Residue distributions and harmonic analysis on reductive groups

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- Let V be an oriented Euclidean vector space of dimension n, with complexification $V_{\mathbb{C}}$.
- Let $\mathcal{A}$ be a finite arrangement of affine hyperplanes $H \subset V$, with complexification $\mathcal{A}_{\mathbb{C}}$.
- Let $P(V_{\mathbb{C}})$ denote the space of Paley-Wiener functions on $V_{\mathbb{C}}$, that is $\varphi \in P(V_{\mathbb{C}})$ iff there exists a constant $R > 0$, and for every $N \in \mathbb{N}$ a constant $C_N > 0$ such that for all $z \in V_{\mathbb{C}}$ we have $|\varphi(z)| \leq C_N(1 + \|z\|)^{-N} e^{R\|\mathrm{Re}(z)\|}$.

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- Let $b \in V \setminus \cup_{H \in \mathcal{A}} H$ and let $X^{\omega,b} : P(V_c) \to \mathbb{C}$ be the linear functional on $P(V_{\mathbb{C}})$ defined by

$$X^{\omega,b}(\varphi) := \int_{\mathrm{Re}(z)=b} \varphi(z)\omega(z).$$

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- Such linear functionals $X^{\omega,b}$ (or slight variations thereof) often arise in harmonic analysis on reductive groups, in the study of "residual contributions" to the spectrum. Our first goal is a basic decomposition theorem for $X^{\omega,b}$ in terms of tempered distributions with certain support conditions.

- For $H \in \mathcal{A}_{\mathbb{C}}$, let $n_H \in \mathbb{Z}$ denote the order of ω along H. For $L \in L(\mathcal{A})$, the intersection semilattice of $\mathcal{A}$, we define

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- Examples:
  (1) V itself is a residual subspace.
  (2) If $H \in \mathcal{A}$ with $n_H < 0$ then H is residual ($o_H = n_H + 1 \leq 0$).

- If $L \in L(\mathcal{A})$ is residual, then we define $V_L \subset V$ as the linear subspace underlying the affine subspace $L \subset V$, and $V^L = (V_L)^\perp$ (the subspace spanned by the lines orthogonal to the hyperplanes of poles $H \in \mathcal{A}$ such that $L \subset H$).

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Proposition[Heckman, O.]

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(a) $\mathrm{Supp}(X_c^b) \subset \bigcup_{L \text{ residual } : c_L = c} L^{temp}$.

(b) For all $\varphi \in P(V_{\mathbb{C}})$ we have: $X^{\omega,b}(\varphi) = \sum_{c \in \mathcal{C}} X_c^b(\varphi|_{c+iV})$.

- Observe that $\varphi|_{c+iV} \in \mathcal{S}(c + iV)$, hence the expression $X_c^b(\varphi|_{c+iV})$ is meaningful.

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- Example: Let $V = \mathbb{R}$ and $\omega = \frac{dx}{x-c}$ with $c \in \mathbb{R}$.
  - If $c \neq 0$ then $X_c^b = \mathrm{sign}(c)2\pi i\delta_c$ if c separates b and 0, and $X_c^b = 0$ otherwise. Moreover $X_0^b = (x-c)^{-1}|_{i\mathbb{R}}$.

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  - If $c = 0$ and $\pm b > 0$ then $X_0^b = \mathrm{Pf}(x^{-1}|_{i\mathbb{R}}) \pm \pi i \delta_0$.
- Example: More generally: Let $V = \mathbb{R}^n$. Let $\alpha_1, \ldots, \alpha_n$ be a basis of V, and let $c \in V$. Let $\mathcal{A} = \{H_i\}_{i=1}^n$ with $H_i = \{x \mid (\alpha_i, x) = (\alpha_i, c)\}$. Let $\omega = \frac{dx_1 \wedge \cdots \wedge dx_n}{\prod_{i=1}^n ((\alpha_i, x) - (\alpha_i, c))}$. Then $X^{\omega, b}$ only depends on the chamber C of $V \setminus \cup_i H_i$ such that $b \in C$. Assume that $0 \in V$ is regular in the dual chamber decomposition with center c. Then $X_c^b \neq 0$ iff $0 \in -C^*$, the anti-dual chamber of C, and

$$X_c^b = \pm \frac{(2\pi i)^n}{\det(\alpha_i, \alpha_j)^{1/2}} \delta_c$$

Two cases of interest

- Now let $G \supset B \supset T$ be a connected reductive group over $\mathbb{C}$, with Borel subgroup B and maximal torus T. Let $V \subset \mathfrak{g}$ be the real span of the cocharacter lattice X of T. Let $\Sigma^\vee$ be the root system of G.

