtain that $\langle x\rangle^{-k+\varepsilon}g(x) \in L_\infty$. Thus we get (3.7) with $g \in L_\infty^{-k+\varepsilon}$, which implies $u \in L_\infty^{-k+\varepsilon}$. Next we get $u \in L_\infty^r$ with $r = \min(-k + 2\varepsilon, 0)$, and so on. Finite repetitions of this argument show that $u \in L_\infty$. Q.E.D.

<u>Completion of the proof of Theorem 1.3</u>. We have already proved the theorem in the case that $n \neq 2$ and there exists a bounded solution g such that $g(x) \to 0$ as $|x| \to \infty$ if $n \geq 3$. First, let $n = 2$. Let u be a positive solution. By virtue of Lemma 3.5, u can be written as (2.9) with $V + W$ replaced by V. Thus

$$u(x) = \mu \log|x| + O(1) \quad \text{as} \quad |x| \to \infty,$$

where $\mu = -c_2 \int Vu(y)dy$. We first treat the case $\mu \neq 0$. Then the same argument as in the proof of Theorem 1.2 shows that there is no nontrivial bounded solution, which implies the existence of the positive solution χ. We have that $H(u - \mu\chi) = 0$ and $u - \mu\chi$ is bounded. Hence $u = \mu\chi$. Next, assume that $\mu = 0$. Then we get (3.1) with f replaced by u, which implies that u is a constant multiple of ψ. This proves the theorem in the case $n = 2$. Let $n \geq 3$, and assume that there exists no nontrivial solution which goes to zero as $|x| \to \infty$. By Lemma 3.5, a positive solution u can be written as

$$u(x) = \lambda - c_n \int |x - y|^{2-n} Vu(y)dy$$

for some constant λ. We see that $\lambda \neq 0$, since $u(x) \to 0$ as $|x| \to \infty$ if $\lambda = 0$. We have that $H(u - \lambda\chi) = 0$ and $(u - \lambda\chi)(x) \to 0$ as $|x| \to \infty$. Hence $u = \lambda\chi$.

Q.E.D.

4. Remark

In this section we show that the equation

$$(4.1) \qquad (-\Delta + \langle x\rangle^{-\delta})u = 0 \quad \text{on} \quad R^n$$

with $n \geq 2$ and $0 \leq \delta < 2$ admits infinitely many linearly independent positive solutions (see also [2] for the case $\delta = 0$).

We use a separation of variables technique in order to construct solutions. Let Y_{jk} $(k = 1,\ldots,N_j)$ be orthonormalized spherical harmonics of degree j. For $j = 0,1,\ldots,$ let v_j be the smooth solution of the equation

$$(4.2) \qquad v''(r) + (n - 1)r^{-1}v'(r) - (\langle r\rangle^{-\delta} + j(j + n - 2)r^{-2})v_j(r) = 0$$

on $[0,\infty)$ with initial condition $v_j(r) = r^j + O(r^{j+1})$ as $r \to 0$. We claim that $v_j(r) > 0$ for $r > 0$ and $v_j(r) \to \infty$ as $r \to \infty$. It is known (see [3, p. 320]) that the equation (4.2) has two solutions $w_\pm$ near infinity such that

$$(4.3) \qquad w_\pm(r) \sim f_j^{-1/4}(r) \exp \pm \int_1^r f_j^{1/2}(t)dt \quad \text{as} \quad r \to \infty ,$$

where $f_j(r) = \langle r\rangle^{-\delta} + j(j + n - 2)r^{-2}$. Thus concerning asymptotic behavior of $v_j(r)$ as $r \to \infty$ we might have three cases: (i) $v_j(r) \to \infty$; (ii) $v_j(r) \to -\infty$; and (iii) $v_j(r) \to 0$. Note that $v_j(r) > v_j(0) \geq 0$ for sufficiently small $r > 0$. Thus in the cases (ii) and (iii) v_j must attain the positive maximum at a point in $(0,\infty)$, which contradicts the maximum principle. Hence $v_j(r) \to \infty$ as $r \to \infty$, which together with the maximum principle implies that $v_j(r) > 0$ for $r > 0$. This proves the claim. Then we see from (4.3) that

$$(4.5) \qquad u_j(x) = v_0(|x|) + \sum_{j=1}^{\infty} \sum_{k=1}^{N_j} c_{jk}\, v_j(|x|)\, Y_{jk}(x/|x|)$$

is a positive solution of (4.1) if the constants c_{jk} are sufficiently small. This completes the proof.

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