Order in Disorder: Modeling the Crumpling Dynamics of Thin Sheets

Jovana Andrejevic
Rycroft Group, Harvard University
IMA Data Science Seminar
May 4, 2021
Crumpling is a geometric deformation brought about by **stress focusing** in thin materials:

In elastoplastic sheets, high, localized stress leads to **permanent deformation** (creases/crease networks):

Background

Crumpling is prevalent across a wide range of length scales:

μm


cm

Candy wrapper.

km

Engineering applications of crumpling

- Highly porous
- Compression-resistant
- Shapeable and robust to deformation
- Lightweight
Engineering applications of crumpling

Crumpled graphene electrodes for batteries:
- Large accessible electrochemical surface area
- Resist aggregation/surface area loss

Engineering applications of crumpling


Suitable for flexible, wearable devices
Engineering applications of crumpling

- compression-resistant
- Shapeable and robust to deformation
- highly porous
- lightweight

Lightweight packing
Durable construction and design

Crinkled Aluminum Furnishings – Fredrikson Stallard
Engineering applications of crumpling

Potential for more applications with **controlled** crumpling; requires predicting crumpling behavior.
Rich behavior of crumpled systems

Disordered process, but order emerges in several ways:

Logarithmic relaxation

Memory effect and aging

Evolution of damage networks


Broad question: How does damage accumulate under repeated loading?
An $L_o \times L_o$ Mylar sheet is compressed uniaxially to a compaction ratio $\Delta$.

The local curvature is calculated from the sheet's topography, producing a map of crease locations.

Mountain and valley creases are identified from the crease pattern, and total crease length measured.

$n$ repetitions

Logarithmic evolution of total crease length $\ell$:

$$\ell(n, \Delta) = c_1 (1 - \Delta) \log \left( 1 + c_2 \frac{n}{\Delta} \right)$$

$n$ – crumpling repetitions
$\Delta$ – compaction ratio

Question 1: Can we find a physical justification for this robust scaling?
History independence

Vary compaction protocols

strong crumpled

weak crumpled

History independence

Vary compaction protocols

History independence

Vary compaction protocols

strong crumples

weak crumples

switch protocols

History independence

Evolution of total crease length $\ell$ is insensitive to the spatial details of the crease network.

History independence

Total crease length $\ell$ acts as a state variable.

Vary compaction protocols

Strong crumples

Switch protocols

Weak crumples

Vary initial state

Crumpled initial state

Ordered initial state

Question 2: Can we explain the observed history independence?

Outline

Fragmentation-based model for crumpling
Can a coarse-grained physical model elucidate the logarithmic growth and history independence of crease length?

Mechanical model of thin sheets
How well does a simple mechanical model replicate our experimental observations?

Future directions: augmenting data-driven studies
Can we make predictions about the spatial structure of damage networks?
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Can we make predictions about the spatial structure of damage networks?
Facet segmentation

Segment the crease pattern into its flat facets

Facet segmentation

Segment the crease pattern into its flat facets

Resemblance to fragmentation processes:

Tracy Lundgren. https://pixabay.com

Trinity Alps, California

Facet segmentation

Expand facets to fill all space

Kinetics of fragmentation

\[ \frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_{x}^{\infty} r(y)c(y, t)f(x|y)dy \]

- $x$ – fragment area, $c(x, t)$ – fragment population with area $x$

Kinetics of fragmentation

\[
\frac{\partial c(x, t)}{\partial t} = -r(x) c(x, t) + \int_x^\infty r(y) c(y, t) f(x|y) dy
\]

depletion

\[x\] – fragment area, \(c(x, t)\) – fragment population with area \(x\)

Kinetics of fragmentation

\[
\frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_x^\infty r(y)c(y, t)f(x|y)dy
\]

accumulation

\(x\) – fragment area, \(c(x,t)\) – fragment population with area \(x\)

Breakup rates

\[
\frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_x^\infty r(y)c(y, t)f(x|y)dy
\]

overall breakup rate  conditional breakup probability

\(x\) – fragment area, \(c(x, t)\) – fragment population with area \(x\)

Breakup rates

\[
\frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_{x}^{\infty} r(y)c(y, t)f(x|y)dy
\]

overall breakup rate  \hspace{1cm} \text{conditional breakup probability}

\( x \) – fragment area, \( c(x,t) \) – fragment population with area \( x \)

