Fast Phase Retrieval in Two Dimensions

Brian Preskitt

University of California, San Diego

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Outline

1. Overview
2. 1-Dimensional Case
3. Extension to Two Dimensional Signals
4. Numerical Results

General setting (for 1D signal $x \in \mathbb{C}^n$)

- Measurement data: $y_i = |\langle a_i, x\rangle|^2 + \eta_i, i = 1, \ldots m$
- Unknown noise $\eta_i \in \mathbb{R}$
- Known vectors $a_i \in \mathbb{C}^n$
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Problem Statement

Given the *magnitudes only* of (potentially noisy) linear measurements of $x \in \mathbb{C}^n$, under what conditions and how can we recover x?

Applications

- X-ray crystallography
- Electron microscopy
- Diffraction imaging
- Ptychographic imaging

Considerations

- Sampling Complexity, $\frac{m}{n}$
- Runtime
- Assumptions on A
- Robustness
- Convergence rate (for iterative methods)

Approaches

- Alternating Projections
  - Gerchberg-Saxton (Error Reduction) [5]
  - Hybrid Input-Output (Fienup [4])
- Convex Relaxations
  - PhaseLift (Candès [3])
  - PhaseCut (Waldspurger [11])
- Non-convex Optimization
  - Wirtinger Flow (Candès et. al [2])
- Graph-based methods
  - Block PR (Iwen, Viswanathan, Wang [7])
  - Salanevich [9], Fickus and Mixon [1], Singer [10]
  - Marchesini [8]

BlockPR – IPSV 2016

Local correlation measurements (IVW '15 [7])

- e.g. Ptychography, STFT
- Window size δ
- Masks $m_i, \mathrm{supp}(m_i) \subset [\delta]$
- Shift operator $(S_\ell \mathbf{x})_j = \mathbf{x}_{j-\ell}$

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Our solution method:

1. Invert this linear system to obtain a projection $\mathcal{P}(\mathbf{xx}^*)$
2. Leverage structure in this projection to determine $\mathbf{x}$.

The linear system can recover *at most* a circulant band of width $2\delta - 1$.

$$\langle S_0 m_j m_j^* S_0^*, \mathbf{xx}^* \rangle = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

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$$\langle S_1 m_j m_j^* S_1^*, \mathbf{x}\mathbf{x}^* \rangle = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

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$$\langle S_6 m_j m_j^* S_6^*, \mathbf{x}\mathbf{x}^* \rangle = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

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$$\langle S_d m_j m_j^* S_d^*, \mathbf{x}\mathbf{x}^* \rangle = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix}$$

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$$\mathcal{P}(\mathbf{x}\mathbf{x}^*) = \begin{pmatrix} * & * & * & & & & * & * \\ * & * & * & * & & & & * \\ * & * & * & * & * & & & \\ & * & * & * & * & * & & \\ & & * & * & * & * & * & \\ & & & * & * & * & * & * \\ * & & & & * & * & * & * \\ * & * & & & & * & * & * \end{pmatrix}$$

$$(T_\delta(X))jk = \begin{cases} X_{jk} & , \quad |j-k| \mod n < \delta \\ 0 & , \quad \text{otherwise} \end{cases}$$

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With $2\delta - 1$ correlation masks m_i, we may invert

$$(y_i)_l = \langle S_\ell m_i m_i^* (S_\ell)^*, \mathbf{xx}^* \rangle$$

to obtain $Y := T_\delta(\mathbf{xx}^*)$

- Diagonal gives magnitudes $|x_i|^2$ exactly.

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- Off diagonals give relative phases

$$\widetilde{X} := \frac{Y}{|Y|}$$

$$\widetilde{X}_{jk} = \begin{cases} \mathrm{e}^{\mathrm{i}(\arg(x_j) - \arg(x_k))}, & |j-k| \mod n < \delta \\ 0, & \text{otherwise} \end{cases}$$

Angular Synchronization (Singer [10])

1. First accomplished with a greedy algorithm (IVW [7])

$$\varphi_{(j+k \mod n)} \leftarrow \varphi_j + \arg(\widetilde{X}_{j,j+k}).$$

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2. Spectral Method (ABFM [1], IPSV [6])

$$\phi = \arg \max_{z \in (S_1)^n} z^* \widetilde{X} z$$

Spectral Method

In noiseless case, $\widetilde{X} = T_\delta(\tilde{x}_0\tilde{x}_0^*)$

Set $D = \mathrm{diag}(\tilde{x}_0)$, then D is unitary and

$$\widetilde{X} = DT_\delta(\mathbb{1}\mathbb{1}^*)D^*$$

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- F is (unitary) discrete Fourier matrix
- $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$, with

$$\lambda_j = K_{\delta-1}\left(\frac{2\pi j}{n}\right), \qquad K_{\delta-1}(z) = \frac{\sin(z \cdot (\delta - 1/2))}{\sin(z/2)}$$

