Using *Macaulay2* effectively in practice

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Macaulay2: at a glance

- Project started in 1993, Dan Grayson and Mike Stillman.
- Open source.
- Key computations: Gröbner bases, free resolutions, Hilbert functions **and applications of these**. Rings, Modules and Chain Complexes are first class objects.
- Language which is comfortable for mathematicians, yet powerful, expressive, and fun to program in.

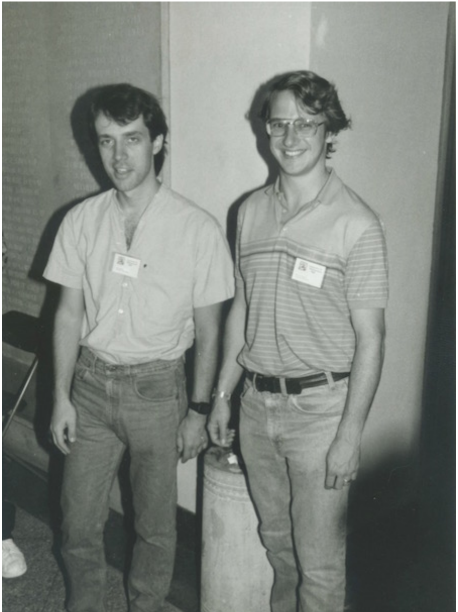
Now a community project

- Journal of Software for Algebra and Geometry (started in 2009. Now we handle: Macaulay2, Singular, Gap, Cocoa) (original editors: Greg Smith, Amelia Taylor).
- Strong community: including about 2 workshops per year.
- User contributed packages (about 200 so far). Each has doc and tests, is tested every night, and is distributed with M2.
- Lots of activity
- Over 2000 math papers refer to Macaulay2.

History: 1976-1978 (My undergrad years at Urbana)

- E. Graham Evans: asked me to write a program to compute syzygies, from Hilbert's algorithm from 1890.
- Really didn't work on computers of the day (probably might still be an issue!).
- Instead: Did computation degree by degree, no finishing condition. Used Buchsbaum-Eisenbud "What makes a complex exact" (by hand!) to see if the resulting complex was exact.
- Winfried Bruns was there too. Very exciting time.

History: 1978-1983 (My grad years, with Dave Bayer, at Harvard)



History: 1978-1983 (My grad years, with Dave Bayer, at Harvard)

- I tried to do “real mathematics”
- but Dave Bayer (basically) rediscovered Groebner bases, and saw that they gave an algorithm for computing all syzygies.
- I got excited, dropped what I was doing, and we programmed (in Pascal), in less than one week, the first version of what would be Macaulay.
- It was enormously fast: computing the free resolution of the rational quartic took less than 30 seconds.
- Exciting time at Brandeis: Eisenbud, Weyman, Schreyer, Kollar, Buchweitz, were all there in (circa) 1983. Weyman: very skeptical, as he could do many of these resolutions faster than the computer.
- Schreyer: had come up with standard bases on his own as well. Had (and still has) extremely good ideas for best ways of computing syzygies.

History: 1980's (Macaulay classic)

- Apple Macintosh came out, Dave and I each got a 512K version (about 1984-1985).
- Rewrote Macaulay (not called that yet) into C. Good move!
- It did: "standard" (Groebner bases), free resolutions, for homogeneous ideals or modules, over a prime field (default $\mathbb{Z}/31991$).
- with David Eisenbud: wrote scripts for computing many things in algebraic geometry.
- Language was "baroque", more like an assembly language.
- Great time: no one knew what in algebraic geometry could be computed.
- Gianni-Trager-Zacharias paper on computing primary decompositions.

History: 1990's (Macaulay2, Dan Grayson)

- Dave wanted to do other things, I needed to get tenure.
- Dan Grayson: at Urbana, one of original authors of Mathematica, but had falling out with Wolfram.
- Eisenbud: suggested to Dan Grayson and me: get together to write a successor system to Macaulay.
- Macaulay2: main idea: we designed a much nicer to use interpreted language, and had an "engine" (in C++) handling Groebner bases, free resolutions, Hilbert functions, etc.
- 1996: First version made public.
- Old school: no tests, documentation extremely limited, source not readily available yet.
- 1999: Macaulay2 book (still relevant, code still mostly works, we keep revised code for chapters in the M2 distribution)

History: 2000-2019, community!

