Residue distributions and harmonic analysis on reductive groups

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The basic residue lemma

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The basic residue lemma

- Let $V$ be an oriented Euclidean vector space of dimension $n$, with complexification $V_{\mathbb{C}}$.
- Let $\mathcal{A}$ be a finite arrangement of affine hyperplanes $H \subset V$, with complexification $\mathcal{A}_{\mathbb{C}}$.
- Let $P(V_{\mathbb{C}})$ denote the space of Paley-Wiener functions on $V_{\mathbb{C}}$, that is $\phi \in P(V_{\mathbb{C}})$ iff there exists a constant $R > 0$, and for every $N \in \mathbb{N}$ a constant $C_N > 0$ such that for all $z \in V_{\mathbb{C}}$ we have $|\phi(z)| \leq C_N(1 + \|z\|)^{-N}e^{R\|\text{Re}(z)\|}$. 
Let $\omega$ be a rational $(n,0)$-form on $V_\mathbb{C}$ whose singular locus and zero locus is contained in $A_\mathbb{C}$.
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Let $b \in V \setminus \bigcup_{H \in A} H$ and let $X^{\omega,b} : P(V_\mathbb{C}) \to \mathbb{C}$ be the linear functional on $P(V_\mathbb{C})$ defined by

$$X^{\omega,b}(\varphi) := \int_{\text{Re}(z)=b} \varphi(z)\omega(z).$$
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Such linear functionals $X^{\omega, b}$ (or slight variations thereof) often arise in harmonic analysis on reductive groups, in the study of “residual contributions” to the spectrum. Our first goal is a basic decomposition theorem for $X^{\omega, b}$ in terms of tempered distributions with certain support conditions.
For \( H \in \mathcal{A}_C \), let \( n_H \in \mathbb{Z} \) denote the order of \( \omega \) along \( H \). For \( L \in L(\mathcal{A}) \), the intersection semilattice of \( \mathcal{A} \), we define

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We call an affine subspace $L \subset V$ \textit{\omega-residual} if

1. $L = \bigcap_{H \in A : L \subset H \text{ and } n_H < 0} H$ (intersection of the pole hyperplanes containing $L$).
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2. We have $o_L := n_L + \text{codim}(L) \leq 0.$
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Examples:

1. $V$ itself is a residual subspace.
2. If $H \in \mathcal{A}$ with $n_H < 0$ then $H$ is residual ($o_H = n_H + 1 \leq 0$).
If $L \in L(\mathcal{A})$ is residual, then we define $V_L \subset V$ as the linear subspace underlying the affine subspace $L \subset V$, and $V^L = (V_L)^\perp$ (the subspace spanned by the lines orthogonal to the hyperplanes of poles $H \in \mathcal{A}$ such that $L \subset H$).
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We define $c_L = V^L \cap L$, the center of $L$ (the point in $L$ with the shortest distance to $0 \in V$). Let $C \subset V$ be the (finite) set of centers of the $\omega$-residual subspaces.
If \( L \in L(\mathcal{A}) \) is residual, then we define \( V_L \subset V \) as the linear subspace underlying the affine subspace \( L \subset V \), and \( \mathcal{V}^{\mathcal{L}} = (V_L)^\perp \) (the subspace spanned by the lines orthogonal to the hyperplanes of poles \( H \in \mathcal{A} \) such that \( L \subset H \)).

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\( L^{\text{temp}} := c_L + iV_L \subset c_L + iV \subset V_\mathbb{C} \), the tempered form of \( L \).
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**Proposition [Heckman, O.]**

There exists a unique collection of tempered distributions \( X^b_c \in S'(c + iV) \) with \( c \in C \) such that
If \( L \in L(\mathcal{A}) \) is residual, then we define \( V_L \subset V \) as the linear subspace underlying the affine subspace \( L \subset V \), and \( V^L = (V_L)^\perp \) (the subspace spanned by the lines orthogonal to the hyperplanes of poles \( H \in \mathcal{A} \) such that \( L \subset H \)).

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(a) \( \text{Supp}(X^b_c) \subset \bigcup_{L \text{ residual}} :c_L=^c L^{\text{temp}}. \)
If $L \in L(A)$ is residual, then we define $V_L \subset V$ as the linear subspace underlying the affine subspace $L \subset V$, and $V^L = (V_L)^\perp$ (the subspace spanned by the lines orthogonal to the hyperplanes of poles $H \in A$ such that $L \subset H$).

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(a) $\text{Supp}(X^b_c) \subset \bigcup L_{\text{residual}} : c_L = c \quad L^{\text{temp}}$.

(b) For all $\varphi \in P(V_C)$ we have: $X^{\omega, b}(\varphi) = \sum_{c \in C} X^b_c(\varphi|_{c + iV})$. 
Observe that $\varphi|_{c+iV} \in S(c+iV)$, hence the expression $X^b_c(\varphi|_{c+iV})$ is meaningful.
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Observe that $\varphi|_{c+iv} \in S(c+iv)$, hence the expression $X^b_c(\varphi|_{c+iv})$ is meaningful.

**Example:** Let $V = \mathbb{R}$ and $\omega = \frac{dx}{x-c}$ with $c \in \mathbb{R}$.

- If $c \neq 0$ then $X^b_c = \text{sign}(c)2\pi i\delta_c$ if $c$ separates $b$ and 0, and $X^b_c = 0$ otherwise. Moreover $X^b_0 = (x-c)^{-1}|_{i\mathbb{R}}$. 

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- If $c = 0$ and $\pm b > 0$ then $X^b_0 = \text{Pf}(x^{-1}|_{i\mathbb{R}}) \pm \pi i\delta_0$. 
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- If $c \neq 0$ then $X_c^b = \text{sign}(c)2\pi i\delta_c$ if $c$ separates $b$ and $0$, and $X_c^b = 0$ otherwise. Moreover $X_0^b = (x - c)^{-1}|_{i\mathbb{R}}$.
- If $c = 0$ and $\pm b > 0$ then $X_0^b = \text{Pf}(x^{-1}|_{i\mathbb{R}}) \pm \pi i\delta_0$.