Contributions

- Deterministic measurements!
- Low runtime ($\sim n \log^c n$)
- Non-iterative

BlockPR – Our Result

Proposition (IPSV [6])

There exist vectors $m_1, \ldots, m_{2\delta-1} \in \mathbb{C}^d$ such that, for any $x_0 \in \mathbb{C}^d$, our algorithm produces an estimate x satisfying

$$\min_{\theta\in[0,2\pi]} \|x_0 - e^{i\theta}x\|_2 \leq C\left(\frac{||x_0||_\infty}{(x_0)^2_{\min}}\right)\left(\frac{d}{\delta}\right)^2 \|n\|_2 + C(d\delta)^{1/4}\sqrt{\|n\|_2}$$

Ptychographic Measurements

Proposition (IVW [7])

Suppose $2\delta - 1$ divides d and that $a \in [4, \infty)$. Then the collection

$$(m_j)_k = \frac{\mathrm{e}^{-k/a} \cdot \mathrm{e}^{\frac{2\pi \mathrm{i} k}{2\delta - 1}}}{\sqrt[4]{2\delta - 1}} \cdot \mathbb{1}_{k \leq \delta}$$

spans $T_\delta(\mathbb{C}^{d \times d})$.

These measurements correspond to the system

$$|\mathcal{F}(S_\ell m \cdot \mathbf{x})|^2, \text{ for } m_k = \frac{\mathrm{e}^{-k/a}}{\sqrt[4]{2\delta - 1}} \cdot \mathbb{1}_{k \leq \delta},$$

i.e. *Fourier transforms* of a single mask shifted across the sample.

Two Dimensional Signal

Want to determine $Q \in \mathbb{C}^{d\times d}$ from measurements

$$(y_{(u,v)})_{(\ell,\ell')} \quad = \quad \left|\langle Q, S_\ell A^{(u,v)} S_{\ell'}\rangle_{\mathrm{HS}}\right|^2$$

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\end{aligned}
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\end{aligned}$$

Set $\mathcal{M} : \mathbb{C}^{d^2\times d^2} \to \mathbb{R}^{N^2\times d^2}$ as $\mathcal{M}(\overrightarrow{Q}\,\overrightarrow{Q}^*) = y$ as above.

We require $A^{(u,v)} = m_u m_v$ for $\mathrm{supp}(m_u) \subset [\delta]$ and $u, v \in [N]$.

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$$\langle \overrightarrow{E_{jk}}\overrightarrow{E_{j'k'}}, (S_{\ell'}\overline{m_{j'} m_{j'}} S_{\ell'}) \otimes (S_\ell m_j m_j S_\ell) \rangle_{\mathrm{HS}} = 0$$

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Proposition (IPSV '17)

If $\operatorname{span}\{a_j a_j^*\} = T_\delta(\mathbb{C}^{d\times d})$, then

$$\operatorname{span}\{(a_j \otimes a_{j'})(a_j \otimes a_{j'})^*\} = \mathcal{B}.$$

In particular, Fourier transforms of shifts of the aperture

$$A_{ij} = \frac{e^{-(i+j)/a}}{\sqrt{2\delta - 1}} \cdot \mathbb{1}_{i,j\leq\delta}$$

will span $\mathcal{B}$.

