A REMARK ABOUT THE FINAL APERIODIC REGIME FOR MAPS ON THE INTERVAL

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A Remark about the Final Aperiodic Regime for Maps on the Interval

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ABSTRACT We consider families of maps on the interval with one maximum, and prove the geometric convergence of the bifurcation parameter for the case of superstable periodic orbits converging towards the final aperiodic regime.

Introduction

Iterated maps on the interval are presently of great interest in many applied contexts such as hydrodynamics, electrical circuits, chemical reactors, Hamiltonian mechanics, and particle accelerators. For a general introduction to the subject see the excellent monograph [CE] by Collet and Eckman. Several recent papers in the field, [G], [DT], and [GN], have considered the approach to the final aperiodic regime for various families (f_α) of maps with one extremum, such as the logistic family with

$$f_\alpha(x) = 4\alpha x(1-x),\ x\in[0,1],\ \alpha\in[0,1].$$

Let α_n denote the final value of α with superstable period of length n, i.e. with the extreme point in the n-periodic orbit. For the logistic family, [G] noted that

$$f^2_{\alpha_{n+1}}(1/2) \approx f^2_{\alpha_n}(1/2)/4$$

$$\lim_{n\to\infty} (4-\delta)^n(1-\alpha_n) = 0 \text{ for } \delta > 0.$$

For families on [0,1]

$$f_\alpha = \alpha P + 1$$

with P analytic, [DT] obtained by formal Taylor expansion considerations that

$$\lim_{n\to\infty} (\alpha_n-\alpha_{n-1})/(\alpha_{n+1}-\alpha_n) = D_x f(1)$$

and [GN] obtained the same result numerically for analytic (f_α), promising calculations later. (The latter authors also combine this type of convergence with universal (Feigenbaum) behaviour to obtain a multitude of asymptotic ratios.)

The present paper considers families (f_α) of functions on an interval $[a,b]$, which are either concave and C^2, or C^3 with negative Schwarzian derivative. It contains a rigorous study of the above geometric convergence and its underlying mechanism, with as main result that

$$\lim_{n\to\infty} (\alpha_n-\alpha_\infty)/(\alpha_{n+1}-\alpha_\infty) = D_x f_{\alpha_\infty}(a).$$

In particular it thereby rigorously proves the above-mentioned statements.

Preliminaries

Let f be a function on an interval, which for the present study we take normalized to $[0,1]$,

$$f : [0,1] \to [0,1].$$

Assume that

$$f(x) = f(1-x),\ f(0) = 0,$$

with f strictly increasing (decreasing) on $[0,1/2]$ ($[1/2,1]$).

Let $(f_\alpha)^0_{-\beta}$ be a family of such functions, defined on some interval $-\beta < \alpha \le 0$, with f, $D_x f$ continuously differentiable with respect to x and α,

$$f_0(1/2) = 1,\ D_\alpha f_\alpha(1/2)|_{\alpha=0} > 0,$$

and with each f_α either concave in x, or in C^3 with Schwarzian derivative $Sf < 0$ on $[0,1]$.

If $D_x f_0(0) < 1$, then there are no superstable periods in x for α close to zero, so for the purpose of the present paper we require $D_x f_0(0) > 1$. (The borderline case $D_x f_0(0) = 1$ is not considered.)

These conditions on (f_α) imply the existence of a superstable orbit for some $\alpha<0$ (see e.g. [JR]). But we include a simple proof as part of this introduction of notations.

Use

$$D_\alpha f_\alpha(1/2)|_{\alpha=0} > 0 \ , \ D_x f_0(0) > 1, \ f_0(1/2) = 1,$$

to make the following simple observation; for some

$$a>1, \ b>1/2, \ c>0, \ x_o>0, \text{ and } \alpha_o<0$$

the function

$$f_\alpha(1/2) : [\alpha_o,0] \rightarrow [b,1]$$

is strictly increasing, and

(1) $\quad D_\alpha f_\alpha(1/2) > c, \ D_x f_\alpha(x) \geq a$ when $x\in[0,x_o], \ \alpha\in[\alpha_o,0]$.