- early 2000's: packages were added to Macaulay2.
- Open sourced Macaulay2 (which caused several other systems, including Singular, to do so too).
- Documentation: spent **a lot** of time writing documentation.
- 2007: version 1.0.
- 2007: Macaulay2 google group
- 200x-present: User contributed packages.
- 2008: David Eisenbud joined us on the project.
- 2009: JSAG (Journal of Software for Algebra and Geometry) started.
- 2013: Changed source control from subversion to git.
- 2013: TryM2: web.macaulay2.com goes online.

A few of the larger packages in Macaulay2

- `Dmodules` (Anton Leykin, Harrison Tsai)
- `Polyhedra` (Rene Birkner, now Lars Kastner)
- `NormalToricVarieties` (Greg Smith)
- `NumericalAlgebraicGeometry` (Anton Leykin, and others. Several packages here).
- `Schubert2` (Based on the Maple package: Schubert, Katz and Stromme).

Example 1: Ideals generated by 6 homogeneous quadrics

Problem setup

Let $S = \mathbb{K}[x_1, \ldots, x_n]$, and let

$$I = \langle Q_1, \ldots, Q_6 \rangle \subset S$$

be an ideal generated by 6 homogeneous quadratic polynomials.

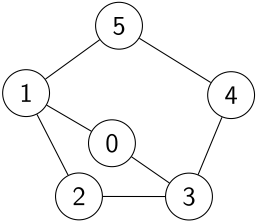
Problem:

- What are the possible regularities of these ideals?
- (graded) Betti numbers?
- Maximum size of a Groebner basis? Maximum degree of a GB generator?

If we restrict to polynomials with 2 terms (binomials), what is the **distribution** of these numbers?

Example 2: Coupled oscillators

The setup: Start with an undirected graph G with vertices $\{0, 1, \ldots, n\}$, e.g.:



- Let $a_{ij} = 1$ if (i, j) is an edge in G, otherwise $a_{ij} = 0$.
- For each vertex, we consider an oscillator $\theta_i = \theta_i(t)$.
- Consider the dynamical system:

$$\dot{\theta}_i = \sum_{j \neq i} a_{ij} \sin(\theta_j - \theta_i) \quad \text{for } i = 0, \ldots, n,$$

Our goal: For a specific G, understand the structure of the equilibrium points (stable and unstable) of this system, i.e. the solutions of the right hand sides of these equations.

Our plan: Translate this to a problem in alegbra.

Example 2: reduction to algebra

- $x_i := \cos\theta_i, \quad y_i := \sin\theta_i, \quad$ for $0 \leq i \leq n$.
- Set $\theta_0 = 0$, that is, $x_0 = 1$ and $y_0 = 0$.
- Set $\sin(\theta_j - \theta_i) = x_j y_i - x_i y_j$.
- Plug those in to the RHS, and add the $x_i^2 + y_i^2 = 1$, for $1 \leq i \leq n$.

Equations

Let $\mathbb{K}$ be a field. Usually $\mathbb{K} = \mathbb{C}, \mathbb{Q}$ or a finite field $\mathbb{F}_p$
Let

$$f_i := \sum_{j=0}^{n} a_{ij}(x_j y_i - x_i y_j).$$

The equilibrium solutions correspond to the real zeros of the ideal $I_G \subset S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$:

$$I_G := \langle f_0, \; \ldots, f_n, \; x_1^2 + y_1^2 - 1, \ldots, \; x_n^2 + y_n^2 - 1 \rangle$$

Note that $f_0 + f_1 + \cdots + f_n = 0$, so we can remove any one of the f_i, if we feel the need.

Example 2: Analyze this example

Macaulay2 example to do now

Find the irreducible components, analyze their singularities, intersections, and non-reducedness structure.

Specifically:

- Construct the ideal in Macaulay2.
- Can we compute its minimal primes? primary decomposition? Over $\mathbb{Q}$, or easier, over $\mathbb{F}_p$?
- What about numerically, over $\mathbb{C}$?

Example 3: Noncommutative Gröbner bases

Frank Moore: has written a package `NCAlgebras`, which calls the program `Bergman` for computation of Gröbner bases.

We are creating new non-commutative ring types in Macaulay2, which compute Gröbner bases too. Our naive implementation appears to be 100's times as fast as Bergman.