Smooth Hilbert Schemes

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The **Hilbert scheme** $\mathrm{Hilb}^q(\mathbb{P}^n)$ parametrizes all closed $X \subset \mathbb{P}^n$ (over $\mathbb{C}$) with Hilbert polynomial $q \in \mathbb{Q}[t]$.

REEVES–STILLMAN (1997): Each $\mathrm{Hilb}^q(\mathbb{P}^n)$ has a "distinguished" smooth point.

VAKIL (2006): Every singularity type appears in a $\mathrm{Hilb}^q(\mathbb{P}^n)$.

GOAL: Identify the smooth Hilbert schemes and understand their geometry.

Combinatorial Encoding

MACAULAY (1926): $\mathrm{Hilb}^q(\mathbb{P}^n) \neq \varnothing$ if and only if there exists an integer partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{N}^r$ such that $n \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ and $q(t) = \sum_{i=1}^{r} \binom{t+\lambda_i-i}{\lambda_i-1}$.

EXAMPLE: A closed $X \subset \mathbb{P}^n$ is a (λ_1-1)-dimensional linear space if and only if $q(t) = \binom{t+\lambda_1-1}{\lambda_1-1}$. Thus, we have

$$r = 1 \iff \mathrm{Hilb}^q(\mathbb{P}^n) = \mathbb{Gr}(\lambda_1-1, \mathbb{P}^n).$$

EXAMPLE: A closed $X \subset \mathbb{P}^n$ is a hypersurface of degree r if and only if $q(t) = \binom{t+n}{n} - \binom{t+n-r}{n} = \sum_{i=1}^{r} \binom{t+n-i}{n-1}$, so $\lambda = (n^r) = (n, n, \ldots, n)$ implies that $\mathrm{Hilb}^q(\mathbb{P}^n) = \mathbb{P}^{\binom{r+n}{n}-1}$.

GOTZMANN (1978): For all $X \subset \mathbb{P}^n$ with Hilbert polynomial $q(t) = \sum_{i=1}^{r} \binom{t+\lambda_i-i}{\lambda_i-1}$, the associated saturated homogeneous ideal I_X has Castelnuovo–Mumford regularity at most r.

Characterizing Smoothness

Theorem (Skjelnes–Smith, 2019): The Hilbert scheme $\mathrm{Hilb}^q(\mathbb{P}^n)$ is smooth if and only if

I. $n \leqslant 2$,

II. $\lambda_r \geqslant 2$,

III. $r \leqslant 1$ or $\lambda = \left(n^{r-2}, \lambda_{r-1}^{1}, 1^{1}\right)$ for all $r \geqslant 2$,

IV. $\lambda = \left(n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 2^{0}, 1^{1}\right)$ for all $0 \leqslant s \leqslant r-3$ and all $r \geqslant 3$,

V. $\lambda = \left(n^{r-s-5}, 2^{s+4}, 1^{1}\right)$ for all $0 \leqslant s \leqslant r-5$ and all $r \geqslant 5$, or

VI. $\lambda = \left(n^{r-3}, 1^{3}\right)$ for all $r \geqslant 3$.

EXAMPLE: When $\lambda = (2^2, 1^1)$ or $q = 2t + 2$, $\mathrm{Hilb}^q(\mathbb{P}^n)$ has two components: the general points correspond to two skew lines or a plane conic union a point.

EXAMPLE: When $\lambda = (2^3, 1^1)$ or $q = 3t + 1$, $\mathrm{Hilb}^q(\mathbb{P}^n)$ also has two components: the general points correspond to a twisted cubic curve or a plane cubic union a point.

I. FOGARTY (1968): For all $n \leqslant 2$, $\mathrm{Hilb}^q(\mathbb{P}^n)$ is smooth.

II—III. STAAL (2017): $\mathrm{Hilb}^q(\mathbb{P}^n)$ has a unique Borel-fixed point in precisely these two situations, so it is smooth.

IV—VI. LEMMA: $\mathrm{Hilb}^q(\mathbb{P}^n)$ has exactly two Borel-fixed points in these three situations.

LEMMA: Whenever $\mathrm{Hilb}^q(\mathbb{P}^n)$ has more than one Borel-fixed point, we can explicitly describe two in terms of λ.

OUTLINE OF PROOF: Compute the dimension of the tangent space at our explicit Borel-fixed points. They agree if and only if we are in one of IV—VI or $\lambda = (n^{r-s-4}, 1^{s+4})$ for all $0 \leqslant s \leqslant r-4$ and all $r \geqslant 4$. In the last case, identify a third Borel-fixed point with a higher-dimensional tangent space.