For later use we also require that

$$f_\alpha(1/2) > 1 - x_o \quad \text{when } \alpha\in[\alpha_o,0],$$

and that

(2) $\quad D_x f_\alpha(x)/ D_x f_0(0) > 1/2$ when $x\in[0,x_o], \ \alpha\in[\alpha_o,0]$.

From $Sf_\alpha < 0$, or from the convexity, it now follows that

(3) $\quad f_\alpha(x) > x$ for $x\in(0,1/2], \ \alpha\in[\alpha_o,0]$.

With f_α^{-1} the inverse of

$$f_\alpha : [0,1/2] \rightarrow [0,1],$$

this implies that for some $\bar{k} > 0$

$$f_\alpha^{-\bar{k}}(1/2) < x_o \ \text{ for } \ \alpha\in[\alpha_o,0].$$

For some $\alpha_1\in[\alpha_o,0]$

$$f_\alpha^2(1/2) \in (0,1/2) \text{ when } \alpha\in[\alpha_1,0).$$

This follows since $f_0^2(1/2) = 0$. So by (3), for some $\bar{n} > 2$

(4) $\quad f_{\alpha_1}^{\bar{n}}(1/2) \geq 1/2$.

Again by continuity (since $f_0^{\bar{n}}(1/2) = 0$), there is $\alpha_2\in[\alpha_1,0)$ with

$$f_{\alpha_2}^{\bar{n}}(1/2) = 1/2.$$

and for the same reason there is a largest $\alpha \equiv \alpha_{\bar{n}} \in [\alpha_2, 0)$, such that

$$f_{\alpha_{\bar{n}}}^{\bar{n}}(1/2) = 1/2.$$

Analogously once we find some parameter value $\alpha<0$ with f_α superstable of any period n, then we have also proved the existence of a largest $\alpha_n < 0$ with f_{α_n} superstable of that period.

The approach to the final aperiodic regime

In the following lemma some properties are collected together about the parameter values corresponding to superstable periodic orbits.

Lemma For large $n \in N$

i) there exists a largest $\alpha_n < 0$ such that $f_{\alpha_n}^{n}(1/2) = 1/2$,

ii) $\alpha_n \nearrow 0$ $(n \to \infty)$

iii) $n\alpha_n \to 0$ $(n \to \infty)$

iv) $\displaystyle\sum_{j=0}^{n-k-2} f_{\alpha_n}^{n-k-j}(1/2) \le a\, f_{\alpha_n}^{n-k}(1/2)/(a-1)$

for $k \geq \bar{k}$, and with a given by (1).

Proof i),ii) are consequences of the general theory (see e.g. [JR]). But let us include the simple argument. The case $n=\bar{n}$ was proved above, together with the remark that for i) and any n it is enough to find some superstable orbit of that period. Now

$$f_{\alpha_{\bar{n}}}^{\bar{n}+1}(1/2) = f_{\alpha_{\bar{n}}}(1/2) > 1/2 \text{ by (3), and } f_0^{\bar{n}+1}(1/2)=0.$$

So by continuity

$$f_{\alpha}^{\bar{n}+1}(1/2) = 1/2$$

has a solution in $[\alpha_{\bar{n}}, 0)$. By induction i) follows with (α_n) strictly increasing. If

$$\lim_{n\to\infty} \alpha_n = \bar{\alpha} < 0 ,$$

then similarly to the proof of (4)

$$f^{n'}_{\bar{\alpha}}(1/2) \geq 1/2$$

for some $n' \geq \bar{n}$. Thus $\alpha_{n'} \geq \bar{\alpha}$, which contradicts the definition of $\bar{\alpha}$. So $\bar{\alpha}=0$.

iii) For the family (f_α) it follows, e.g. from the general ordering properties of full families, that

$$0 < f^2_{\alpha_n}(1/2) < f^3_{\alpha_n}(1/2) <\ldots< f^{n-1}_{\alpha_n}(1/2) < f^n_{\alpha_n}(1/2) = 1/2.$$