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- Define a rational function on V by $c(\lambda) = \prod_{\alpha \in \Sigma_+^\vee} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$.

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- Define a rational function on V by $c(\lambda) = \prod_{\alpha \in \Sigma_+^\vee} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$.
- We would like to understand the following functionals: For $\varphi \in P(V_{\mathbb{C}})$ and b deep in the Weyl chamber, define:

$$X^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda)\omega^X(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(-\lambda)}$$

and

$$Y^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda)\omega^Y(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

Positivity

Special properties of Y^b:

Theorem

For all $L \subset V$, affine subspace, let $o_L^Y = n_L^Y + \text{codim}(L)$ with n_L^Y the pole order of ω^Y along L. Then $o_L^Y \geq 0$. In particular, L is ω^Y-residual iff $o_L^Y = 0$, or equivalently:

$$|\{\alpha \in \Sigma \mid \alpha^\vee|_L = 1\}| = |\{\alpha \in \Sigma \mid \alpha^\vee|_L = 0\}| + \text{codim}(L)$$

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Theorem[Heckman, O.]

Let $\mathcal{C}^Y \subset V$ denote the set of centers of ω^Y-residual subspaces. For all $c \in \mathcal{C}^Y$, Y_c^b is a nonnegative measure supported by $\cup_{L:c_L=c} L^{temp} \subset c + iV$.

The support theorem

Algebraic description of the support of the Y_c:

Theorem

For all $c \in \mathcal{C}^Y$, there exists $w \in W$ such that $Y^b_{wc} \neq 0$. In this case $w(c)$ is in the dual chamber of V_+.

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The support of Y^b in terms of nilpotent orbits

The set $\mathcal{C}^Y$ is W-invariant, and $c \in \mathcal{C}^Y \cap V_+$ iff there exists a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ such that $c = \lambda_{\mathcal{O}}$, where $\lambda_{\mathcal{O}}$ is half the weighted Dynkin diagram of $\mathcal{O}$. Hence there is a canonical bijection between $W \backslash \mathcal{C}^Y$ and the set of nilpotent orbits of $\mathfrak{g}$.

Symmetrization of the X-distribution

Using the uniqueness of residue distributions it is easy to express the local X-distributions in terms of those of Y:

Theorem

Let $f \in P(V_{\mathbb{C}})$. For every $c \in V_+$ and $w \in W$ we have

$$X^b_{wc}(f|_{wc+iV}) = Y^b_c((A_{wc}(f) \circ w)|_{c+iV})$$

where $A_{wc}(f) \in P(V_{\mathbb{C}})$ is defined by (for $\lambda \in V^{reg}_{\mathbb{C}}$):
$A_{wc}(f)(\lambda) = \frac{1}{|W_{wc}|} \sum_{u \in W_{wc}} c(u\lambda) f(u\lambda)$ (the symmetrization operator).

Spectral decomposition of affine Hecke algebras

We now consider a variation of the previous, closely linked to harmonic analysis on p-adic reductive groups. This helps to understand the distributions X and Y better, both their properities and their applications.

- Let k be a p-adic field, and let $G'(\supset B' \subset T')$ be the group of k-points of a connected split reductive group defined over k with based root datum dual to G. Let $\mathbb{I} \subset G'$ be an Iwahori subgroup for an alcove C of the apartment of T', and let $\mathcal{H}_{\mathbb{I}} = C_c^\infty(G'//\mathbb{I})$ the Iwahori Hecke algebra.

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- $\mathcal{H}_{\mathbb{I}}$ is an affine Hecke algebra for the (extended) affine Weyl group $W^e \simeq W \ltimes X^*(T)$, with parameter $q = v^2$, the cardinality of the residue field of k.

