Not Knot

Jennifer Schultens

How low-dimensional topologists view knots

1-dimension: The line $\{x\}$, $\mathbb{R}$
2-dimensions: The plane $\{(x, y)\}$, $\mathbb{R}^2$
3-dimensions: The plane $\{(x, y, z)\}$, $\mathbb{R}^3$

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Spheres in every dimension:

$$\mathbb{S}^0 = \{x \in \mathbb{R} | x^2 = 1\},$$
$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\},$$
$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\},$$
$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}.$$

Spheres have hemispheres.

Definition

Let $f : X \to Y$ be an injective continuous function. An *isotopy* of f is a continuous function $F : X \times [0,1] \to Y$ such that

- $F(x,0) = f(x)\ \forall x \in X$;
- $F(-,t) \to Y$ is injective $\forall t \in [0,1]$.

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Knots

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Definition

A smooth function is an *embedding* if it is injective and its derivative is injective at everywhere.

Definition

A *knot* is a smooth isotopy class of smooth embeddings of $\mathbb{S}^1$ into $\mathbb{S}^3$.

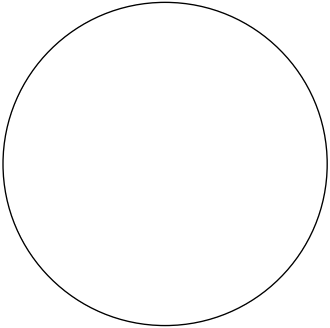


Figure: The unknot

Knots

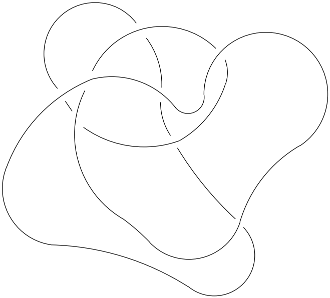


Figure: Also the unknot

Knots

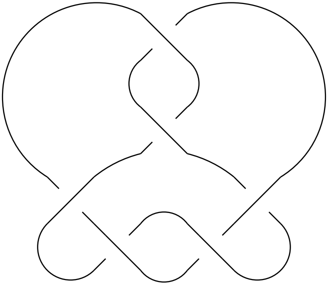


Figure: A knot

Knots

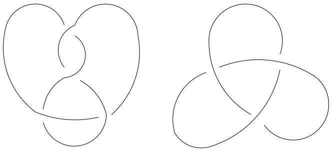


Figure: Two knots (the Figure 8 knot and the trefoil)

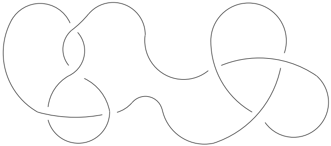


Figure: The connected sum of two knots

Not knot

Definition

Let K be a knot. The *complement* of K is

$$C(K) = \mathbb{S}^3 - \eta(K)$$

where $\eta(K)$ is a regular neighborhood of K.

Not knot

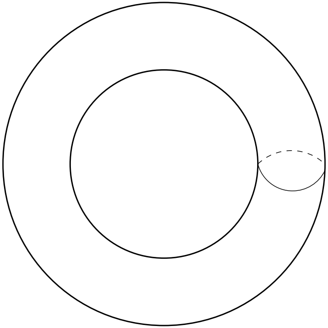


Figure: The complement of the unknot

Not knot

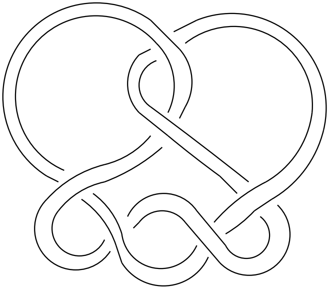


Figure: The complement of a knot

Not knot

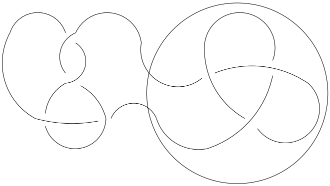


Figure: A decomposing annulus in the connected sum of two knots

The torus

Torus knots

The torus

$$\mathbb{T}^2 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 = \frac{1}{2}, w^2 + z^2 = \frac{1}{2}\}$$

$$= \mathbb{S}^1 \times \mathbb{S}^1$$

$$\mathbb{R}^2 / \sim \quad \textit{where} \;\; (x, y) \sim (x + 1, y) \;\; \textit{and} \;\; (x, y) \sim (x, y + 1)$$

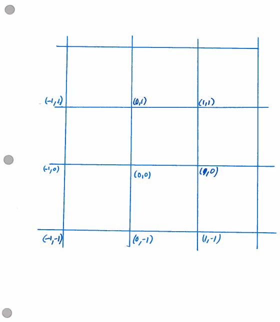
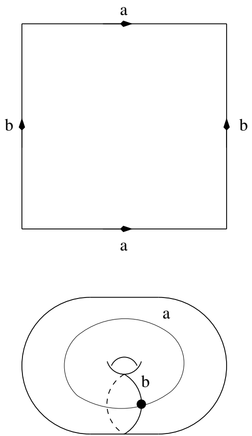


