Graph structure and algorithm complexity

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We are interested in problems on graphs, such as

- Graph coloring,
- Shortest paths/spanning trees,
- Isomorphism testing,
- Largest independent sets/matchings,
- ...

We can sort problems into types, based on the complexity of solving them on a general graph.
The class of all problems for which there exists an algorithm with run time $f(n) < n^c$ for some constant $c$ is called $P$: problems solvable in polynomial time.

There is a wider class of problems that are solvable in polynomial time if we have nondeterministic computers that can make “lucky” guesses. This is the class $NP$: problems nondeterministically solvable in polynomial time.
For a given problem that is \textbf{NP}-complete on graphs in general, there may be sets of graphs for which the problem is in \textbf{P}.

Example: given two graphs $G$ and $H$, is there an \textbf{isomorphism} $\sigma : V(G) \rightarrow V(H)$?

The Graph Isomorphism problem is in \textbf{NP}, but is not known to be \textbf{NP}-complete or in \textbf{P}. (Although it is close to \textbf{P}.)

However, if we restrict ourselves to \textbf{planar} graphs, the problem can be solved in $O(n)$ time.
Forbidden pattern characterizations

We often characterize important classes of graphs in terms of forbidden substructures, such as

- induced subgraphs (hereditary class)
  obtained by: vertex deletion
- subgraphs (monotone class)
  obtained by: vertex deletion, edge deletion
- minors (minor-closed class)
  obtained by: vertex deletion, edge deletion, edge contraction
Forbidden pattern characterizations

Both forbidden-minor and forbidden-subgraph characterizations may be useful in applications.

For example:

- **forbidden subgraphs:**
  
  \( k \)-partite graphs, bounded degree, bounded girth, comparability graphs, triangle-free or bounded triadic closures, co-graphs, . . .

- **forbidden minors:**
  
  planar graphs (and other topological surfaces), bounded crossing numbers, graphs of bounded treewidth/treedepth, . . .
Forbidden pattern characterizations

Finite set of forbidden subgraphs $\rightarrow$ local properties.

Finite set of forbidden minors $\rightarrow$ global properties.
**Theorem (Robertson-Seymour theorem)**

*If $\mathcal{F}$ is a minor-closed class of graphs, it has the nice property that there is a finite set of minor-minimal graphs $X$ that is not in $\mathcal{F}$.***
Minor-closed classes

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For example, the planar graphs are the $\{K_5, K_{3,3}\}$-minor-free graphs.
Minor-closed classes

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For example, the planar graphs are the \( \{K_5, K_{3,3}\} \)-minor-free graphs.

Consequently, every set of minor-closed classes is well-quasi-ordered with respect to the containment relation:

- must have a minimal element (finitely many),
- no infinite descending chains,
- no infinite anti-chains.
Treewidth

Graph treewidth is intricately connected to both computational complexity and graph minors. Intuitively, the treewidth of a graph tells you how hard you have to squeeze to make it look like a tree.
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Treewidth

Many $\textbf{NP}$-complete graph problems are in $\textbf{P}$ for graphs of bounded treewidth:

- independent set,
- clique,
- proper coloring,
- max-cut,
- vertex covers,
- cycle packing,
- ...
Minor-closed classes

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If \( Y \) is a minor-closed class of graphs, restricting the independent set problem to \( Y \) is in \( P \) if and only if at least one planar graph is not in \( Y \).
Minor-closed classes

For such problems, in terms of minor-closed classes of graphs, the result that the problem is \( NP \)-hard for planar graphs is “best possible”:

If \( Y \) is a minor-closed class of graphs, restricting the independent set problem to \( Y \) is in \( P \) if and only if at least one planar graph is not in \( Y \).

Because: planar graphs are the unique minimal minor-closed class of graphs of unbounded treewidth.
Hereditary classes

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They are not well-founded with respect to the containment relation.

To tackle this, Alekseev\textsuperscript{1} introduced the notion of a \textit{boundary class}.

\textsuperscript{1}V.E. Alekseev. \textit{On easy and hard hereditary classes of graphs with respect to the independent set problem}. Discrete Applied Mathematics, 132:17–26, 2003
Boundary classes

Let $\mathcal{F}$ be a family of hereditary classes closed under taking subclasses.

Suppose $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$ is an infinite descending anti-chain of hereditary graph classes outside $\mathcal{F}$ and let $Y = \bigcap_n Y_n$.

Then $Y$ is a limit class for $\mathcal{F}$, and a minimal limit class is a boundary class.

A finitely defined hereditary graph class $X$ is outside $\mathcal{F}$ if and only if it contains a boundary class for $\mathcal{F}$.
Boundary classes

Example:

Let $\mathcal{F}$ be the family of hereditary graph classes for which the independent set problem is in $P$.

Let $Y_j$ be the class of $(C_3, \ldots, C_j)$-free graphs, and let $Y = \bigcap Y_j$.

Then, $Y_k$ is the class of graphs with no cycles of length $\leq k$, and $Y$ is the class of trees, which is a limit class for 3-coloring.

Note that, in this case, $Y \in \mathcal{F}$.
Coupon coloring

A $k$-coupon coloring of a graph is a vertex $k$-coloring, such that every vertex sees at least one vertex of each color in their neighborhood.

For example\(^2\): large multi-robot network with local communication. Each robot has a single sensor that measures either temperature, light, or humidity, and it collects remaining data from its neighbors.

3-coupon coloring $\Rightarrow$ every robot is fully informed!

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\(^2\)W. Abbas, M. Egerstedt, Chun-Hung Liu, Robin Thomas, and Peter Whalen, *Deploying Robots with two sensors in $K_{1,6}$-free graphs*. arxiv.org/abs/1308.5450.
Coupon coloring

$\mathcal{F}_k$: forests of degree $\leq k$.

**Theorem**

*For prime $k$, $\mathcal{F}_k$ is a boundary class for the $k$-coupon coloring problem.*

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Coupon coloring

Proof idea: “graph implantation”.

\[\text{Diagram 1}\]

\[\text{Diagram 2}\]
Coupon coloring

Proof idea: “graph implantation”.

![Diagram of coupon coloring proof idea]
Coupon coloring

Proof idea: “graph implantation”.

Every $k$-regular graph yields an arbitrarily high-girth $k$-regular graph with equivalent coupon-coloring.
Theorem

For prime $k$, there exists a family of bipartite, high-girth, $k$-regular, $k$-coupon colorable graphs.

This implies the $k$-coupon coloring problem is $NP$-hard for $(K_{1,k+1}, C_3, C_4, \ldots, C_j)$-free graphs.

If we call this class $Y_j$, then

$$\mathcal{F}_k = \bigcap_j Y_j,$$

is a limit class.

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Recap

Hereditary and monotone graph classes are of interest in algorithmic settings (as well as minor-closed classes).

It is natural to characterize forbidden graph classes by their minimal forbidden subclasses, but this is only guaranteed in the minor-closed case.

Boundary classes give us minimal forbidden subclasses for the finitely defined classes in a family.

These provide new ways to characterize polynomially solvable cases for $NP$-hard graph problems.
THANK YOU!