The affine Hecke algebra as a Hilbert algebra

- Denote by $S^a \subset W^e$ the set of simple affine reflections defined by the alcove C. $\mathcal{H}_{\mathbb{I}}$ has a Coxeter basis $\{T_w\}_{w \in W^e}$ such that for all $s \in S^a$:

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  - $T_s T_w = T_{sw}$ if $l(sw) = l(w) + 1$.
  - $T_s T_w = (v - v^{-1}) T_w + T_{sw}$ else.
- We equip $\mathcal{H}_{\mathbb{I}}$ with the trace $\tau \in (\mathcal{H}_{\mathbb{I}})^*$ defined by $\tau(T_w) = \delta_{w,e} \mathrm{vol}(\mathbb{I})^{-1}$ and the anti-linear anti-isomorphism $* : \mathcal{H}_{\mathbb{I}} \to \mathcal{H}_{\mathbb{I}}$ given by $T_w^* = T_{w^{-1}}$. Then the Hermitian inner product on $\mathcal{H}_{\mathbb{I}}$ defined by $(x, y) = \tau(x^* y)$ is positive, and its basis $\{T_w\}_{w \in W}$ is orthonormal.

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- This gives the $*$-algebra $\mathcal{H}_{\mathbb{I}}$ the structure of a Hilbert algebra of type I with trace τ.

Plancherel measure and affine Hecke algebras

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- It was shown by Bushnell, Henniart and Kutzko (much more generally, in the context of types) that ν_{Pl} corresponds to the Plancherel measure of G' restricted to the Borel component of Iwahori-spherical representations.

Bernstein's basis of the Hecke algebra

- Let $\mathcal{A} = \mathbb{C}[T]$ and let $\mathcal{H}_0 = C_c^\infty(\mathbb{K}//\mathbb{I})$ (the finite Hecke algebra of the Coxeter group (W, S) and parameter $q = v^2$). Bernstein's basis theorem: There exist injective algebra homomorphisms $\mathcal{A} \to \mathcal{H}_\mathbb{I}$ and $\mathcal{H}_0 \to \mathcal{H}_\mathbb{I}$ such that the multiplication maps $\mathcal{H}_0 \otimes \mathcal{A} \to \mathcal{H}_\mathbb{I}$ and $\mathcal{A} \otimes \mathcal{H}_0 \to \mathcal{H}_\mathbb{I}$ are isomorphisms of vector spaces.

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- A well known Corollary is $Z(\mathcal{H}_\mathbb{I}) = \mathcal{A}^W = \mathbb{C}[T]^W$.
- Define $\Delta \in \mathcal{A} = \mathbb{C}[T]$ by $\Delta(t) = \prod_{\alpha\in\Sigma}(1 - \alpha^\vee(t)^{-1})$, and its "$q$-deformation" $\Delta_q(t) = \prod_{\alpha\in\Sigma}(1 - q^{-1}\alpha^\vee(t)^{-1})$.

Two functionals on $\mathcal{A}$

Definition

We define two functionals for $\varphi \in \mathcal{A} = \mathbb{C}[T]$, and $b \in T_{v,-}$ (deep in the negative chamber):

$$\mathcal{X}^b(\varphi) = \int_{t \in bT_u} \varphi(t) \frac{\Delta(t)}{q(w_0)\Delta_q(t)} dt$$

and

$$\mathcal{Y}^b(\varphi) = \int_{t \in bT_u} \varphi(t)\mu(t)dt := \int_{t \in bT_u} \varphi(t) \frac{\Delta(t)\Delta(t^{-1})}{q(w_0)\Delta_q(t)\Delta_q(t^{-1})} dt,$$

where $q(w_0) = q^{|\Sigma_+|}$.

The relation between $\mathcal{X}^b$ and $\mathcal{Y}^b$ is given by a process of symmetrization.

Residual cosets

Definition

Let $\eta(t) = \mu(t)dt = \frac{\Delta(t)\Delta(t^{-1})}{q(w_0)\Delta_q(t)\Delta_q(t^{-1})}dt$. For any coset $L = pT_L$ of a subtorus $T_L \subset T$, let $n_L^{\mathcal{Y}}$ be the pole order of μ along L. We call L η-residual if

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- L is a connected component of $\cap_{\alpha \in \Sigma | L \subset H_\alpha} H_\alpha$, with $H_\alpha = \{t \in T \mid \alpha^\vee(t) = q\}$.
- We have $o_L^{\mathcal{Y}} := n_L^{\mathcal{Y}} + \text{codim}(L) \leq 0$.