**Example**: More generally: Let $V = \mathbb{R}^n$. Let $\alpha_1, \ldots, \alpha_n$ be a basis of $V$, and let $c \in V$. Let $A = \{H_i\}_{i=1}^n$ with $H_i = \{x \mid (\alpha_i, x) = (\alpha_i, c)\}$. Let $\omega = \frac{dx_1 \wedge \cdots \wedge dx_n}{\prod_{i=1}^n((\alpha_i, x) - (\alpha_i, c))}$. Then $X_{\omega, b}$ only depends on the chamber $C$ of $V \setminus \bigcup_i H_i$ such that $b \in C$. Assume that $0 \in V$ is regular in the dual chamber decomposition with center $c$. Then $X_c^b \neq 0$ iff $0 \in -C^*$, the anti-dual chamber of $C$, and

$$X_c^b = \pm \frac{(2\pi i)^n}{\det(\alpha_i, \alpha_j)^{1/2}} \delta_c$$
Two cases of interest

Now let $G \supset B \supset T$ be a connected reductive group over $\mathbb{C}$, with Borel subgroup $B$ and maximal torus $T$. Let $V \subset g$ be the real span of the cocharacter lattice $X$ of $T$. Let $\Sigma^\vee$ be the root system of $G$. 
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- Define a rational function on $V$ by $c(\lambda) = \prod_{\alpha \in \Sigma^\vee} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$. 
Two cases of interest

Now let $G \supset B \supset T$ be a connected reductive group over $\mathbb{C}$, with Borel subgroup $B$ and maximal torus $T$. Let $V \subset g$ be the real span of the cocharacter lattice $\chi$ of $T$. Let $\Sigma^\vee$ be the root system of $G$.

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We would like to understand the following functionals: For $\varphi \in P(V_\mathbb{C})$ and $b$ deep in the Weyl chamber, define:

$$X^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda) \omega^X(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(-\lambda)}$$

and

$$Y^b(\varphi) = \int_{\lambda \in b+iV} \varphi(\lambda) \omega^Y(\lambda) := (2\pi i)^{-n} \int_{\lambda \in b+iV} \varphi(\lambda) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$
Positivity

Special properties of $Y^b$:

**Theorem**

For all $L \subset V$, affine subspace, let $o_L^Y = n_L^Y + \text{codim}(L)$ with $n_L^Y$ the pole order of $\omega^Y$ along $L$. Then $o_L^Y \geq 0$. In particular, $L$ is $\omega^Y$-residual iff $o_L^Y = 0$, or equivalently:

$$|\{\alpha \in \Sigma \mid \alpha^\vee|_L = 1\}| = |\{\alpha \in \Sigma \mid \alpha^\vee|_L = 0\}| + \text{codim}(L)$$
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**Theorem [Heckman, O.]**
Let $C^Y \subset V$ denote the set of centers of $\omega^Y$-residual subspaces. For all $c \in C^Y$, $Y^b_c$ is a nonnegative measure supported by $\bigcup_{L: c_L = c} L^{\text{temp}} \subset c + iV$. 
The support theorem

Algebraic description of the support of the $Y_c$:

**Theorem**

For all $c \in C^Y$, there exists $w \in W$ such that $Y^b_{wc} \neq 0$. In this case $w(c)$ is in the dual chamber of $V_+$. 
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The support of $Y^b$ in terms of nilpotent orbits

The set $C^Y$ is $W$-invariant, and $c \in C^Y \cap V_+$ iff there exists a nilpotent orbit $O \subset g$ such that $c = \lambda_O$, where $\lambda_O$ is half the weighted Dynkin diagram of $O$. Hence there is a canonical bijection between $W \backslash C^Y$ and the set of nilpotent orbits of $g$. 
Symmetrization of the $X$-distribution

Using the uniqueness of residue distributions it is easy to express the local $X$-distributions in terms of those of $Y$:

**Theorem**

Let $f \in P(V_C)$. For every $c \in V_+$ and $w \in W$ we have

$$X_{wc}^b(f|_{wc+iV}) = Y_{c}^b((A_{wc}(f) \circ w)|_{c+iV})$$

where $A_{wc}(f) \in P(V_C)$ is defined by (for $\lambda \in V_C^{reg}$):

$$A_{wc}(f)(\lambda) = \frac{1}{|W_{wc}|} \sum_{u \in W_{wc}} c(u\lambda)f(u\lambda) \text{ (the symmetrization operator)}.$$
Spectral decomposition of affine Hecke algebras

We now consider a variation of the previous, closely linked to harmonic analysis on $p$-adic reductive groups. This helps to understand the distributions $X$ and $Y$ better, both their properties and their applications.

Let $k$ be a $p$-adic field, and let $G'(\supset B' \subset T')$ be the group of $k$-points of a connected split reductive group defined over $k$ with based root datum dual to $G$. Let $\mathcal{I} \subset G'$ be an Iwahori subgroup for an alcove $C$ of the apartment of $T'$, and let $\mathcal{H}_\mathcal{I} = C^\infty_c(G'/\mathcal{I})$ the Iwahori Hecke algebra.
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- $\mathcal{H}_{\mathbb{I}}$ is an affine Hecke algebra for the (extended) affine Weyl group $W^e \simeq W \rtimes X^*(T)$, with parameter $q = v^2$, the cardinality of the residue field of $k$. 

