Classical knot invariants

Jennifer Schultens

Knots

Definition

A knot K in $\mathbb{S}^3$ is a smooth isotopy class of smooth embeddings of $\mathbb{S}^1$ into $\mathbb{S}^3$.

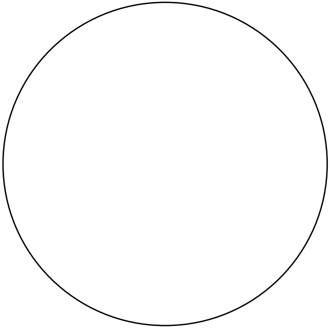
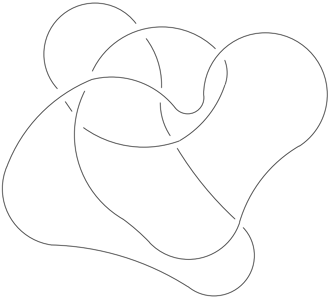


Figure: The unknot

Knots



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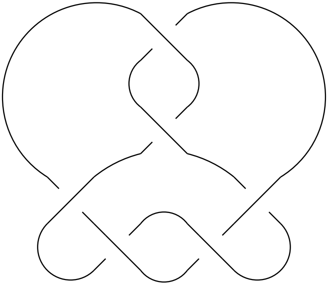

Figure: A knot

Knots

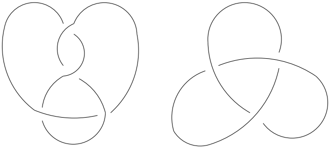

Figure: Two knots

Knots

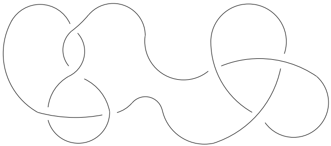

Figure: The connected sum of two knots

Crossing number

Definition

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Crossing numbers of knots

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Example 1: The unknot has crossing number 0.

Crossing numbers of knots

Example 2: The trefoil has crossing number 3.

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This can be seen by checking all diagrams with crossing number $0, 1, 2$ and noting that they do not give the trefoil. (Except that you need to know that the trefoil is truly knotted.)

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Theorem

A reduced alternating diagram of a knot K realizes the crossing number of K.

Fun interlude

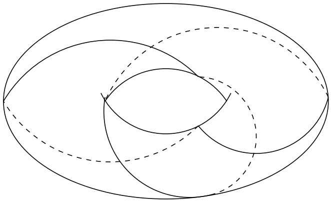

Figure: T(2, 3), also known as the trefoill

Question 1: What is the crossing number of a torus knot, $T(p, q)$?

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Theorem:

$$c(T(p,q)) = min\{q(p-1), p(q-1)\}$$

Fun interlude

Question 2: What is known about the crossing number of a satellite knot?

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Nothing!

Bridge number

Bridge numbers of knots

Definition

A *height function* on $\mathbb{R}^3$ (or $\mathbb{S}^3$) is a smooth function

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(or $h : \mathbb{S}^3 \to \mathbb{R}$) without critical points.

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Let K be a knot. The *bridge number*, $b(K)$, of K, is the least possible number of maxima of K with respect to a height function.

Bridge numbers of knots

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Theorem

(Schubert): $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$

Bridge numbers of knots

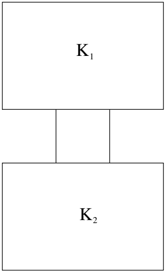
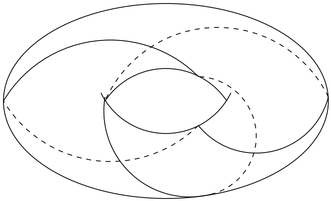


Figure: Schematic for bridge number inequality

Fun interlude



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Question 2: What can we say about the bridge number of a satellite knot?

Answer: It's at least the bridge number of the companion knot times the wrapping number of the pattern.

Tunnel number

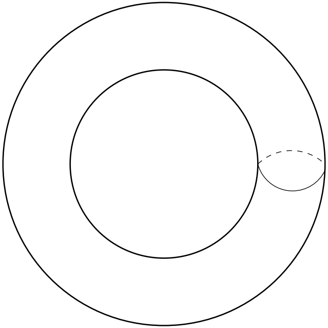

Figure: A handlebody

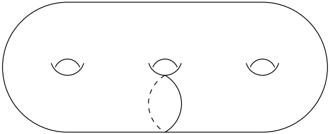

Figure: Another handlebody

Tunnel numbers of knots

Definition

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Let K be a knot. A *tunnel system* for K is a collection of arcs $a_1, \ldots, a_n$ properly embedded in $(\mathbb{S}^3, K)$ such that $\mathbb{S}^3 - \eta(K \cup a_1 \cup \cdots \cup a_n)$ is a handlebody.

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Definition

The *tunnel number* of K is the least number of arcs in a tunnel system for K.

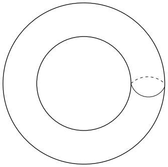
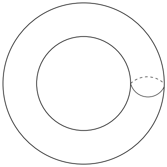


Figure: The complement of the unknot

The complement of the unknot is a solid torus, which is a 3-dimensional regular neighborhood of the circle (a graph), hence a handlebody.



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Why?

Preliminaries for tunnel number

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$$\eta(unknot) \approx \{(x, y, z, w) | x^2 + y^2 = \frac{1}{2}, z > \frac{1}{2\sqrt{2}}, w < \frac{1}{2\sqrt{2}}\} \in \mathbb{S}^3$$

$$C(K) = \mathbb{S}^3 - \eta(unknot)$$

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Example 3: Every 2-bridge knot has a tunnel number 1.

Tunnel numbers of knots

Observation: $t(K) \leq b(K) - 1$

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Question: What happens to tunnel number under the operation of connected sum?

Preliminaries for tunnel number

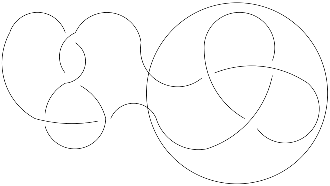

Figure: The decomposing annulus

Tunnel numbers of knots

Theorem

$$t(K\#K') \leq t(K) + t(K') + 1$$

Tunnel numbers of knots

Theorem

(Morimoto) There is a knot, K, such that for any 2-bridge knot K',

$$t(K\#K') = t(K')$$

$$"2 + 1 = 1"$$

Theorem

(Morimoto-Sakuma-Yokota) There are knots, K_1, K_2, such that

$$t(K\#K') = t(K) + t(K') + 1$$

$$"1 + 1 = 3"$$

Tunnel numbers of knots

Theorem

(Morimoto-S) If K_1, K_2 *and* K_2 *are small, then*

$$t(K_1 \# K_2) \geq t(K_1) + t(K_2)$$

Tunnel numbers of knots

Theorem

(Scharlemann-S) If $K_1, \ldots, K_n$ are knots and no K_i is a 2-bridge knot, then

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(Scharlemann-S) If $K_1, \ldots, K_n$ are knots such that $K_1, \ldots, K_p$ are 2-bridge knots and $K_{p+1}, \ldots, K_n$ are not, then

$$t(K_1 \# \ldots \# K_n) \geq \frac{p}{5} + \frac{2}{5}(t(K_{p+1}) + \cdots + t(K_n))$$

Unknotting numbers of knots

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Unknotting numbers of knots

Theorem

$$u(K) \leq c(K)$$

Theorem

(Scharlemann) Unknotting number 1 *knots are prime.*

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The (nonorientable) genus of the surface is exactly the unknotting number of K.

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Answer: Nobody knows.