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Theorem

$L = pT_L$ is η-residual iff $o_L = 0$.

The tempered form of a residual coset

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Let $L \subset T$ be η-residual. Let $T^L \subset T$ be the subtorus such that Lie(T^L) is spanned by the $\alpha \in \Sigma$ such that $\alpha^\vee|_L = q$.

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Theorem

Let $\mathcal{C}^{\mathcal{Y}} \subset T_v$ denote the set of centers of the $\eta = \mu dt$-residual cosets. For each $c \in \mathcal{C}^{\mathcal{Y}}$ there exists a unique distribution $\mathcal{Y}_c^b$ on cT_u with $\mathrm{supp}(\mathcal{Y}_c^b) \subset \cup_{L:c_L=c} L^{temp}$ such that: $\forall a \in \mathcal{A} = \mathbb{C}[T]$:

$$\tau(a) = \mathcal{Y}^b(a) = \sum_{c \in \mathcal{C}^{\mathcal{Y}}} \mathcal{Y}_c^b(a|_{cT_u})$$

The Plancherel measure and residue distributions

Theorem (The trace of $\mathcal{H}_{\mathbb{I}}$ expressed in terms of $\mathcal{Y}^b$)

There exists a holomorphic family $T \ni t \to E(\cdot, t) \in (\mathcal{H}_{\mathbb{I}})^*$ of matrix coefficients $E(\cdot, t)$ of the minimal principal series $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}_{\mathbb{I}}}(t)$ of $\mathcal{H}_{\mathbb{I}}$ such that, for b deep in $T_{v,-}$: $\tau(h) = \mathcal{Y}^b(\frac{E(\cdot,h)}{q(w_0)\Delta(\cdot)})$.

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Restriction of τ to $\mathcal{A}$ and the center $Z(\mathcal{H}_{\mathbb{I}}) \subset \mathcal{A}$

For $a \in \mathcal{A}$ we have $\tau(a) = \mathcal{Y}^b(a)$. (Note that $Z(\mathcal{H}_{\mathbb{I}})$ is a $*$-subalgebra, but in general $\mathcal{A}$ is not a $*$-subalgebra).

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Restriction of τ to $\mathcal{H}_{\mathbb{I}}^{gen} := e_-\mathcal{H}_{\mathbb{I}}e_- = e_-Z(\mathcal{H}_{\mathbb{I}})$

Let $P_W(q) = \sum_{w \in W} q^{l(w)}$. Let $e_- \in \mathcal{H}_0$ be the idempotent of the sign character. On the abelian $*$-subalgebra $\mathcal{H}_{\mathbb{I}}^{gen} \simeq e_-\mathcal{A}e_-$ we have, for all $a \in \mathcal{A} = \mathbb{C}[T]$: $P_W(q^{-1})\tau(ae_-) = \mathcal{X}^b(a)$.

Residue distributions and Plancherel measures

Corollary

These interpretations of $\mathcal{X}^b, \mathcal{Y}^b$ as functionals representing the positive trace τ on various $*$-subalgebras of $\mathcal{H}_{\mathbb{I}}$, have strong implications: The "local distributions" $\mathcal{Y}_c^b$ are positive measures.

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Remark

The $\{\mathcal{Y}_c^b\}_{c \in \mathcal{C}^{\mathcal{Y}}}$ are instrumental to compute ν_{Pl}. Conversely $\nu_{Pl}^{gen} := \nu_{Pl}|_{\widehat{\mathcal{H}_{\mathbb{I}}^{gen}}}$ determines $\{\mathcal{X}_c^b\}_{c \in \mathcal{C}^{\mathcal{X}}}$ and $\{\mathcal{Y}_c^b\}_{c \in \mathcal{C}^{\mathcal{Y}} \cap T_{v,-}}$

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Remark

The functional Y^b can be computed explicitly by a limit process for $q \to 1$ of $\mathcal{Y}^b$ and a twist by an involution.