Facet area distribution

Rate equation

\[ \frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_x^\infty r(y)c(y, t)f(x|y)dy \]

fragment area \( x \)
fragment population \( c(x,t) \)

Solved with ansatz

\[ c(x, t) = \frac{1}{s(t)} \phi(\xi) \]

mean area \( s(t) \)
scaled area distribution \( \phi \)
scaled area \( \xi = x/s \)

Towards a model for re-fragmentation

segment length $y \sim \sqrt{x}$, facet area $x$

total crease length $\ell$

\[
f(y) = \frac{t}{\Gamma(a)} (yt)^{a-1} e^{-yt}
\]

gamma distribution

\[
\ell \sim \bar{\ell} \times n_f \times \frac{n_e}{2} \sim \frac{t}{a+1}
\]

Average edge length $\bar{\ell} = a/t$

Total number of facets $n_f = 1/s = t^2/(a(a+1))$

Number of edges per facet $n_e/2$

Missing a connection between $t$ and crumpling iterations $n$. 

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Towards a model for re-fragmentation

- Each vertical transect can be regarded as a random walk.
- One axial crumple = ensemble of random walks, some which escape geometric confinement.
- $t$ advances in proportion to likelihood that random walk “escapes”.

$$\frac{\partial t}{\partial n} \sim S(w; a, t)$$

Asymptotic approximation for large $t$:

$$t(n, \tilde{\Delta}) = \tilde{c}_1(\tilde{\Delta}) \log \left( 1 + \tilde{c}_2(\tilde{\Delta}) n \right)$$

Towards a model for re-fragmentation

Empirical model$^1$

Fragmentation-based analytical model$^2$


History independence

Steady-state distribution is a strong attractor – details of initial preparation are rapidly washed out.

Can **fragmentation theory** help us to understand this?
History independence

Diffusion equation

\[
\frac{\partial c(x, t)}{\partial t} = D \frac{\partial^2 c(x, t)}{\partial x^2}
\]
History independence

Diffusion equation

\[
\frac{\partial c(x, t)}{\partial t} = D \frac{\partial^2 c(x, t)}{\partial x^2}
\]

Fragmentation rate equation

\[
\frac{\partial c(x, t)}{\partial t} = -r(x)c(x, t) + \int_{x}^{\infty} r(y)c(y, t)f(x|y)dy
\]

Fragmentation causes **non-local changes** in the size distribution.
Summary

Fragmentation-based model for crumpling

• Captures statistical properties well across many samples
• Offers a physical basis for logarithmic scaling
• Explains observed history independence
• Unclear how mechanical properties enter in
• Arguments are tailored to specific geometry
Outline

Fragmentation-based model for crumpling

Can a coarse-grained physical model elucidate the logarithmic growth and history independence of crease length?

Mechanical model of thin sheets

How well does a simple mechanical model replicate our experimental observations?

Future directions: augmenting data-driven studies

Can we make predictions about the spatial structure of damage networks?
Mechanical model for thin sheets

\[ F_i = \sum_{j \in N_i} \nabla x_i E_s(r_{ij}) - \sum_{(j,k,l) \in T_i} \nabla x_i E_b(\kappa_{ijkl}) + F_d + F_r + F_e \]

Mechanical model for thin sheets

\[ F_i = -\sum_{j \in N_i} \nabla x_i E_s(r_{ij}) - \sum_{(j,k,l) \in T_i} \nabla x_i E_b(\kappa_{ijkl}) + F_d + F_r + F_e \]

- **Self-avoidance**
- **Stretching**
- **Bending**
- **Damping**
- **External forces**

\[ E_s(r_{ij}) = \frac{1}{2} k_s \left( \| r_{ij} \| - \ell_{ij} \right)^2 \]

- **In-plane interactions**

\[ F_d = -d_{iso} v_i - d_{int} \sum_{j \in N_i} \frac{d}{dt} \left( r_{ij} - \ell_{ij} \hat{r}_{ij} \right) \]

- **Isotropic drag**
- **Internal damping**

\[ E_b(\kappa_{ijkl}) \sim k_b \left( \kappa_{ijkl} \right)^2 \]