Spectral decomposition of affine Hecke algebras
The affine Hecke algebra as a Hilbert algebra

Denote by $S^a \subset W^e$ the set of simple affine reflections defined by the alcove $C$. $\mathcal{H}_I$ has a Coxeter basis $\{T_w\}_{w \in W^e}$ such that for all $s \in S^a$:
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- $T_s T_w = T_{sw}$ if $l(sw) = l(w) + 1$. 
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- \( T_s T_w = T_{sw} \) if \( l(sw) = l(w) + 1 \).
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We equip \( \mathcal{H}_I \) with the trace \( \tau \in (\mathcal{H}_I)^* \) defined by \( \tau(T_w) = \delta_{w,evol(I)}^{-1} \) and the anti-linear anti-isomorphism \( \ast : \mathcal{H}_I \rightarrow \mathcal{H}_I \) given by \( T_w^* = T_{w^{-1}} \). Then the Hermitian inner product on \( \mathcal{H}_I \) defined by \( (x, y) = \tau(x^* y) \) is positive, and its basis \( \{ T_w \}_{w \in W} \) is orthonormal.
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- This gives the $\ast$-algebra $\mathcal{H}_I$ the structure of a Hilbert algebra of type $I$ with trace $\tau$. 
There exists a unique positive measure $\nu_{Pl}$ on $\hat{H}_I$ such that

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It was shown by Bushnell, Henniart and Kutzko (much more generally, in the context of types) that $\nu_{Pl}$ corresponds to the Plancherel measure of $G'$ restricted to the Borel component of Iwahori-spherical representations.
Bernstein’s basis of the Hecke algebra

Let $\mathcal{A} = \mathbb{C}[T]$ and let $\mathcal{H}_0 = C_\infty_c(\mathbb{K}/\mathbb{I})$ (the finite Hecke algebra of the Coxeter group $(\mathcal{W}, S)$ and parameter $q = \nu^2$). **Bernstein’s basis theorem**: There exist injective algebra homomorphisms $\mathcal{A} \to \mathcal{H}_\|_\|$ and $\mathcal{H}_0 \to \mathcal{H}_\|$ such that the multiplication maps $\mathcal{H}_0 \otimes \mathcal{A} \to \mathcal{H}_\|$ and $\mathcal{A} \otimes \mathcal{H}_0 \to \mathcal{H}_\|$ are isomorphisms of vector spaces.
Let $\mathcal{A} = \mathbb{C}[T]$ and let $\mathcal{H}_0 = C_c^\infty(\mathbb{K}/\mathbb{I})$ (the finite Hecke algebra of the Coxeter group $(\mathcal{W}, S)$ and parameter $q = v^2$). **Bernstein’s basis theorem**: There exist injective algebra homomorphisms $\mathcal{A} \rightarrow \mathcal{H}_I$ and $\mathcal{H}_0 \rightarrow \mathcal{H}_I$ such that the multiplication maps $\mathcal{H}_0 \otimes \mathcal{A} \rightarrow \mathcal{H}_I$ and $\mathcal{A} \otimes \mathcal{H}_0 \rightarrow \mathcal{H}_I$ are isomorphisms of vector spaces.

A well known Corollary is $Z(\mathcal{H}_I) = \mathcal{A}^W = \mathbb{C}[T]^W$. 
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A well known Corollary is $Z(\mathcal{H}_\mathcal{I}) = \mathcal{A}^\mathcal{W} = \mathbb{C}[T]^\mathcal{W}$.

Define $\Delta \in \mathcal{A} = \mathbb{C}[T]$ by $\Delta(t) = \prod_{\alpha \in \Sigma} (1 - \alpha^\vee(t)^{-1})$, and its “$q$-deformation” $\Delta_q(t) = \prod_{\alpha \in \Sigma} (1 - q^{-1}\alpha^\vee(t)^{-1})$. 
Two functionals on $\mathcal{A}$

**Definition**

We define two functionals for $\varphi \in \mathcal{A} = \mathbb{C}[T]$, and $b \in T_{v,-}$ (deep in the negative chamber):

$$\chi^b(\varphi) = \int_{t \in bT_u} \varphi(t) \frac{\Delta(t)}{q(w_0)\Delta_q(t)} \, dt$$

and

$$\gamma^b(\varphi) = \int_{t \in bT_u} \varphi(t) \mu(t) \, dt := \int_{t \in bT_u} \varphi(t) \frac{\Delta(t)\Delta(t^{-1})}{q(w_0)\Delta_q(t)\Delta_q(t^{-1})} \, dt,$$

where $q(w_0) = q|_{\Sigma^+}$.

The relation between $\chi^b$ and $\gamma^b$ is given by a process of symmetrization.
Residual cosets

**Definition**

Let $\eta(t) = \mu(t)dt = \frac{\Delta(t)\Delta(t^{-1})}{q(w_0)\Delta_q(t)\Delta_q(t^{-1})} dt$. For any coset $L = pT_L$ of a subtorus $T_L \subset T$, let $n_L^\gamma$ be the pole order of $\mu$ along $L$. We call $L$ \( \eta \)-residual if
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- $L$ is a connected component of $\bigcap_{\alpha \in \Sigma} L \subset H_\alpha H_\alpha$, with $H_\alpha = \{ t \in T \mid \alpha^\vee(t) = q \}$. 

Theorem

$L = pT_L$ is $\eta$-residual iff $o_L = 0$. 

Residual cosets

**Definition**

Let \( \eta(t) = \mu(t)dt = \frac{\Delta(t)\Delta(t^{-1})}{q(w_0)\Delta_q(t)\Delta_q(t^{-1})} dt \). For any coset \( L = pT_L \) of a subtorus \( T_L \subset T \), let \( n^Y_L \) be the pole order of \( \mu \) along \( L \). We call \( L \) \( \eta \)-residual if

- \( L \) is a connected component of \( \bigcap_{\alpha \in \Sigma} L \subset H_\alpha \), with \( H_\alpha = \{ t \in T \mid \alpha^\vee(t) = q \} \).
- We have \( o^Y_L := n^Y_L + \text{codim}(L) \leq 0 \).
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Theorem
\( L = pT_L \) is \( \eta \)-residual iff \( o_L = 0 \).
The tempered form of a residual coset

**Definition**

Let $L \subset T$ be $\eta$-residual. Let $T^L \subset T$ be the subtorus such that $\text{Lie}(T^L)$ is spanned by the $\alpha \in \Sigma$ such that $\alpha^\vee |_L = q$. 

We call $c_L := \text{Re}(T^L \cap L) \in T_v$ the center of $L$. Define $L_{\text{temp}} := (T^L \cap L) T^L$, $u$, the tempered form of $L$. For any $r_L \in T^L \cap L$ we have:

$L_{\text{temp}} = r_L T^L, u$. 

