Surfaces and more

Jennifer Schultens

Surfaces (2-manifolds)

Definition

A *closed surface* is a compact space S such that every point in S has a neighborhood homeomorphic to $\mathbb{R}^2$.

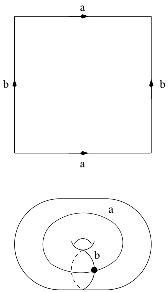

Figure: A polygonal representation of the torus

Surfaces

Theorem

Every closed surface has a polygonal representation.

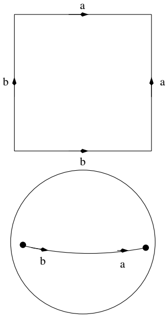

Figure: A polygonal representation of the 2-sphere

Surfaces

Definition

Let S be an orientable closed surface. The *Euler characteristic* of S is the following number:

$$\#vertices - \#edges + \#faces$$

(This number is computed in the surface - the bottom picture - invariant under different choices of polygonal representations.)

Surfaces

Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.

Surfaces

Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.

Definition

Let S be an orientable closed surface. The number of tori in a connected sum representing S is called the *genus* of S. (It is uniquely determined.)

Surfaces

Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.

Definition

Let S be an orientable closed surface. The number of tori in a connected sum representing S is called the *genus* of S. (It is uniquely determined.)

Computation: If S has genus g, then it is the quotient of a $4n$-gon.

$$genus(S) = g \Longrightarrow \chi(S) = 2 - 2g$$

I spy an orientable closed surface with Euler characteristic -2.

I spy an orientable closed surface with Euler characteristic -2.

It's a genus 2 surface!

Recognizing surfaces

I spy an orientable closed surface with Euler characteristic -2.

It's a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

Recognizing surfaces

I spy an orientable closed surface with Euler characteristic -2.

It's a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It's a genus 7 surface!

Recognizing surfaces

I spy an orientable closed surface with Euler characteristic -2.

It's a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It's a genus 7 surface!

I spy an orientable closed surface with Euler characteristic -11.

Recognizing surfaces

I spy an orientable closed surface with Euler characteristic -2.

It's a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It's a genus 7 surface!

I spy an orientable closed surface with Euler characteristic -11.

No you don't!

Digression on geometrization for surfaces

Geometrization for surfaces

Definition

We say that a manifold M exhibits a certain *geometry* if it is the quotient of the geometry by a "nice" group (*i.e.*, a discrete group of isometries).

Digression on geometrization for surfaces

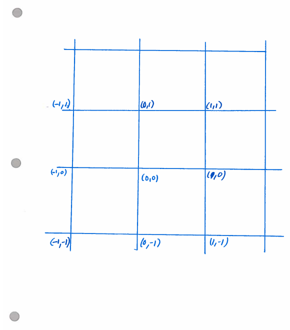

Figure: The torus is a quotient of the Cartesian plane by a translation group

Digression on geometrization for surfaces

Theorem

(Geometrization for surfaces) Any orientable closed surface is either spherical, Euclidean, or hyperbolic.

Circles inscribed in triangles

Circles inscribed in triangles in the hyperbolic plane are *not* arbitrarily large!

The curve complex

The curve complex

The curve complex, defined by Harvey in the 1980s, established itself as a tool for studying 3-manifolds through the work of Howie Masur, Yair Minsky, who showed that it is "quasi-hyperbolic" and of John Hempel who used it to analyze 3-manifolds.

The curve complex

Definition

A *simplicial complex* $\mathfrak{K}$ is a set of simplices that satisfies the following:

- Every face of a simplex of $\mathfrak{K}$ is also in $\mathfrak{K}$;
- The intersection of any two simplices $\sigma_1, \sigma_2 \in \mathfrak{K}$ is either empty or a face of both σ_1 and σ_2.

Definition

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.

The curve complex

Definition

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.

Definition

The k-skeleton of a simplicial complex is the subcomplex consisting of vertices, 1-simplices, . . . , k-simplices.

The curve complex

Definition

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.

Definition

The k-skeleton of a simplicial complex is the subcomplex consisting of vertices, 1-simplices, . . . , k-simplices.

Definition

A simplicial complex $\mathcal{K}$ is *flag* if any simplex whose 1-skeleton lies $\mathcal{K}$ is also in $\mathcal{K}$.

The curve complex

Definition

Let S be an orientable closed surface. The *curve complex* of S, denoted $\mathfrak{K}(S)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in S and whose edges are pairs of distinct vertices that admit disjoint representatives.

The curve complex

Definition

Let S be an orientable closed surface. The *curve complex* of S, denoted $\mathfrak{K}(S)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in S and whose edges are pairs of distinct vertices that admit disjoint representatives.

Definition

The distance between two vertices v, w in the curve complex is the number of edges in an edge-path between v and w.

The curve complex

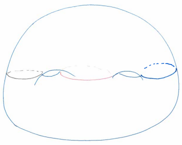

Figure: A simplex in the curve complex of the genus 2 surface

The curve complex

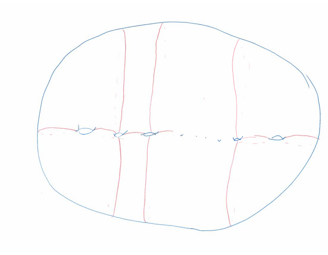

Figure: A maximal simplex in the curve complex of the genus g surface

The curve complex

Theorem

$$dim(\mathfrak{K}(S)) = 3genus(S) - 3$$

Case study: The curve complex of the torus

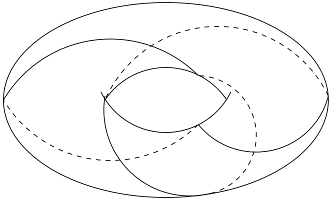

Figure: The torus knot $T(2,3)$

Case study: The curve complex of the torus

Observation 1: Torus knots correspond to rational numbers.