Symmetrization of $\mathcal{X}^b$; relation to $\mathcal{Y}^b$

Let $a \in \mathcal{A}$ and $t \in T$. Let χ_t denote the character of $\mathcal{H}_{\mathbb{I}}$ of the minimal principal series $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}_{\mathbb{I}}}(t)$. Let $\phi_t^-(\cdot) := \chi_t(\cdot e_-) \in (\mathcal{H}_{\mathbb{I}}^{gen})^*$ denote the "elementary skew-spherical function" of $\mathcal{H}_{\mathbb{I}}^{gen} = e_-\mathcal{H}_{\mathbb{I}}e_- = Z(\mathcal{H}_{\mathbb{I}})e_- \simeq Z(\mathcal{H}_{\mathbb{I}})$ at character $Wt \in W\backslash T$. Let $\nu_{WL} = \sum_{L' \in WL} \nu_{L'}$ be the unique W-invariant measure supported on WL^{temp} such that $\tau(ae_-) = \frac{1}{P(q^{-1})}\mathcal{X}^b(a)$ can be rewritten as:

$$\tau(ae_-) = \sum_{c \in \mathcal{C}_-^{\mathcal{Y}}} \frac{1}{|W_c|}\mathcal{Y}_c^b(\phi_t^-(a)) = \sum_{L \text{ residual}} \int_{L^{temp}} \phi_t^-(a) d\nu_L$$

In particular: $\sum_{L \text{ residual}} \nu_L = \nu_{Pl}^{gen} := \nu_{Pl}|_{\widehat{\mathcal{H}_{\mathbb{I}}^{gen}}}$.

The explicit computation of ν_{Pl}

Theorem (The support of $\mathcal{Y}^b$; "no cancellation Theorem")

The support $\mathcal{S}^{\mathcal{Y}} \subset W\backslash T$ of the restriction of $\mathcal{Y}^b = \sum_{c \in \mathcal{C}^\eta} \mathcal{Y}_c^b$ to $Z(\mathcal{H}_{\mathbb{I}}) = \mathbb{C}[T]^W$ is exactly equal to $\mathcal{S}^{\mathcal{Y}} := \cup_{L \text{ residual}} L^{temp}$.

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  - Dirac induction.

Dirac induction; parameter families of discrete series

- Let Q be set of positive real Hecke algebra parameters for $\mathcal{H} = \mathcal{H}(W^e)$, i.e. the real vector group of tuples $(\mathrm{v}_s)_{s \in S^a}$ of positive real parameters v_s such that $\mathrm{v}_s = \mathrm{v}_{s'}$ if $s \sim_{W^e} s'$.

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- Let $R_{\mathbb{Z}}^{ell}(W^e)$ denote the lattice of elliptic characters of W^e with respect to the action of W^e on V, and equipped with the Euler-Poincaré paring.

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- There exists an ON set $B^{mass} = \{b_1, \ldots, b_m\} \subset R_{\mathbb{Z}}^{ell}(W^e)$ of “positive generically massive unit vectors”.

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- $\forall b \in B^{mass}$ we have an orbit of residual points $Wr_b \in W \backslash T$ with values in $\mathbb{C}[v_s^{\pm 1} \mid v_s = v_{s'} \text{ if } s \sim_{W^e} s']$ and an explicit rational function m_{Wr_b} on Q which is regular on Q.

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- $\forall b \in B^{mass}$ and $\mathrm{v} = (\mathrm{v}_s)_{s \in S^a} \in Q$ such that $m_{Wr_b}(\mathrm{v}) \neq 0$ we have a virtual discrete series character $\mathrm{Ind}_D^{an}(b, \mathrm{v})$ of $\mathcal{H}_{\mathrm{v}}$ with central character $Wr_b(\mathrm{v})$ such that $[\lim_{\epsilon \to 0} \mathrm{Ind}_D^{an}(b, \mathrm{v}^{\epsilon})] = b \in B^{mass}$.

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- $\pm \mathrm{Ind}_D^{an}(b, \mathrm{v})$ is irreducible, and this parameterizes the irr. DS characters of $\mathcal{H}_{\mathrm{v}}$ by $\{b \in B^{mass} \mid m_{Wr_b}(\mathrm{v}) \neq 0\}$.