**Theorem**

Let $C_Y \subset T_v$ denote the set of centers of the $\eta = \mu dt$-residual cosets. For each $c \in C_Y$ there exists a unique distribution $Y_{bc}$ on $cT^u$ with $\text{supp}(Y_{bc}) \subset \bigcup L : cL = cL_{\text{temp}}$ such that:

$\forall a \in A = C[T]$:

$\tau(a) = Y_{bc}(a) = \sum_{c \in C_Y} Y_{bc}(a|_{cT^u})$.
The tempered form of a residual coset

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**Theorem**

Let $C^\gamma \subset T_v$ denote the set of centers of the $\eta = \mu dt$-residual cosets. For each $c \in C^\gamma$ there exists a unique distribution $\gamma^b_c$ on $cT_u$ with $\text{supp}(\gamma^b_c) \subset \cup_{L : c_L = c} L^{\text{temp}}$ such that: $\forall a \in A = \mathbb{C}[T]$:

$$ \tau(a) = \gamma^b(a) = \sum_{c \in C^\gamma} \gamma^b_c(a|cT_u) $$
The Plancherel measure and residue distributions

**Theorem (The trace of \( H_I \) expressed in terms of \( \gamma^b \))**

There exists a holomorphic family \( T \ni t \rightarrow E(\cdot, t) \in (H_I)^* \) of matrix coefficients \( E(\cdot, t) \) of the minimal principal series \( \text{ind}_{\mathcal{A}}^{H_I}(t) \) of \( H_I \) such that, for \( b \) deep in \( T_v, -\) : 
\[
\tau(h) = \gamma^b\left(\frac{E(\cdot,h)}{q(w_0)\Delta(\cdot)}\right).
\]
The Plancherel measure and residue distributions

Theorem (The trace of $\mathcal{H}_I$ expressed in terms of $\mathcal{Y}^b$)

There exists a holomorphic family $T \ni t \rightarrow E(\cdot, t) \in (\mathcal{H}_I)^*$ of matrix coefficients $E(\cdot, t)$ of the minimal principal series $\text{ind}_{\mathcal{H}_I}(t)$ of $\mathcal{H}_I$ such that, for $b$ deep in $T_{v,-}$: $\tau(h) = \mathcal{Y}^b\left(\frac{E(\cdot,h)}{q(w_0)\Delta(\cdot)}\right)$.

Restriction of $\tau$ to $\mathcal{A}$ and the center $Z(\mathcal{H}_I) \subset \mathcal{A}$

For $a \in \mathcal{A}$ we have $\tau(a) = \mathcal{Y}^b(a)$. (Note that $Z(\mathcal{H}_I)$ is a $*$-subalgebra, but in general $\mathcal{A}$ is not a $*$-subalgebra).
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**Theorem (The trace of $\mathcal{H}_I$ expressed in terms of $\gamma^b$)**

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For $a \in \mathcal{A}$ we have $\tau(a) = \gamma^b(a)$. (Note that $Z(\mathcal{H}_I)$ is a $\ast$-subalgebra, but in general $\mathcal{A}$ is not a $\ast$-subalgebra).

**Restriction of $\tau$ to $\mathcal{H}_I^{\text{gen}} := e_- \mathcal{H}_I e_- = e_- Z(\mathcal{H}_I)$**

Let $P_W(q) = \sum_{w \in W} q^{l(w)}$. Let $e_- \in \mathcal{H}_0$ be the idempotent of the sign character. On the abelian $\ast$-subalgebra $\mathcal{H}_I^{\text{gen}} \simeq e_- \mathcal{A} e_-$ we have, for all $a \in \mathcal{A} = \mathbb{C}[T]$: $P_W(q^{-1})\tau(ae_-) = \lambda^b(a)$. 
Residue distributions and Plancherel measures

Corollary

These interpretations of $\mathcal{X}_b$, $\gamma^b$ as functionals representing the positive trace $\tau$ on various $*$-subalgebras of $\mathcal{H}_I$, have strong implications: The “local distributions” $\gamma^b_c$ are positive measures.
Corollary

These interpretations of $\mathcal{X}^b, \mathcal{Y}^b$ as functionals representing the positive trace $\tau$ on various $*$-subalgebras of $\mathcal{H}_I$, have strong implications: The "local distributions" $\mathcal{Y}_c^b$ are positive measures.

Remark

The $\{\mathcal{Y}_c^b\}_{c \in \mathcal{C}_\mathcal{Y}}$ are instrumental to compute $\nu_{Pl}$. Conversely $\nu_{Pl}^{gen} := \nu_{Pl}|_{\mathcal{H}_I^{gen}}$ determines $\{\mathcal{X}_c^b\}_{c \in \mathcal{C}_\mathcal{X}}$ and $\{\mathcal{Y}_c^b\}_{c \in \mathcal{C}_\mathcal{Y} \cap \mathcal{T}_{v,-}}$
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These interpretations of $\mathcal{X}^b$, $\mathcal{Y}^b$ as functionals representing the positive trace $\tau$ on various $*$-subalgebras of $\mathcal{H}_I$, have strong implications: The “local distributions” $\mathcal{Y}_c^b$ are positive measures.

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Remark

The functional $Y^b$ can be computed explicitly by a limit process for $q \to 1$ of $\mathcal{Y}^b$ and a twist by an involution.
Symmetrization of $\chi^b$; relation to $\gamma^b$

Let $a \in A$ and $t \in T$. Let $\chi_t$ denote the character of $\mathcal{H}_\Pi$ of the minimal principal series $\text{ind}_{A}^{\mathcal{H}_\Pi}(t)$. Let $\phi_t^{-}(\cdot) := \chi_t(\cdot e_{-}) \in (\mathcal{H}_\Pi^{\text{gen}})^{*}$ denote the "elementary skew-spherical function" of $\mathcal{H}_\Pi^{\text{gen}} = e_{-} \mathcal{H}_\Pi e_{-} = Z(\mathcal{H}_\Pi)e_{-} \simeq Z(\mathcal{H}_\Pi)$ at character $Wt \in W \backslash T$. Let $\nu_{WL} = \sum_{L' \in WL} \nu_{L'}$ be the unique $W$-invariant measure supported on $WL^{\text{temp}}$ such that $\tau(ae_{-}) = \frac{1}{P(q^{-1})} \chi^b(a)$ can be rewritten as:

$$
\tau(ae_{-}) = \sum_{c \in C_{\gamma}} \frac{1}{|W_{c}|} \gamma_{c}^{b}(\phi_{-}^{-}(a)) = \sum_{L \text{ residual}} \int_{L^{\text{temp}}} \phi_{-}^{-}(a) d\nu_{L}
$$

In particular: $\sum_{L \text{ residual}} \nu_{L} = \nu_{PL}^{\text{gen}} := \nu_{PL}\big|_{\mathcal{H}_\Pi^{\text{gen}}}$. 
The explicit computation of $\nu_{Pl}$

**Theorem (The support of $Y^b$; “no cancellation Theorem”)**

The support $S_Y \subset W \setminus T$ of the restriction of $Y^b = \sum_{c \in C} Y^b_c$ to $Z(\mathcal{H}_I) = \mathbb{C}[T]^W$ is exactly equal to $S_Y := \bigcup_{L \text{ residual}} L_{\text{temp}}$. 