Case study: The curve complex of the torus

Observation 1: Torus knots correspond to rational numbers.

Observation 2: Non isotopic torus knots are not disjoint.

Case study: The curve complex of the torus

Observation 1: Torus knots correspond to rational numbers.

Observation 2: Non isotopic torus knots are not disjoint.

Observation 3: $T(p, q)$ and $T(r, s)$ meet in a single point $\Leftrightarrow$

$$ps - qr = \pm 1$$

Definition

The *(modified) curve complex* of the torus, denoted $\mathfrak{K}(\mathbb{T}^2)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in the torus and whose edges are pairs of distinct vertices that admit representatives that meet exactly once non tangentially.

Case study: The curve complex of the torus

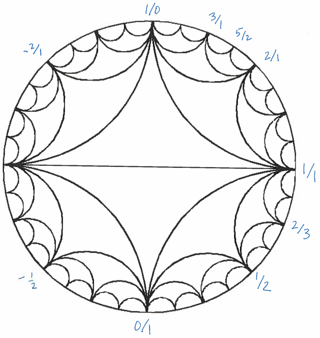

Figure: The Farey graph is the curve complex of the torus

Coarse geometry

Coarse geometry concerns properties of metric spaces from a 'large scale' point of view.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces. A *quasi-isometry* between (X, d_X) and (Y, d_Y) is a (not necessarily continuous) function

$$f : X \to Y$$

such that there are constants A, C with

$$\frac{1}{C} d(x, y) - A \leq d(f(x), f(y)) \leq Cd(x, y) + A$$

If there is a quasi-isometry between (X, d_X) and (Y, d_Y), then we say that (X, d_X) and (Y, d_Y) are *quasi-isometric*.

Example 1: Every compact space is quasi-isometric to a point.

Example 1: Every compact space is quasi-isometric to a point.

Example 2: The ‘integer lattice’ is quasi-isometric to the Cartesian plane.

Example 1: Every compact space is quasi-isometric to a point.

Example 2: The ‘integer lattice’ is quasi-isometric to the Cartesian plane.

Definition

A space is *quasi-Euclidean* if it is quasi-isometric to the Cartesian plane.

Coarse geometry

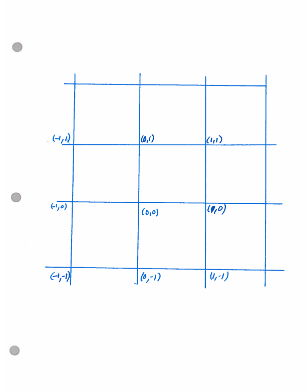

Figure: The Cartesian plane with the integer lattice

Definition

A triangle in a metric space is δ-thin if each edge lies in a δ-neighborhood of the other two.
A (X, d_X) *Gromov hyperbolic* if there exists a δ such that all triangles are δ-thin.

Coarse geometry

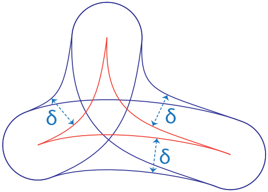

Figure: A δ-thin triangle

The curve complex

Theorem

(Masur-Minsky) The curve complex of an orientable closed surface of genus at least two is Gromov hyperbolic.

Heegaard splittings

Definition

Let M be a 3-manifold. If $M = V \cup W$, where V, W are handlebodies and $\partial V = \partial W$ is an orientable closed surface S, then we call $M = V \cup_S W$ a *Heegaard splitting* of M with *splitting surface* S.

Heegaard splittings

Definition

Let M be a 3-manifold. If $M = V \cup W$, where V, W are handlebodies and $\partial V = \partial W$ is an orientable closed surface S, then we call $M = V \cup_S W$ a *Heegaard splitting* of M with *splitting surface* S.

Definition

The *genus* of a Heegaard splitting is the genus of its splitting surface.

Theorem

(Moise, Bing) Every 3-manifold admits a Heegaard splitting.

Heegaard splittings

Example 1: $\mathbb{S}^3$ has a genus 0 Heegaard splitting.

Heegaard splittings

Example 1: $\mathbb{S}^3$ has a genus 0 Heegaard splitting.

Example 2: $\mathbb{S}^3$ has a genus 1 Heegaard splitting.

Heegaard splittings

Example 1: $\mathbb{S}^3$ has a genus 0 Heegaard splitting.

Example 2: $\mathbb{S}^3$ has a genus 1 Heegaard splitting.

Example 3: $\mathbb{T}^3$ has a genus 3 Heegaard spltiting.

Heegaard splittings

Definition

Let M be a 3-manifold. If $M = V \cup W$, where V, W are handlebodies and $\partial V = \partial W$ is an orientable closed surface S, then we call $M = V \cup_S W$ a *Heegaard splitting* of M with *splitting surface* S.

Definition

Let M be a 3-manifold. If $M = V \cup W$, where V, W are handlebodies and $\partial V = \partial W$ is an orientable closed surface S, then we call $M = V \cup_S W$ a *Heegaard splitting* of M with *splitting surface* S.

Definition

The *genus* of a Heegaard splitting is the genus of its splitting surface.

Heegaard splittings

Definition

The *distance* of a Heegaard splitting $M = V \cup_S W$ is the minimal distance, in the curve complex of S, between a simple closed curve in S that bounds a disk in V and a simple closed curve in S that bounds a disk in W.