Computation with Degree-Rips Bifiltrations
in collaboration with Michael Lesnick

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Definitions

Define the following partial order on $\mathbb{N}^2$: 
$(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \geq x_2$ and $y_1 \leq y_2$.

A bifiltration $B$ is a set of vector spaces $\{B_P\}_{P \in \mathbb{N}^2}$ such that $B_P \subset B_Q$ for all $P \leq Q$.

Given a set of points $S$ and non-negative real number $t \in \mathbb{N}$, the $t$ skeleton of $S$ is the graph whose vertices are the points in $S$ and and edge exists between two vertices if the distance between the two points is $\leq t$.

In the degree-Rips bifiltration of a set of points $S$, a 0-simplex $P$ exists at a point $(d, t)$ if in the $t$ skeleton of $S$, we have $\deg(P) \geq d$. A simplex $\sigma$ exists at a point if all of its points and all of its edges exist.
Example

Diagram of a triangle with labeled edges:

- Edge 1 between points 1 and 2
- Edge 2 between points 2 and 3
- Edge 2 between points 3 and 1
Incomparable Grades

What is the set of all points such that point $A$ appears?
Incomparable Grades

What is the set of all points such that point $A$ appears?

**Proposition**

For any simplex $\sigma$ in a degree-Rips bifiltration of a finite set of points $S$, there exists a finite set of points $\Sigma$ such that $\sigma$ exists at a point $Q$ iff 

$$\exists P \in \Sigma \text{ such that } P \leq Q.$$  

The points in a minimal such $\Sigma$ are called the *grades of appearance* of $\sigma$. 
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Step 1: Deal with 0-Simplices

```
function GenerateVertexMultigrades(d, k)
    dk ← {d(k, j) : j ≠ k}
    Grades_k ← {(0, 0)}
    sort(d_k)
    for i ← 0; i < size(d_k); i ++ do
        while i + 1 < size(d_k) and d_k[i + 1] = d_k[i] do
            i ++
        end while
        Grades_k ← Grades_k ∪ {(i + 1, d_k(i))}
    end for
end function
```
Step 2: Induct to $n$-Simplices

**Proposition**

Let $\sigma$ be a $n$-simplex with $n \geq 1$, $\tau$ be a face of $\sigma$ and $P = \sigma \setminus \tau$. Let $D = \max_{Q \in \tau} d(P, Q)$. Let $S_{\sigma}, S_{\tau}, S_{P}$ be the set of points where $\sigma, \tau, P$ exist respectively. Then $S_{\sigma} = S_{\tau} \cap S_{P} \cap \{(x, y) : y \geq D\}$. 

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Sweeping line from left to right. For each $x \in \mathbb{N}$, maintain $\max(D, \min_{(x, y) \in S_P} y, \min_{(x, y) \in S_\tau} y)$. This maximum changes if and only if we have encountered a grade of appearance for $\sigma$. 
Pseudo-code

function CombineMultigrades($S_P$, $S_\tau$, $D$)
    $G \leftarrow \{\}$
    $i1 \leftarrow 0$, $i2 \leftarrow 0$
    $d = \max(D, S_P[i1].y, S_\tau[i2].y)$
    while $i1 < |S_P|$ and $i2 < |S_\tau|$ do
        $minX \leftarrow \min(S_P[i1].x, S_\tau[i2].x)$
        if $S_P[i1].x == minX$ then
            $i1 ++$
        end if
        if $S_\tau[i2].x == minX$ then
            $i2 ++$
        end if
        $d' = \max(D, S_P[i1].y, S_\tau[i2].y)$
        if $d' > d$ then
            $G \leftarrow G \cup \{(minX, d)\}$
            $d \leftarrow d'$
        end if
    end while
end function
Given a bifiltration $B$, we get a chain complex of 2-D bipersistence modules

$$
C_{j+1} \xrightarrow{d_{j+1}} C_j \xrightarrow{d_j} C_{j-1} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} C_0 \to 0
$$

and

$$H_j(B) \cong \ker d_j / \text{im } d_{j+1}.$$ 

RIVET requires the $C_i$ to be free or one-critical.
Let $M[P]$ be the indecomposable module with $M[P]_Q = \begin{cases} k & \text{if } Q \geq P \\ 0 & \text{otherwise} \end{cases}$.

For any set of points $\Sigma$, let $F(\Sigma) = \bigoplus_{P \in \Sigma} M[P]$.

**Proposition**

For a bipersistence module $N$ of rank 1 with grades of appearance given by the set $G$, let $R$ be the set of points which are least upper bounds of pairs of distinct points in $G$. Then there exists a surjective map $F[G] \to N$ and an injection $F[R] \hookrightarrow F[G]$ whose image is the kernel of the former map.
Example

\[ \cdots \leftarrow k^2 \leftarrow k^2 \leftarrow k \]
\[ \cdots \leftarrow k^2 \leftarrow k \leftarrow 0 \]
\[ \cdots \leftarrow k \leftarrow 0 \leftarrow 0 \]

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Example

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\uparrow & \uparrow & \uparrow & \\
\cdots & k^2 & k & 0 \\
\uparrow & \uparrow & \uparrow & \\
\cdots & k & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \\
\cdots & 0 & 0 & 0 \\
\end{array}
\rightarrow
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\uparrow & \uparrow & \uparrow & \\
\cdots & k^2 & k^2 & k \\
\uparrow & \uparrow & \uparrow & \\
\cdots & k^2 & k & 0 \\
\uparrow & \uparrow & \uparrow & \\
\cdots & k & 0 & 0 \\
\end{array}
\]
Resolving $d_{j+1}$

In general, $C_i = \bigoplus_{\text{i-simplices } \sigma} C[\sigma]$ where $C[\sigma]_P$ is $k$ if $\sigma$ exists at $P$ and 0 otherwise.

Let $G_i$ be the set of all the grades of appearance of all the $i$ simplices, counted with multiplicity, and let $R_i$ be the set of least upper bounds of pairs of distinct points coming from the same simplex in $G_i$.

Then we have two maps

$$F[G_{j+1}] \rightarrow C_{j+1} \rightarrow C_j \quad \text{and} \quad F[R_j] \rightarrow F[G_j].$$
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Claim: We can lift the first map to a map $F[G_{j+1}] \to F[G_j]$. 
Resolving $d_{j+1}$

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$$F[G_{j+1}] \to C_{j+1} \to C_j \quad \text{and} \quad F[R_j] \to F[G_j].$$

Claim: We can lift the first map to a map $F[G_{j+1}] \to F[G_j]$. This gives a map:

$$f : F[G_{j+1}] \oplus F[R_j] \to F[G_j].$$
Resolving $d_j$

For each $j - 1$ simplex $\tau$, let $L[\tau]$ be the greatest upper bound of the grades of appearance of $\tau$ and let $L$ be the set consisting of $L[\tau]$ for all such $\tau$. Then, there exists a canonical injection $C_{j-1} \hookrightarrow F[L]$. 
Resolving $d_j$

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$$g : F[G_j] \rightarrow F[L].$$
Resolving $d_j$

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$$g : F[G_j] \rightarrow F[L].$$

Proposition

In the presentation $F[G_{j+1}] \oplus F[R_j] \xrightarrow{f} F[G_j] \xrightarrow{g} F[L]$, $H_j(B) \cong \ker d_j / \im d_{j+1} \cong \ker g / \im f$. 
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Future Work

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- Clearing
- Dualization