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- $\forall b \in B^{mass}$ there exists $c_b \in \mathbb{Q}^\times$ such that for all $\mathrm{v} \in Q$: $\mathrm{fdeg}(\mathrm{Ind}_D^{an}(b, v)) := \nu_{Pl}(\{\mathrm{Ind}_D^{an}(b, v)\}) = c_b m_{Wr_b}(v)$.

Application 1: The conjecture of Hiraga, Ichino and Ikeda for representations of unipotent reduction

Definition (Tempered unramified Langlands parameters)

Let G' be connected reductive over k, with dual group ${}^LG' = G \rtimes \langle\theta\rangle$. An unramified Langlands parameter of G' is a homomorphism:

$$\varphi : \langle \text{Frob} \rangle \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG' = G \rtimes \langle\theta\rangle$$

satisfying certain natural conditions (depending on G'). We call φ tempered if $\langle \text{Frob} \rangle$ has bounded image. The set of equivalence classes of tempered unramified parameters is denoted by $\Phi_{un}^{temp}(G')$.

Tempered unramified Langlands parameters

Lusztig defined the class of representations of unipotent reduction of G'. For G' simple over k (of adjoint type, split over an unramified extension) he classified such tempered characters in terms of $\Phi_{un}^{temp}(G')$.

The support and tempered unramified Langlands parameters

Let G' be quasisplit. We identify $W\backslash T$ canonically with the set of semisimple G-orbits in $G\theta$. Put $\widetilde{\mathrm{Frob}} := (\mathrm{Frob}, \begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix})$. The map

$$\Phi(G')^{temp} \ni [\varphi] \to W\varphi(\widetilde{\mathrm{Frob}}) \in \mathcal{S}^{\mathcal{Y}} \subset W\backslash T$$

is a bijection $\Phi_{un}^{temp}(G') \xrightarrow{\sim} \mathcal{S}^{\mathcal{Y}}$, with inverse $\chi = Wt \to [\varphi_\chi]$.

Fix an additive character ψ of k, and normalize the Haar measure $\mu^{\psi}_{G'}$ of G' accordingly.

Theorem [Reeder; O.] (The conjecture of HII for representations of G' of unipotent reduction)

Suppose that G' is of adjoint type. Let φ be an discrete unramified Langlands parameter of G', and let $S_\varphi = \{s \in G \mid \mathrm{Ad}(s) \circ \varphi = \varphi\}$. Let $\chi_{G'} \in ({}^L Z)^*$ be the character corresponding to the inner class $[G'] \in H^1(k, (G')^*)$, extended to Z. Put $\Pi(\mathcal{S}_\varphi, \chi_{G'}) = \{\rho \in \mathrm{Irr}(\mathcal{S}_\varphi) \mid \rho|_Z = n\chi_{G'}\}$. Let $\pi_\rho \in \Pi_\varphi$ be the discrete series representation of unipotent reduction of G' according to Lusztig's parameterization. Then

$$\mathrm{fdeg}(\pi_\rho)(= \nu_{Pl}(\{\pi_\rho\})) = \frac{\dim(\rho)}{|\mathcal{S}_\varphi|} |\gamma(0, \mathrm{Ad} \circ \varphi, \psi)| \qquad (1)$$

where $\gamma(0, \mathrm{Ad} \circ \varphi, \psi)$ denotes the γ-factor of $\mathrm{Ad} \circ \varphi$ at $s = 0$.

Uniqueness properties

- Many other partial results known: Silberger-Zink, Hiraga-Saito, Hiraga-Ichino-Ikeda, Beuzart-Plessis, Moeglin-Waldspurger, Ichino-Lapid-Mao, ...

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- Mark Reeder (1994) that for split exceptional groups, the formal degree of a discrete series representation π of G' of unipotent reduction uniquely determines an unramified discrete Langlands parameter such that (1) holds.
- This observation generalizes to general tempered representations of unipotent reduction (with Y. Feng and M. Solleveld). Let G' be reductive over k, and split over an unramified extension of k. There exists an essentially unique map (i.e. up to twisting by automorphisms) $\Pi_{unip}^{temp}(G') \to \Phi_{un}^{temp}(G')$ such that the conjectures of Hiraga-Ichino-Ikeda on ν_{Pl} hold, up to rational constants.