Figure: The torus as a quotient space

The torus



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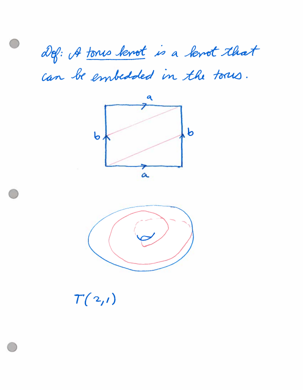

Figure: A torus knot is a knot that can be embedded on the torus

Torus knots

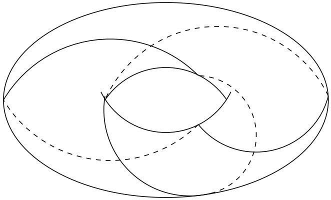

Figure: T(2, 3), also known as the trefoill

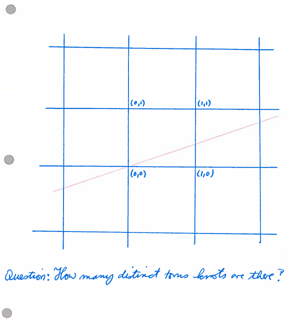

Figure: How many torus knots are there?

Torus knots

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The torus knot K can leave the torus. We imagine the isotopy from $T(p, q)$ to $T(q, p)$ as puncturing the torus, flipping the two wings inside out and closing it up again. The knot has been isotoped as required.

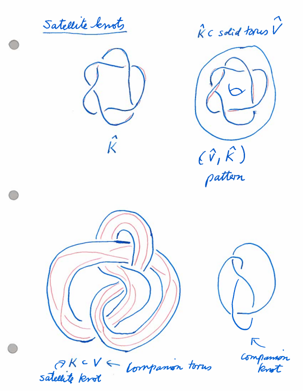

Figure: Satellite knots

Wrapping number

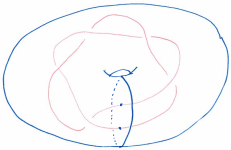

Figure: Wrapping number

Definition

The *wrapping number* of the pattern is the minimal number of times that $\hat{K}$ meets a meridian disk of $\hat{V}$. (Geometric intersection number)

Definition

A *satellite knot* is a knot resulting from this construction when the pattern, the wrapping number, and the companion knot are all non trivial.

Satellite knots

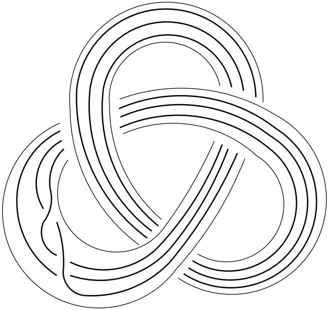

Figure: A satellite knot

Connected sum of knots as a special case of satellite knots

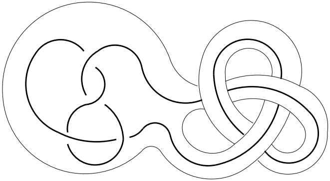

Figure: A swallow-follow torus

Manifolds

Definition

A *homeomorphism* is a continuous map with a continuous inverse. Two spaces are *homeomorphic* if there is a homeomorphism between them.

Definition

A n-dimensional *manifold* is a paracompact Hausdorff space in which every point has a neighborhood homeomorpic to $\mathbb{R}^n$.

Manifolds

Example 1: $\mathbb{R}$ and $\mathbb{S}^1$ are 1-dimensional manifolds.

Example 2: $\mathbb{R}^2$ and $\mathbb{S}^2$ are 2-dimensional manifolds.

Example 3: $\mathbb{R}^3$ and $\mathbb{S}^3$ are 3-dimensional manifolds.

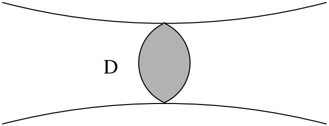

Figure: A “compressible” surface

Definition

A surface S (think torus) in a 3-manifold M (think $C(K)$ is *incompressible* if every simple closed curve in S that bounds a disk in M also bounds a disk in S.

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A surface S (think torus) in a 3-manifold M (think $C(K)$ is *essential* if it is incompressible and not boundary parallel.

Observation: The complement of a satellite knot contains an essential torus.

Fun interlude

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We can project onto the vertical axis. The *bridge number* of a knot is the least possible number of maxima.

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Question 2: What can we say about the bridge number of a satellite knot?

(Both questions are answered by theorems of Schubert.)

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Question 2: What can we say about the bridge number of a satellite knot?

Answer: It's at least the bridge number of the companion knot times the wrapping number of the pattern.

Thurston's theorem

Theorem

(Trichotomy for knots) Let K be a knot in $\mathbb{S}^3$. Then exactly one of the following holds:

- *K is a torus knot;*
- *K is a satellite knot;*
- *K is hyperbolic.*

A knot is hyperbolic, if its complement can be constructed from small pieces of hyperbolic space. By Thurston's theorem, this is equivalent to being anannular and atoroidal.

Thurston's geometrization conjecture

Thurston's theorem concerning the trichotomy for knots served as a precursor and motivation for his geometrization conjecture. Thurston's geometrization conjecture states that every 3-manifold can be cut along essential spheres and tori so that the resulting pieces that have one of 8 possible geometries.

(Spherical, Euclidean, hyperbolic, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, the universal cover of $SL(2, \mathbb{R})$, Nil geometry, Solv geometry.)