- **Out-of-plane rigidity**

\[ \kappa_{ijkl} = -\frac{\hat{n}_{ijk} \cdot r_{il}}{||\hat{n}_{ijk} \cdot r_{il}||} ||\hat{n}_{ikl} - \hat{n}_{ijk}|| \]

- **Effective curvature**

Integration

Explicit integration

\[ \dot{x} = v \]
\[ \dot{v} = F(t, x, v) \]

- 4th order adaptive solver
- Use for resolving events on small time scales (e.g. self-contact)

Implicit integration

\[ \dot{x} = v \]
\[ \dot{v} = F(t, x, v) \]
\[ 0 = F(t, x, v) \]

- 3rd order adaptive solver
- Use for fast integration during smooth deformations
- Solve nonlinear system iteratively using Newton steps

Integrating differential-algebraic equations (DAEs)

A subset of differential algebraic systems are those semi-explicit form,

\[ \dot{y} = f(t, y, z) \]
\[ 0 = g(t, y, z) \]

where the differential variables \( y \) are decoupled from the algebraic variables \( z \).


Thin sheet model:

\[ \dot{x} = v \quad \text{differential} \]
\[ 0 = F(t, x, v) \quad \text{algebraic} \]

Integrating differential-algebraic equations (DAEs)

We can discretize the differential equations using a $p$-order Backward Differentiation Formula (BDF), and solve the system concurrently with algebraic constraints:

\[
\begin{align*}
    r_{k+1}^{\text{diff}} &= y_{k+1} + \sum_{s=0}^{p+1} \alpha_{k-s} y_{k-s} - \beta h f_{k+1} = 0 \\
    r_{k+1}^{\text{alg}} &= -g_{k+1} = 0
\end{align*}
\]

Thin sheet model:

\[
\begin{align*}
    r_{k+1}^{\text{diff}} &= x_{k+1} + \sum_{s=0}^{p+1} \alpha_{k-s} x_{k-s} - \beta h v_{k+1} = 0 \\
    r_{k+1}^{\text{alg}} &= -F(t_{k+1}, x_{k+1}, v_{k+1}) = 0
\end{align*}
\]

Integrating differential-algebraic equations (DAEs)

Root-finding problem solved iteratively via Newton’s method:

\[ J(q_i) \Delta q_i = -r(q_i) \]
\[ q = (y, z) \]
\[ r(q) = (r^{\text{diff}}, r^{\text{alg}}) \]

\[ J(q) = \begin{bmatrix}
1 - \beta h \frac{\partial f}{\partial y} & -\beta h \frac{\partial f}{\partial z} \\
-\frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z}
\end{bmatrix} \]

Thin sheet model:

\[ J(x, v) = \begin{bmatrix}
1 & -\beta hl \\
H_x & H_v
\end{bmatrix} \]

\[ H_x = -\nabla_x F \]
\[ H_v = -\nabla_v F \]
Integrating differential-algebraic equations (DAEs)

Root-finding problem solved iteratively via Newton’s method:

\[ J(q_i) \Delta q_i = -r(q_i) \]
\[ q = (y, z) \]
\[ r(q) = (r^{\text{diff}}, r^{\text{alg}}) \]
\[ J(q) = \begin{bmatrix} 1 - \beta h \frac{\partial f}{\partial y} & -\beta h \frac{\partial f}{\partial z} \\ -\frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z} \end{bmatrix} \]

Thin sheet model:

\[ (H_x + \frac{1}{\beta h} H_v) \Delta x = F - \frac{1}{\beta h} H_v r^{\text{diff}} \]
\[ \Delta v = \frac{1}{\beta h} (r^{\text{diff}} + \Delta x) \]

Linear solve

Explicit solution

Integrating differential-algebraic equations (DAEs)

Root-finding problem solved iteratively via Newton’s method:

\[ J(q_i) \Delta q_i = -r(q_i) \]
\[ q = (y, z) \]
\[ r(q) = (r^{\text{diff}}, r^{\text{alg}}) \]

\[ J(q) = \begin{bmatrix} 1 - \beta h \frac{\partial f}{\partial y} & -\beta h \frac{\partial f}{\partial z} \\ -\frac{\partial g}{\partial y} & -\frac{\partial g}{\partial z} \end{bmatrix} \]

