Harish-Chandra characters and the local Langlands correspondence

Tasho Kaletha

University of Michigan

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Global Langlands correspondence

Galois representations

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Application

- Langlands 1980: Proves many cases of the 2-dimensional Artin conjecture

General reductive groups

Global Langlands correspondence

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The groups

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Local representation theory

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$G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ ought to behave similarly

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π admissible $\leftrightarrow (M, \sigma, \nu)$, $M \subset G$ Levi, $\sigma \in \mathrm{Irr}(M)$ tempered

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- Langlands: φ

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3. $\Theta_\pi(s) = (-1)^{q(G)} \sum_{w \in N(S,G)(\mathbb{R})/S(\mathbb{R})} \frac{\theta(\gamma^w)}{\prod_{\alpha>0}(1-\alpha(\gamma^w)^{-1})}$

Highest weight theory

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Local Langlands correspondence

- Langlands: $\varphi \to (S, B, \theta)/G(\mathbb{C}) \to \{\pi_1, \dots, \pi_k\}$
- $\Theta_{\pi_1} + \dots + \Theta_{\pi_k}$ conjugation invariant under $G(\mathbb{C})$

Endoscopy: $G(\bar{F})$-conjugacy vs. $G(F)$-conjugacy

Geometric side

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Supercuspidal representations

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Yu's construction 2001

$$\left\{ \begin{array}{c} (G^0 \subset G^1 \subset \cdots \subset G^d = G) \\ \pi_{-1} \\ (\phi_0, \phi_1, \ldots, \phi_d) \end{array} \right\} \xrightarrow{\text{J.K.Yu}} \{\text{irred. s.c reps of } G(F)\}$$

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4. Fintzen 2018: Surjective for $p \nmid |W|$ and in positive characteristic.

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3. S_φ no longer abelian, structure of $\Pi_\varphi(G)$ more complex

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From $\varphi : W_F \to {}^L G$ discrete and regular we obtain *canonically*

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Each $j : S \to G$ provides unique irrep π_j of $G(F)$ by

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From $\varphi : W_F \to {}^L G$ discrete and regular we obtain *canonically*

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- Main Challenge: Construct $\boxed{\mathrm{Irr}(S_\varphi) \leftrightarrow \Pi_\varphi(G)}$

Geometric intertwining operators

Reductions

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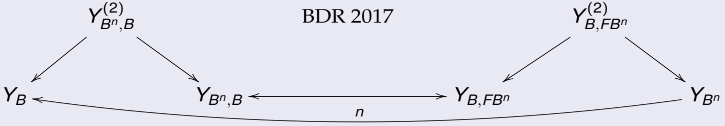
BDR 2017



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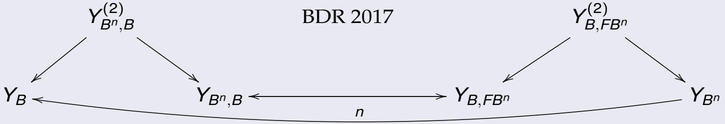

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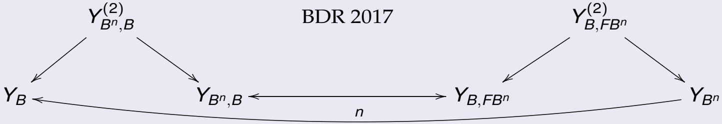

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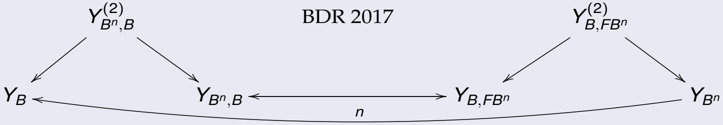

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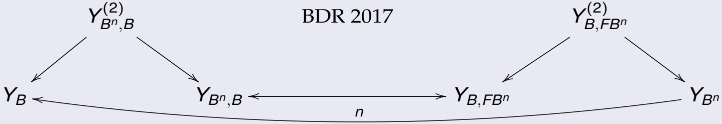

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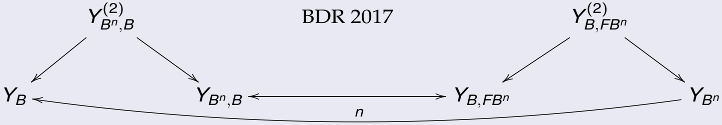

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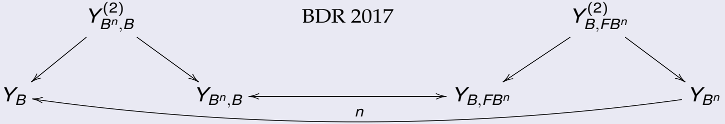

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Real discrete series

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3. K.-Weinstein 2017: Partial results.

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Let $G' \subset G$ be an elliptic twisted Levi subgroup, $\varphi : W_F \to {}^LG'$.
Expect for $\gamma \in G'(F)$ elliptic that $S\Theta_{\varphi,G}(\gamma)$ equals

$$e(G')e(G)\epsilon(X^*(T)_\mathbb{C} - X^*(T')_\mathbb{C}, \Lambda) \sum_{w \in W(G',G)(F)} \Delta_{II}^{G/G'}(\gamma^w) S\Theta_{\varphi,G'}(\gamma^w).$$