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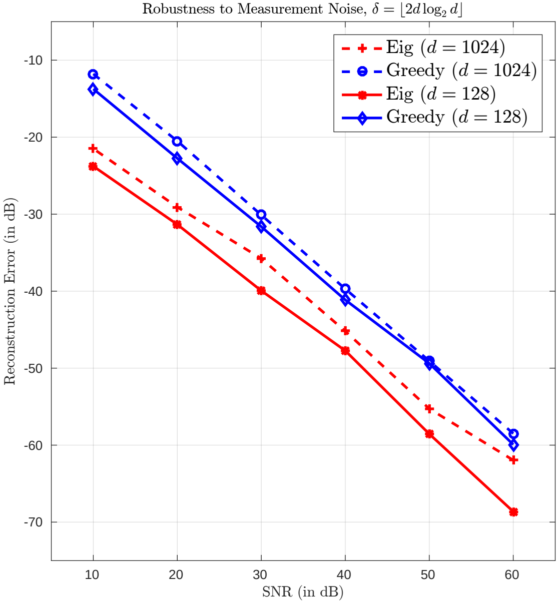
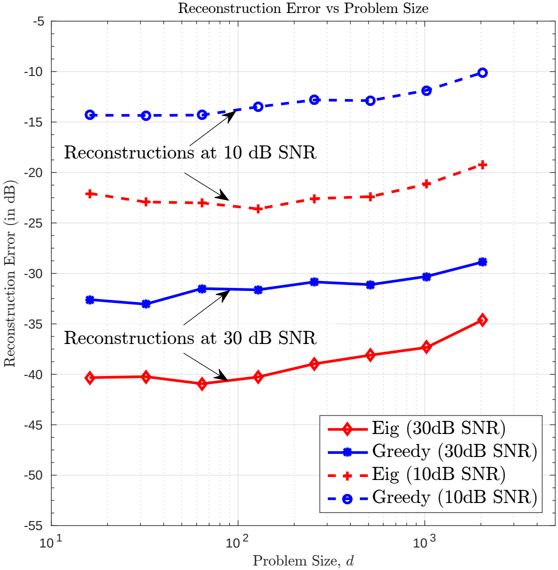
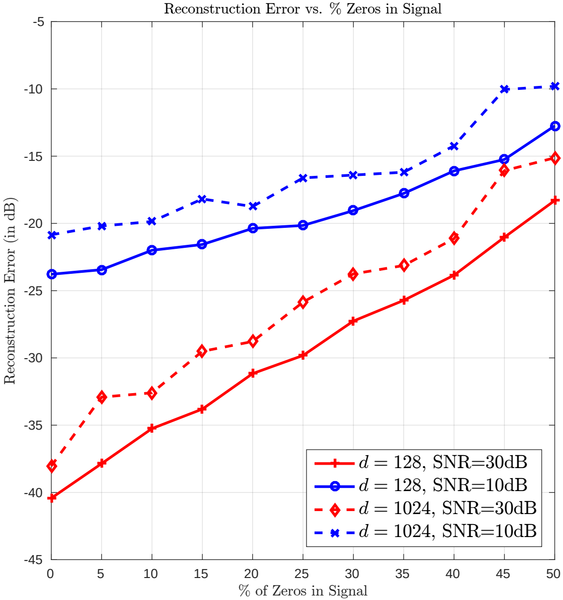
Proposition (IPSV '17)

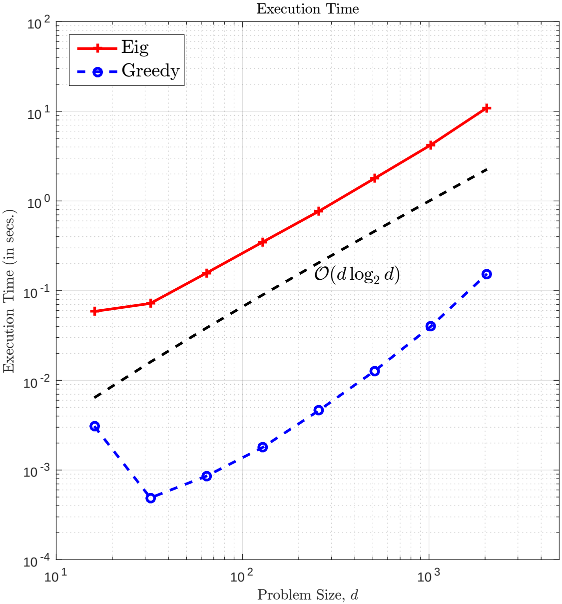
$$\mathcal{B}(\mathbb{1}\mathbb{1}^*) = (F \otimes F)(\Lambda \otimes \Lambda)(F^* \otimes F^*)$$

In other words, the spectral technique for phase synch. and its guarantees extend to two dimensions.


Robustness to Measurement Noise, $\delta = \lfloor 2d \log_2 d \rfloor$
Eig ($d = 1024$)
Greedy ($d = 1024$)
Eig ($d = 128$)
Greedy ($d = 128$)
Reconstruction Error (in dB)
-10
-20
-30
-40
-50
-60
-70
10
20
30
40
50
60
SNR (in dB)


Receonstruction Error vs Problem Size
Reconstruction Error (in dB)
Problem Size, d
Reconstructions at 10 dB SNR
Reconstructions at 30 dB SNR
Eig (30dB SNR)
Greedy (30dB SNR)
Eig (10dB SNR)
Greedy (10dB SNR)


Reconstruction Error vs. % Zeros in Signal
Reconstruction Error (in dB)
% of Zeros in Signal
-5
-10
-15
-20
-25
-30
-35
-40
-45
0
5
10
15
20
25
30
35
40
45
50
d = 128, SNR=30dB
d = 128, SNR=10dB
d = 1024, SNR=30dB
d = 1024, SNR=10dB


Execution Time
Eig
Greedy
$\mathcal{O}(d \log_2 d)$
Execution Time (in secs.)
Problem Size, d
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{1}
10^{2}
10^{3}

Thank you!

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