Thin sheet model:

\[ \left( H_x + \frac{1}{\beta h} H_v \right) \Delta x = F - \frac{1}{\beta h} H_v r^{\text{diff}} \]

\[ \frac{1}{\beta h} H_v = \frac{d_{\text{int}}}{k_s \beta h} H_x + \frac{d_{\text{iso}}}{\beta h} l \]

Linear solve

Regulate conditioning with adaptive step size and damping

Preconditioning

Accelerate iterative solve of large, sparse system $Ax = b$ through preconditioning

Goal is to find a preconditioner $M$ so that $M^{-1}A$ is better conditioned, and such that $M$ is not expensive to invert.

$$M^{-1}Ax = M^{-1}b$$

Block preconditioner:

Matrix sparsity pattern

Preconditioning

node indexing  
y-coordinate  
Reverse Cuthill-McKee  
Hilbert space-filling curve

Hilbert space-filling curve gives us the densest blocks to invert.

Plasticity

\[ E_b(\kappa_{ijkl}) \sim k_b \left( \kappa_{ijkl} - p_{ik} \right)^2 \]

out-of-plane rigidity

\[ \kappa_{ijkl} = - \frac{\hat{n}_{ijk} \cdot r_{il}}{||\hat{n}_{ijk} \cdot r_{il}||} ||\hat{n}_{ikl} - \hat{n}_{ijk}|| \]

effective curvature

\[ \dot{x} = v \]
\[ \dot{p} = G(t, x, p) \]
\[ 0 = F(t, x, v, p) \]

equations of motion

Radial crumpling simulation

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young's modulus: 1 GPa
Bending rigidity: 9.3750e-5 N*m
Axial crumpling simulation

1\textsuperscript{st} crumple

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young’s modulus: 0.5 GPa
Bending rigidity: 4.6875e-5 N*m
Axial crumpling simulation

2\textsuperscript{nd} crumple

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young’s modulus: 0.5 GPa
Bending rigidity: 4.6875e-5 N*m
Axial crumpling simulation

3rd crumple

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young’s modulus: 0.5 GPa
Bending rigidity: $4.6875 \times 10^{-5}$ N*m
Axial crumpling simulation

4\textsuperscript{th} crumple

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young's modulus: 0.5 GPa
Bending rigidity: 4.6875e-5 N*m
Axial crumpling simulation

5th crumple

Dimensions: 10 cm x 10 cm
Thickness: 0.1 mm
Young’s modulus: 0.5 GPa
Bending rigidity: 4.6875e-5 N*m
Facet segmentation

1\textsuperscript{st} crumple

2\textsuperscript{nd} crumple

3\textsuperscript{rd} crumple

1\textsuperscript{st} crumple
2\textsuperscript{nd} crumple
3\textsuperscript{rd} crumple
Summary

Mechanical model of thin sheets

- Flexibly captures different geometries
- Preliminary agreement with facet statistics and logarithmic scaling
- Plasticity model needs calibration with experiment
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Can we make predictions about the spatial structure of damage networks?
Data-driven studies

Total crease length and fragmentation-based model provide a coarse-grained description of crumpled states.

**Broad question:** Can we make predictions about the spatial evolution of damage networks?

**Simpler sub-problem:** Are there correlations between creases in the network?

Case study in machine learning: network completion problem predicting mountain folds given the valleys.

Augmenting experiment with simulation

Attempt to learn from crumpling data alone unsuccessful - augment data with simulated flat-folding

Augmenting experiment with simulation

Extend data augmentation approach with simulated crumpling:

- improve current studies of network reconstruction
- explore other spatial trends such as energy distribution over time
Summary

Fragmentation-based model for crumpling

- Proposed a fragmentation-based model to describe crumpling as a breakup process.
- Offered a physical basis for logarithmic scaling and observed history independence.

Mechanical model of thin sheets

- Developed an efficient framework for thin sheet simulation.
- Preliminary agreement with facet statistics and logarithmic scaling.

Future directions: augmenting data-driven studies

- Continued effort to probe spatial structure in crease networks.
Thank you!

Principal investigators

Chris Rycroft
Harvard University

Shmuel Rubinstein
The Hebrew University of Jerusalem

Collaborators

Madelyn Leembruggen
Jordan Hoffmann
Lisa Lee
Shruti Mishra
Omer Gottesman
Yohai Bar-Sinai
Arshad Kudrolli

The Rycroft Group