We will soon see that $S_Y \cong \Phi_{\text{temp}}(G')$, the set of equivalence classes of tempered unramified Langlands parameters of $G'$. 

The Plancherel measure $\nu_{Pl}$ has been computed explicitly for all affine Hecke algebras with (unequal) parameters $v_s > 0$, with $s \in S_a$ (the affine simple reflections). 

The main techniques which have been used for this: 

- Deformation in the Hecke algebra parameters $v_s$. 
- Dirac induction.
The explicit computation of $\nu_{Pl}$

**Theorem (The support of $\mathcal{Y}^b$; “no cancellation Theorem”)**

The support $S^\mathcal{Y} \subset W \setminus T$ of the restriction of $\mathcal{Y}^b = \sum_{c \in \mathcal{C}} \mathcal{Y}_c^b$ to $Z(\mathcal{H}_\Pi) = \mathbb{C}[T]^W$ is exactly equal to $S^\mathcal{Y} := \bigcup_{\text{residual }L} L_{\text{temp}}$.

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The explicit computation of $\nu_{Pl}$

**Theorem (The support of $\mathcal{Y}^b$; “no cancellation Theorem”)**

The support $S^\mathcal{Y} \subset W \backslash T$ of the restriction of $\mathcal{Y}^b = \sum_{c \in C^n} \mathcal{Y}^b_c$ to $Z(\mathcal{H}_I) = \mathbb{C}[T]^W$ is exactly equal to $S^\mathcal{Y} := \bigcup_{L \text{ residual}} L^{\text{temp}}$.

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- The main techniques which have been used for this:
  - Deformation in the Hecke algebra parameters $\nu_s$.
  - Dirac induction.
Dirac induction; parameter families of discrete series

Let $Q$ be set of positive real Hecke algebra parameters for $\mathcal{H} = \mathcal{H}(W^e)$, i.e. the real vector group of tuples $(v_s)_{s \in S^a}$ of positive real parameters $v_s$ such that $v_s = v_{s'}$ if $s \sim W e s'$. 

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Theorem [Ciubotaru-O.] (Dirac induction)

There exists an ON set $B^{mass} = \{ b_1, \ldots, b_m \} \subset \mathbb{R}^\ell_{\mathbb{Z}}(W^e)$ of "positive generically massive unit vectors".
Theorem [Ciubotaru-O.] (Dirac induction)

- There exists an ON set $B^{mass} = \{b_1, \ldots, b_m\} \subset R_{\mathbb{Z}}(W^e)$ of "positive generically massive unit vectors".

- For all $b \in B^{mass}$, we have an orbit of residual points $W_{rb} \in W \setminus T$ with values in $\mathbb{C}[\nu_{s}^{\pm 1} \mid \nu_s = \nu_{s'} \text{ if } s \sim_{W^e} s']$ and an explicit rational function $m_{Wrb}$ on $Q$ which is regular on $Q$. 

∀ $b \in B^{mass}$ there exists $c_b \in Q \times$ such that for all $v \in Q$:

$f_{deg}(\text{Ind}an D(b, v)) := \nu_{Pl}(\{\text{Ind}an D(b, v)\}) = c_bm_{Wrb}(v)$. 

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- \( \forall b \in B^{mass} \) and \( v = (v_s)_{s \in S^a} \in Q \) such that \( m_{Wr_b}(v) \neq 0 \) we have a virtual discrete series character \( \text{Ind}^\text{an}_D(b, v) \) of \( \mathcal{H}_v \) with central character \( Wr_b(v) \) such that \( \lim_{\epsilon \to 0} \text{Ind}^\text{an}_D(b, v^\epsilon) = b \in B^{mass} \).
Theorem [Ciubotaru-O.] (Dirac induction)

- There exists an ON set $B^{mass} = \{b_1, \ldots, b_m\} \subset R^e(Z(W^e))$ of "positive generically massive unit vectors".
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- For all $b \in B^{mass}$ and $\nu = (\nu_s)_{s \in S_a} \in \mathbb{Q}$ such that $m_{W_r b}(\nu) \neq 0$ we have a virtual discrete series character $\text{Ind}_{\nu}^{an}(b, \nu)$ of $\mathcal{H}_{\nu}$ with central character $W_r b(\nu)$ such that $\lim_{\epsilon \to 0} \text{Ind}_{\nu}^{an}(b, \nu^\epsilon) = b \in B^{mass}$.
- $\pm \text{Ind}_{\nu}^{an}(b, \nu)$ is irreducible, and this parameterizes the irr. DS characters of $\mathcal{H}_{\nu}$ by $\{b \in B^{mass} \mid m_{W_r b}(\nu) \neq 0\}$. 