Let $\bar{n}$, $\bar{k}$, a, and c be defined as in the previous section. Then

$$(5)\quad 1/2 > f^{-k}_{\alpha_n}(1/2) = f^{n-k}_{\alpha_n}(1/2) = f_{\alpha_n}(f^{n-k-1}_{\alpha_n}(1/2)) \geq a\, f^{n-k-1}_{\alpha_n}(1/2) \geq$$

$$\ldots \geq a^{n-k-2} f^2_{\alpha_n}(1/2) = a^{n-k-2}[f_{\alpha_n}(f_{\alpha_n}(1/2))-f_{\alpha_n}(1)] \geq$$

$$\geq a^{n-k-1}[1-f_{\alpha_n}(1/2)] = a^{n-k-1}[f_0(1/2)-f_{\alpha_n}(1/2)] \geq c\, a^{n-k-1}|\alpha_n|$$

for $n>\bar{n}$ and $n-3 \geq k \geq \bar{k}$. In particular this implies

$$0 \leq n|\alpha_n| \leq n\, a^{k+1-n}/(2c) \to 0 \quad (n\to\infty),$$

which proves iii)

iv) By (5) we also get for $n> \bar{n}$ and $n-3 \geq k > \bar{k}$ that

$$\sum_{j=0}^{n-k-2} f^{n-k-j}_{\alpha_n}(1/2) \leq f^{n-k}_{\alpha_n}(1/2) \sum_{j=0}^{n-k-2} a^{-j} \leq a\, f^{n-k}_{\alpha_n}(1/2)/(a-1),$$

which is iv).

Using this lemma we finally prove the following theorem, which is the main result of this paper.

Theorem $\lim \alpha_n / \alpha_{n+1} = D_x f_0(0)$.

Proof We can write

$$\frac{f^{-k}_{\alpha_n}(1/2)}{f^{-k}_{\alpha_{n-1}}(1/2)} = \frac{f^{n-k}_{\alpha_n}(1/2)}{f^{n-k-1}_{\alpha_{n-1}}(1/2)} = \frac{D_x f_{\alpha_n}(a_1) f^{n-k-1}_{\alpha_n}(1/2)}{D_x f_{\alpha_{n-1}}(b_1) f^{n-k-2}_{\alpha_n}(1/2)} =\ldots=$$

$$= \prod_{1}^{n-k-3} \frac{D_x f_{\alpha_n}(a_j)}{D_x f_{\alpha_{n-1}}(b_j)} : \frac{f^3_{\alpha_n}(1/2)}{f^2_{\alpha_{n-1}}(1/2)}$$

with some (a_j), (b_j) such that

$$0 \le a_j \le f_{\alpha_n}^{n-k-j}(1/2) \ , \ 0 \le b_j \le f_{\alpha_{n-1}}^{n-k-j-1}(1/2),$$

and with

$$k \ge 1 \ ; \quad n \ge \max(\bar{n}+1, k+4).$$

Set

$$t_j = D_x f_{\alpha_n}(a_j)/D_x f_0(0), \ s_j = D_x f_{\alpha_{n-1}}(b_j)/D_x f_0(0).$$

By (2) t_j, $s_j \ge 1/2$ for $k \ge \bar{k}$. This implies

(6) $$\exp\Big(-2\sum_{1}^{n-k-3}(|t_j-1|+|s_j-1|)\Big) \le \prod_{1}^{n-k-3} t_j/s_j \le \exp\Big(2\sum_{1}^{n-k-3}(|t_j-1|+|s_j-1|)\Big),$$

since

$$1+y \le \exp y \quad \text{for } y>0, \exp(-2y) \le 1-y \text{ for } 0 \le y \le 1/2.$$