Application 2: Residues of unramified Eisenstein series

- This is joint work in progress with M. De Martino and V. Heiermann (see our preprint arXiv:1512.08566).

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- There is unfortunately a gap in arXiv:1512.08566. Consequently, what we can prove at present is weaker than what was announced in the preprint. We are still working on proving the stronger result. Let me describe the current state of affairs.

Application: Residues of unramified Eisenstein series

Let G be split connected reductive over a number field F. Let $K \subset G(\mathbb{A})$ be maximal compact, and $B = TU$ an F-Borel subgroup. In view of the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$ we have a left $B(F)$ and right K invariant map $m_B : G(\mathbb{A}) \to T(\mathbb{A})^1 \backslash T(\mathbb{A}) \simeq X_*(T) \otimes \mathbb{R}_+$. Put $\mathfrak{a}^*_{\mathbb{C}} = X^*(T) \otimes \mathbb{C}$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $g \in G(\mathbb{A})$ one defines:

$$\mathcal{E}(\lambda, g) = \sum_{\gamma \in B(F) \backslash G(F)} m_B(\gamma g)^{\lambda+\rho},$$

the Borel Eisenstein series.

Residues of unramified Eisenstein series

Theorem[Langlands]

- Absolutely convergent if $\mathrm{Re}(\lambda - \rho) > 0$, $\in A(G(F)\backslash G(\mathbb{A}))^K$.

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- Absolutely convergent if $\mathrm{Re}(\lambda - \rho) > 0$, $\in A(G(F)\backslash G(\mathbb{A}))^K$.
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- Put Λ for the completed Dedekind zeta function of F, and $\rho(s) = s(s-1)\Lambda(s)$ (entire, zeroes in critical strip). For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we put $r(\lambda) = \prod_{\alpha\in\Sigma_+} \rho(\alpha^\vee(\lambda))$ and $c(\lambda) = \prod_{\alpha\in\Sigma_+} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$. Then for all $w \in W$ we have:

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$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}\mathcal{E}(\lambda, g)$$
- $\mathcal{E}(\lambda, \cdot)$ is an $\mathcal{H}(G'(\mathbb{A})//K)$-eigenform with eigenvalue χ_λ.

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- Put Λ for the completed Dedekind zeta function of F, and $\rho(s) = s(s-1)\Lambda(s)$ (entire, zeroes in critical strip). For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we put $r(\lambda) = \prod_{\alpha\in\Sigma_+} \rho(\alpha^\vee(\lambda))$ and $c(\lambda) = \prod_{\alpha\in\Sigma_+} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$. Then for all $w \in W$ we have:

$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}\mathcal{E}(\lambda, g)$$

- $\mathcal{E}(\lambda, \cdot)$ is an $\mathcal{H}(G'(\mathbb{A})//K)$-eigenform with eigenvalue χ_λ.
- For $f \in P(\mathfrak{a}^*_{\mathbb{C}})$, the Pseudo-Eisenstein series $\theta_f := \int_{\mathrm{Re}(\lambda)=b>>0} f(\lambda)\mathcal{E}(\lambda, \cdot)d\lambda \in L^2(G(F)\backslash G(\mathbb{A}))^K$.

Residues of unramified Eisenstein series

Definition

- Define the normalized Eisenstein series $\mathcal{E}_0(\lambda, g) := \frac{1}{|W|} c(-\lambda) r(-\lambda) \mathcal{E}(\lambda, g)$. Then $\mathcal{E}_0$ extends to a holomorphic, W-invariant function of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

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- We define $L^2(G(F)\backslash G(\mathbb{A}))^K_{[T,1]}$ (or simply $L^{2,K}_{[T,1]}$) as the closure in $L^2(G(F)\backslash G(\mathbb{A}))^K$ of the span of the pseudo-Eisenstein series $\{\theta_f \mid f \in P(\mathfrak{a}_{\mathbb{C}}^*)\}$.

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- Basic challenge: The spectral decomposition of the unitary representation $L^{2,K}_{[T,1]}$ of the abelian $*$-algebra $\mathcal{H}(G'(\mathbb{A})//K)$.