∀ $b \in B^{mass}$ there exists $c_b \in \mathbb{Q} \times \mathbb{Q}$ such that for all $\nu \in \mathbb{Q}$:

$$f_{deg}(\text{Ind}_{\nu}^{an}(b, \nu)) := \nu_{Pl}(\{\text{Ind}_{\nu}^{an}(b, \nu)\}) = c_b m_{W_r b}(\nu).$$
Theorem [Ciubotaru-O.] (Dirac induction)

- There exists an ON set $B^{mass} = \{b_1, \ldots, b_m\} \subset R^e_{\mathbb{Z}}(W^e)$ of “positive generically massive unit vectors”.
- $\forall b \in B^{mass}$ we have an orbit of residual points $W_{rb} \in W \setminus T$ with values in $\mathbb{C}[v_s^{\pm 1} \mid v_s = v_{s'} \text{ if } s \sim W^e s']$ and an explicit rational function $m_{Wrb}$ on $Q$ which is regular on $Q$.
- $\forall b \in B^{mass}$ and $v = (v_s)_{s \in S^a} \in Q$ such that $m_{Wrb}(v) \neq 0$ we have a virtual discrete series character $\text{Ind}_{D}^{an}(b, v)$ of $H_v$ with central character $W_{rb}(v)$ such that $[\lim_{\epsilon \to 0} \text{Ind}_{D}^{an}(b, v^\epsilon)] = b \in B^{mass}$.
- $\pm \text{Ind}_{D}^{an}(b, v)$ is irreducible, and this parameterizes the irr. DS characters of $H_v$ by $\{b \in B^{mass} \mid m_{Wrb}(v) \neq 0\}$.
- $\forall b \in B^{mass}$ there exists $c_b \in \mathbb{Q}^\times$ such that for all $v \in Q$: $fdeg(\text{Ind}_{D}^{an}(b, v)) := \nu_{Pl}(\{\text{Ind}_{D}^{an}(b, v)\}) = c_b m_{Wrb}(v)$.
Application 1: The conjecture of Hiraga, Ichino and Ikeda for representations of unipotent reduction

Definition (Tempered unramified Langlands parameters)

Let $G'$ be connected reductive over $k$, with dual group $^LG' = G \rtimes \langle \theta \rangle$. An unramified Langlands parameter of $G'$ is a homomorphism:

$$\varphi : \langle \text{Frob} \rangle \times \text{SL}_2(\mathbb{C}) \rightarrow ^LG' = G \rtimes \langle \theta \rangle$$

satisfying certain natural conditions (depending on $G'$). We call $\varphi$ tempered if $\langle \text{Frob} \rangle$ has bounded image. The set of equivalence classes of tempered unramified parameters is denoted by $\Phi_{un}^{temp}(G')$. 
Lusztig defined the class of representations of unipotent reduction of $G'$. For $G'$ simple over $k$ (of adjoint type, split over an unramified extension) he classified such tempered characters in terms of $\Phi_{\text{un}}^{\text{temp}}(G')$.

The support and tempered unramified Langlands parameters

Let $G'$ be quasisplit. We identify $W \setminus T$ canonically with the set of semisimple $G$-orbits in $G\theta$. Put $\widetilde{\text{Frob}} := (\text{Frob}, \begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix})$.

The map

$$\Phi(G')^{\text{temp}} \ni [\varphi] \to W\varphi(\widetilde{\text{Frob}}) \in S^\vee \subset W \setminus T$$

is a bijection $\Phi_{\text{un}}^{\text{temp}}(G') \sim S^\vee$, with inverse $\chi = Wt \to [\varphi\chi]$.
Fix an additive character $\psi$ of $k$, and normalize the Haar measure $\mu_{G'}^{\psi}$ of $G'$ accordingly.

Theorem [Reeder; O.] (The conjecture of HII for representations of $G'$ of unipotent reduction)

Suppose that $G'$ is of adjoint type. Let $\varphi$ be an discrete unramified Langlands parameter of $G'$, and let $S_\varphi = \{ s \in G \mid \text{Ad}(s) \circ \varphi = \varphi \}$. Let $\chi_{G'} \in (L^1 Z)^*$ be the character corresponding to the inner class $[G'] \in H^1(k, (G')^*)$, extended to $Z$. Put $\Pi(S_\varphi, \chi_{G'}) = \{ \rho \in \text{Irr}(S_\varphi) \mid \rho|_Z = n\chi_{G'} \}$. Let $\pi_\rho \in \Pi_{\varphi}$ be the discrete series representation of unipotent reduction of $G'$ according to Lusztig’s parameterization. Then

$$fdeg(\pi_\rho)(= \nu_{Pl}(\{ \pi_{\rho} \})) = \frac{\dim(\rho)}{|S_\varphi|} |\gamma(0, \text{Ad} \circ \varphi, \psi)| \quad (1)$$

where $\gamma(0, \text{Ad} \circ \varphi, \psi)$ denotes the $\gamma$-factor of $\text{Ad} \circ \varphi$ at $s = 0$. 
Many other partial results known: Silberger-Zink, Hiraga-Saito, Hiraga-Ichino-Ikeda, Beuzart-Plessis, Moeglin-Waldspurger, Ichino-Lapid-Mao, ...
Uniqueness properties

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- Mark Reeder (1994) that for split exceptional groups, the formal degree of a discrete series representation $\pi$ of $G'$ of unipotent reduction uniquely determines an unramified discrete Langlands parameter such that (1) holds.

- This observation generalizes to general tempered representations of unipotent reduction (with Y. Feng and M. Solleveld). Let $G'$ be reductive over $k$, and split over an unramified extension of $k$. There exists an essentially unique map (i.e. up to twisting by automorphisms) $\Pi_{\text{temp}}^{\text{unip}}(G') \rightarrow \Phi_{\text{un}}^{\text{temp}}(G')$ such that the conjectures of Hiraga-Ichino-Ikeda on $\nu_{P_l}$ hold, up to rational constants.
Application 2: Residues of unramified Eisenstein series

This is joint work in progress with M. De Martino and V. Heiermann (see our preprint arXiv:1512.08566).
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- This is joint work in progress with M. De Martino and V. Heiermann (see our preprint arXiv:1512.08566).
- The goal is to use our knowledge on residue distributions to handle residues of Eisenstein series. This theme has of course appear been studied a lot in the work of Jacquet, Langlands, Moeglin, Waldspurger, Kim, and more recently S. Miller.
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- This is joint work in progress with M. De Martino and V. Heiermann (see our preprint arXiv:1512.08566).
- The goal is to use our knowledge on residue distributions to handle residues of Eisenstein series. This theme has of course appear been studied a lot in the work of Jacquet, Langlands, Moeglin, Waldspurger, Kim, and more recently S. Miller.
- There is unfortunately a gap in arXiv:1512.08566. Consequently, what we can prove at present is weaker than what was announced in the preprint. We are still working on proving the stronger result. Let me describe the current state of affairs.
Application: Residues of unramified Eisenstein series