Now

$$D_x f_0(0)|t_j-1| = |D_x f_{\alpha_n}(a_j) - D_x f_0(0)| =$$
$$= |D_x f_{\alpha_n}(a_j) - D_x f_{\alpha_n}(0) + D_x f_{\alpha_n}(0) - D_x f_0(0)| \le C(a_j + |\alpha_n|)$$

with e.g. $C = \sup|D_x^2 f_\alpha| + \sup|D_\alpha D_x f_\alpha|$. Analogously

$$D_x f_0(0)|s_j-1| \le C(b_j+|\alpha_{n-1}|).$$

So

$$\sum_{1}^{n-k-3}\Big(|t_j-1| + |s_j-1|\Big) \le C\Big[\sum_{1}^{n-k-3}(a_j+b_j) + (n-k-3)(|\alpha_n| + |\alpha_{n-1}|)\Big] \le$$

$$\le C\Big[\sum_{1}^{n-k-3} f_{\alpha_n}^{n-k-j}(1/2) + \sum_{1}^{n-k-3} f_{\alpha_{n-1}}^{n-k-j-1}(1/2) + n|\alpha_n| + (n-1)|\alpha_{n-1}|\Big].$$

By the lemma this implies

$$0 \le \sum_{1}^{n-k-3}(|t_j - 1| + |s_j - 1|) \le C[a(f_{\alpha_n}^{-k}(1/2) + f_{\alpha_{n-1}}^{-k}(1/2))/(a-1) +$$

$$+ n|\alpha_n| + (n-1)|\alpha_{n-1}|] \to 2aC\, f_0^{-k}(1/2)/(a-1) \qquad (n \to \infty).$$

Hence by (6)

$$(7) \quad \exp(-4aC\, f_0^{-k}(1/2)/(a-1)) \le \varliminf_{n\to\infty} \prod_{1}^{n}(t_j/s_j) \le \varlimsup \prod_{1}^{n}(t_j/s_j) \le$$

$$\le \exp(4aC\, f_0^{-k}(1/2)/(a-1)).$$

Here the limits of the right and left members both are 1 when $k \to \infty$.

Now

$$(8) \quad 1 = \lim_{n\to\infty} \frac{f_{\alpha_n}^{-k}(1/2)}{f_{\alpha_{n-1}}^{-k}(1/2)} = \lim_{n\to\infty} \left[\left(\prod_{1}^{n-k-3} t_j/s_j\right) \frac{f_{\alpha_n}^{3}(1/2)}{f_{\alpha_{n-1}}^{2}(1/2)}\right].$$

Also for some $0<\theta_n<1$, $f_{\alpha_n}(1/2)<\omega_n<1$, $\alpha_n<\beta_n<0$,

$$(9) \quad \frac{\alpha_{n-1}}{\alpha_n} \frac{f_{\alpha_n}^{3}(1/2)}{f_{\alpha_{n-1}}^{2}(1/2)} = \frac{\alpha_{n-1}}{\alpha_n} D_x f_{\alpha_n}(\theta_n f_{\alpha_n}^2(1/2)) \frac{(f_{\alpha_n}(f_{\alpha_n}(1/2)) - f_{\alpha_n}(1))}{(f_{\alpha_{n-1}}(f_{\alpha_{n-1}}(1/2)) - f_{\alpha_{n-1}}(1))}$$

$$= D_x f_{\alpha_n}(\theta_n f_{\alpha_n}^2(1/2)) \frac{D_x f_{\alpha_n}(\omega_n)}{D_x f_{\alpha_{n-1}}(\omega_{n-1})} \frac{(f_{\alpha_n}(1/2) - 1)}{(f_{\alpha_{n-1}}(1/2) - 1)} \frac{\alpha_{n-1}}{\alpha_n} =$$

$$= D_x f_{\alpha_n}(\theta_n f_{\alpha_n}^2(1/2)) \frac{D_x f_{\alpha_n}(\omega_n)}{D_x f_{\alpha_{n-1}}(\omega_{n-1})} \frac{D_\alpha f_{\beta_n}(1/2)}{D_\alpha f_{\beta_{n-1}}(1/2)} \to D_x f_0(0) \text{ as } n \to \infty.$$

Finally the theorem is an immediate consequence of (7)-(9).

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