Residues of unramified Eisenstein series

Theorem[Langlands]

For $f, g \in P(\mathfrak{a}_{\mathbb{C}}^*)$ one has the inner product formula

$$(\theta_f, \theta_g) := X^b(gR_f)$$

with $R_f(\lambda) := \sum_{w \in W} c(-w\lambda) \frac{r(\lambda)}{r(w\lambda)} f^-(-w\lambda)$, and $f^-(\lambda) := \overline{f(\overline{\lambda})}$.

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Remark

We would like to apply our previous results for X^b, Y^b etc., to get an explicit spectral decomposition of $L^{2,K}_{[T,1]}$, but a priori we can only do this for a certain subspace $L^{2,K}_{[T,1],n}$ of $L^{2,K}_{[T,1]}$. The complication here is the potential interference from poles induced from the (critical) zeroes of $\rho(s)$.

Unramified anti-tempered global Arthur parameters

Let C_F denote the Idèle class group of F. Define:

$$AP^{su} := \{\psi : C_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee \mid \begin{array}{l}(a)\ \psi|_{C_F} \text{ is bounded.} \\ (b)\ \psi|_{C_F} \text{ factors through } \|\cdot\|. \\ (c)\ \psi|_{\mathrm{SL}_2(\mathbb{C})} \text{ is algebraic.}\end{array} \}$$

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Remark

Let $\overline{AP}^{su}$ be the set of equivalence classes in AP^{su}. Given $\psi \in AP^{su}$ we can choose $\psi' \in AP^{su}$ with $\psi' \sim \psi$ such that:

- For all $\xi \in C_F$, $\psi'(\xi) = \|\xi\|^{\nu'} \in T^\vee$ for a (unique) $\nu' \in i\mathfrak{a}^*$,
- For all $a \in \mathbb{C}^\times$, $\psi'(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) \in T^\vee$.

Arthur parameters and residual spaces

Proposition[De Martino, Heiermann, O.]

Define

$$D : \overline{AP}^{su} \to W \backslash \mathfrak{a}_{\mathbb{C}}^{*}$$
$$\overline{\psi} \to \nu' + d\psi'(\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix})$$

where $\psi' \sim \psi$ and ν' are as above. Then D defines a bijection between $\overline{AP}^{su}$ and $\Xi := W \backslash \bigcup_{L \text{ residual}} (L^{temp}) \subset W \backslash \mathfrak{a}_{\mathbb{C}}^{*}$.

Theorem[De Martino, Heiermann, O.]

The Hilbert subspace $L^{2,K}_{[T,1],n} \subset L^{2,K}_{[T,1]}$ is isomorphic to the space $L^2(\Xi, \mu_n)$ for an explicitly known absolutely continuous positive measure μ_n on Ξ. Hence $L^{2,K}_{[T,1],n}$ is spanned by wave packets of normalized Eisenstein series $\mathcal{E}_0(\lambda, \cdot)$ with $\lambda \in \Xi$.

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Corollary[De Martino, Heiermann, O.]

For any distinguished nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}^\vee$, the normalized Eisenstein series $\mathcal{E}_0(\lambda_{\mathcal{O}}, \cdot)$ is a nonzero element in $L^{2,K}_{[T,1],n}$, with explicit L^2-norm.

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Corollary[De Martino, Heiermann, O.]

The correponding local representations $\pi_{\nu,\lambda_\mathcal{O}}$ of $G(F_\nu)$ are unitarizable at all local places ν of F.

The complementary subspace

In arXiv:1512.08566 we also claimed to have proven the expected equality $L^{2,K}_{[T,1],n} = L^{2,K}_{[T,1]}$, but here the argument was incomplete. For classical groups this equality was shown by Moeglin, by a very intricate and beautiful combinatorial method. For G_2 this was known already since the works of Langlands and Kim. Proving such equality amounts to showing that certain potential residues cancel. So far we have not been able to see the reason behind these cancellations in general, unfortunately. We are still working on this.

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Remark

The above results can be extended to G a unramified groups G, and more general cuspidal support $[T,\chi]$ where $\chi \in \mathrm{Hom}_c(T(F)\backslash T(\mathbb{A}), \mathbb{C})^{T(\mathbb{A})\cap K}$ an unramified character.

Many congratulations!