Let $G$ be split connected reductive over a number field $F$. Let $K \subset G(\mathbb{A})$ be maximal compact, and $B = TU$ an $F$-Borel subgroup. In view of the Iwasawa decomposition $G(\mathbb{A}) = B(\mathbb{A})K$ we have a left $B(F)$ and right $K$ invariant map $m_B : G(\mathbb{A}) \to T(\mathbb{A})^1 \setminus T(\mathbb{A}) \simeq X^*(T) \otimes \mathbb{R}_+$. Put $a^*_C = X^*(T) \otimes \mathbb{C}$. For $\lambda \in a^*_C$ and $g \in G(\mathbb{A})$ one defines:

$$\mathcal{E}(\lambda, g) = \sum_{\gamma \in B(F) \setminus G(F)} m_B(\gamma g)^{\lambda + \rho},$$

the Borel Eisenstein series.
Residues of unramified Eisenstein series

Theorem[Langlands]

Absolutely convergent if Re($\lambda - \rho$) > 0, $\in A(G(F) \backslash G(\mathbb{A}))^K$. 
Residues of unramified Eisenstein series

Theorem[Langlands]

- Absolutely convergent if \( \text{Re}(\lambda - \rho) > 0, \quad \in A(G(F) \backslash G(\mathbb{A}))^K \).
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Residues of unramified Eisenstein series

**Theorem [Langlands]**

- Absolutely convergent if $\text{Re}(\lambda - \rho) > 0, \in A(G(F) \backslash G(\mathbb{A}))^K$.
- Has meromorphic continuation to $a^*_C$ as function of $\lambda$.
- Put $\Lambda$ for the completed Dedekind zeta function of $F$, and $\rho(s) = s(s - 1)\Lambda(s)$ (entire, zeroes in critical strip). For $\lambda \in a^*_C$ we put $r(\lambda) = \prod_{\alpha \in \Sigma^+} \rho(\alpha^\vee(\lambda))$ and $c(\lambda) = \prod_{\alpha \in \Sigma^+} \frac{\alpha^\vee(\lambda) + 1}{\alpha^\vee(\lambda)}$. Then for all $w \in W$ we have:

$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}\mathcal{E}(\lambda, g)$$
Residues of unramified Eisenstein series

**Theorem [Langlands]**

- Absolutely convergent if $\text{Re}(\lambda - \rho) > 0 , \in A(G(F) \backslash G(\mathbb{A}))^K$.
- Has meromorphic continuation to $a^*_C$ as function of $\lambda$.
- Put $\Lambda$ for the completed Dedekind zeta function of $F$, and $\rho(s) = s(s - 1)\Lambda(s)$ (entire, zeroes in critical strip). For $\lambda \in a^*_C$ we put $r(\lambda) = \prod_{\alpha \in \Sigma_+} \rho(\alpha^\vee(\lambda))$ and $c(\lambda) = \prod_{\alpha \in \Sigma_+} \frac{\alpha^\vee(\lambda)+1}{\alpha^\vee(\lambda)}$. Then for all $w \in W$ we have:

$$\mathcal{E}(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}\mathcal{E}(\lambda, g)$$

- $\mathcal{E}(\lambda, \cdot)$ is an $\mathcal{H}(G'(\mathbb{A})//K)$-eigenform with eigenvalue $\chi_\lambda$. 

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Residue distributions

Reducive groups and Hecke algebras

Applications
Residues of unramified Eisenstein series

**Theorem [Langlands]**
- Absolutely convergent if \( \text{Re}(\lambda - \rho) > 0, \in A(G(F) \backslash G(\mathbb{A}))^K \).
- Has meromorphic continuation to \( \mathfrak{a}_C^* \) as function of \( \lambda \).
- Put \( \Lambda \) for the completed Dedekind zeta function of \( F \), and \( \rho(s) = s(s - 1)\Lambda(s) \) (entire, zeroes in critical strip). For \( \lambda \in \mathfrak{a}_C^* \) we put \( r(\lambda) = \prod_{\alpha \in \Sigma_+} \rho(\alpha^\vee(\lambda)) \) and \( c(\lambda) = \prod_{\alpha \in \Sigma_+} \frac{\alpha^\vee(\lambda) + 1}{\alpha^\vee(\lambda)} \). Then for all \( w \in W \) we have:

\[
E(w\lambda, g) = \frac{c(w\lambda)r(w\lambda)}{c(\lambda)r(\lambda)}E(\lambda, g)
\]

- \( E(\lambda, \cdot) \) is an \( \mathcal{H}(G'(\mathbb{A})//K) \)-eigenform with eigenvalue \( \chi_\lambda \).
- For \( f \in P(\mathfrak{a}_C^*) \), the **Pseudo-Eisenstein series**

\[
\theta_f := \int_{\text{Re}(\lambda)=b>0} f(\lambda)E(\lambda, \cdot) d\lambda \in L^2(G(F) \backslash G(\mathbb{A}))^K.
\]
Residues of unramified Eisenstein series

**Definition**

Define the normalized Eisenstein series
\[ \mathcal{E}_0(\lambda, g) := \frac{1}{|W|} c(-\lambda) r(-\lambda) \mathcal{E}(\lambda, g). \]
Then \( \mathcal{E}_0 \) extends to a holomorphic, \( W \)-invariant function of \( \lambda \in a^*_C \).
Residues of unramified Eisenstein series

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Then \( \mathcal{E}_0 \) extends to a holomorphic, \( W \)-invariant function of \( \lambda \in a_{\mathbb{C}}^* \).

We define \( L^2(G(F) \backslash G(\mathbb{A}))_{[T,1]}^K \) (or simply \( L^2_{[T,1]} \)) as the closure in \( L^2(G(F) \backslash G(\mathbb{A}))^K \) of the span of the pseudo-Eisenstein series \( \{ \theta_f \mid f \in P(a_{\mathbb{C}}^*) \} \).
Residues of unramified Eisenstein series

**Definition**

- Define the **normalized Eisenstein series**
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  Then \( \mathcal{E}_0 \) extends to a holomorphic, \( W \)-invariant function of \( \lambda \in a_C^* \).

- We define \( L^2(G(F) \setminus G(\mathbb{A}))/K \) (or simply \( L^2_{[T,1]} \)) as the closure in \( L^2(G(F) \setminus G(\mathbb{A}))/K \) of the span of the pseudo-Eisenstein series \( \{ \theta_f \mid f \in P(a_C^*) \} \).

- **Basic challenge:** The spectral decomposition of the unitary representation \( L^2_{[T,1]} \) of the abelian \(*\)-algebra \( \mathcal{H}(G'(\mathbb{A})//K) \).
Residues of unramified Eisenstein series

**Theorem [Langlands]**

For \( f, g \in P(a_C^*) \) one has the inner product formula

\[
(\theta_f, \theta_g) := X^b(gR_f)
\]

with \( R_f(\lambda) := \sum_{w \in W} c(-w\lambda) \frac{r(\lambda)}{r(w\lambda)} f^-( -w\lambda ) \), and \( f^-(\lambda) := \overline{f(\lambda)} \).
Residues of unramified Eisenstein series

**Theorem [Langlands]**

For $f, g \in P(a^*_{\mathbb{C}})$ one has the inner product formula

$$(\theta_f, \theta_g) := X^b(gR_f)$$

with $R_f(\lambda) := \sum_{w \in W} c(-w\lambda) \frac{r(\lambda)}{r(w\lambda)} f^*(-w\lambda)$, and $f^*(\lambda) := \overline{f(\overline{\lambda})}$.

**Remark**

We would like to apply our previous results for $X^b$, $Y^b$ etc., to get an explicit spectral decomposition of $L^{2,K}_{[T,1]}$, but a priori we can only do this for a certain subspace $L^{2,K}_{[T,1],n}$ of $L^{2,K}_{[T,1]}$. The complication here is the potential interference from poles induced from the (critical) zeroes of $\rho(s)$. 
Unramified anti-tempered global Arthur parameters

Let $C_F$ denote the Idèle class group of $F$. Define:

$$AP^{su} := \{ \psi : C_F \times SL_2(\mathbb{C}) \to G^\vee \mid (a) \ \psi|_{C_F} \text{ is bounded.}$$

(b) $\psi|_{C_F}$ factors through $\| \cdot \|$.

(c) $\psi|_{SL_2(\mathbb{C})}$ is algebraic. \}
Unramified anti-tempered global Arthur parameters

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$$(c) \ \psi|_{\text{SL}_2(\mathbb{C})} \text{ is algebraic.} \}$$

Remark

Let $\overline{AP}^{su}$ be the set of equivalence classes in $AP^{su}$. Given $\psi \in AP^{su}$ we can choose $\psi' \in AP^{su}$ with $\psi' \sim \psi$ such that:

- For all $\xi \in C_F$, $\psi'(\xi) = \|\xi\|^{\nu'} \in T^\vee$ for a (unique) $\nu' \in i\mathfrak{a}^*$,
- For all $a \in \mathbb{C}^\times$, $\psi'(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) \in T^\vee$. 
Proposition[De Martino, Heiermann, O.]

Define

\[ D : \overline{AP}^{su} \rightarrow W \backslash \mathfrak{a}_C^* \]

\[ \overline{\psi} \rightarrow \nu' + d\psi' \left( \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right) \]

where \( \psi' \sim \psi \) and \( \nu' \) are as above. Then \( D \) defines a bijection between \( \overline{AP}^{su} \) and \( \Xi := W \backslash \bigcup_{L \text{ residual}} (L^{\text{temp}}) \subset W \backslash \mathfrak{a}_C^* \).
### Theorem [De Martino, Heiermann, O.]

The Hilbert subspace \( L_{[T,1],n}^{2,K} \subset L_{[T,1]}^{2,K} \) is isomorphic to the space \( L^2(\Xi, \mu_n) \) for an **explicitly known** absolutely continuous positive measure \( \mu_n \) on \( \Xi \). Hence \( L_{[T,1],n}^{2,K} \) is spanned by wave packets of **normalized** Eisenstein series \( E_0(\lambda, \cdot) \) with \( \lambda \in \Xi \).
Theorem[De Martino, Heiermann, O.]

The Hilbert subspace $L^{2,K}_{[T,1],n} \subset L^{2,K}_{[T,1]}$ is isomorphic to the space $L^2(\Xi, \mu_n)$ for an explicitly known absolutely continuous positive measure $\mu_n$ on $\Xi$. Hence $L^{2,K}_{[T,1],n}$ is spanned by wave packets of normalized Eisenstein series $E_0(\lambda, \cdot)$ with $\lambda \in \Xi$.

Corollary[De Martino, Heiermann, O.]

For any distinguished nilpotent orbit $\mathcal{O}$ of $g^\vee$, the normalized Eisenstein series $E_0(\lambda_\mathcal{O}, \cdot)$ is a nonzero element in $L^{2,K}_{[T,1],n}$, with explicit $L^2$-norm.
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Corollary[De Martino, Heiermann, O.]

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Corollary[De Martino, Heiermann, O.]

The corresponding local representations $\pi_{\nu,\lambda_{\mathcal{O}}}$ of $G(F_{\nu})$ are unitarizable at all local places $\nu$ of $F$. 
In arXiv:1512.08566 we also claimed to have proven the expected equality $L^{2,K}_{[T,1],n} = L^{2,K}_{[T,1]}$, but here the argument was incomplete. For classical groups this equality was shown by Moeglin, by a very intricate and beautiful combinatorial method. For $G_2$ this was known already since the works of Langlands and Kim. Proving such equality amounts to showing that certain potential residues cancel. So far we have not been able to see the reason behind these cancellations in general, unfortunately. We are still working on this.
The complementary subspace

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Remark

The above results can be extended to $G$ a unramified groups $G$, and more general cuspidal support $[T, \chi]$ where $\chi \in \text{Hom}_c(T(F) \backslash T(\mathbb{A}), \mathbb{C})^{T(\mathbb{A}) \cap K}$ an unramified character.
Many congratulations!