Computation with Degree-Rips Bifiltrations

in collaboration with Michael Lesnick

Roy Zhao

August 15, 2018

Table of Contents

1 Background
- Degree-Rips Bifiltration
- Difficulties

2 Computing Bifiltrations
- Grades of Appearance
- Resolving Relations

3 Comments

Definitions

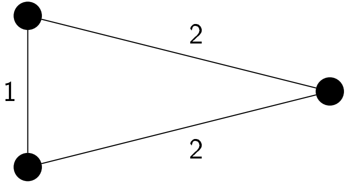
Define the following partial order on $\mathbb{N}^2$:
$(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \geq x_2$ and $y_1 \leq y_2$.

A bifiltration B is a set of vector spaces $\{B_P\}_{P \in \mathbb{N}^2}$ such that $B_P \subset B_Q$ for all $P \leq Q$.

Given a set of points S and non-negative real number $t \in \mathbb{N}$, the t skeleton of S is the graph whose vertices are the points in S and and edge exists between two vertices if the distance between the two points is $\leq t$.

In the *degree-Rips bifiltration* of a set of points S, a 0-simplex P exists at a point (d, t) if in the t skeleton of S, we have $\deg(P) \geq d$.
A simplex σ exists at a point if all of its points and all of its edges exist.

Example



Incomparable Grades

What is the set of all points such that point A appears?

Incomparable Grades

What is the set of all points such that point A appears?

Proposition

For any simplex σ in a degree-Rips bifiltration of a finite set of points S, there exists a finite set of points Σ such that σ exists at a point Q iff $\exists P \in \Sigma$ such that $P \leq Q$.

The points in a minimal such Σ are called the *grades of appearance* of σ.

Table of Contents

1. Background
   - Degree-Rips Bifiltration
   - Difficulties
2. Computing Bifiltrations
   - Grades of Appearance
   - Resolving Relations
3. Comments

Step 1: Deal with 0-Simplices

function GenerateVertexMultigrades(d, k)
    $d_k \leftarrow \{d(k,j) : j \neq k\}$
    $Grades_k \leftarrow \{(0,0)\}$
    $sort(d_k)$
    for $i \leftarrow 0; i < size(d_k); i++$ **do**
        while $i + 1 < size(d_k)$ and $d_k[i+1] = d_k[i]$ **do**
            $i++$
        end while
        $Grades_k \leftarrow Grades_k \cup \{(i+1, d_k(i))\}$
    end for
end function

Step 2: Induct to n-Simplices

Proposition

Let σ be a n-simplex with $n \geq 1$, τ be a face of σ and $P = \sigma \backslash \tau$. Let $D = \max_{Q \in \tau} d(P, Q)$. Let S_σ, S_τ, S_P be the set of points where σ, τ, P exist respectively. Then $S_\sigma = S_\tau \cap S_P \cap \{(x, y) : y \geq D\}$.

Step 2: Induct to n-Simplices

Proposition

Let σ be a n-simplex with $n \geq 1$, τ be a face of σ and $P = \sigma \backslash \tau$. Let $D = \max_{Q \in \tau} d(P, Q)$. Let S_σ, S_τ, S_P be the set of points where σ, τ, P exist respectively. Then $S_\sigma = S_\tau \cap S_P \cap \{(x, y) : y \geq D\}$.

Sweeping line from left to right. For each $x \in \mathbb{N}$, maintain $\max(D, \min_{(x,y) \in S_P} y, \min_{(x,y) \in S_\tau} y)$.
This maximum changes if and only if we have encountered a grade of appearance for σ.

Pseudo-code

function CombineMultigrades(S_P, S_τ, D)

    $G \leftarrow \{\}$

    $i1 \leftarrow 0, i2 \leftarrow 0$

    $d = \max(D, S_P[i1].y, S_\tau[i2].y)$

    while $i1 < |S_P|$ and $i2 < |S_\tau|$ **do**

        $minX \leftarrow \min(S_P[i1].x, S_\tau[i2].x)$

        if $S_P[i1].x == minX$ **then**

            $i1++$

        end if

        if $S_\tau[i2].x == minX$ **then**

            $i2++$

        end if

        $d' = \max(D, S_P[i1].y, S_\tau[i2].y)$

        if $d' > d$ **then**

            $G \leftarrow G \cup \{(minX, d)\}$

            $d \leftarrow d'$

        end if

Mathematical Background

Given a bifiltration B, we get a chain complex of 2-D bipersistence modules

$$C_{j+1} \xrightarrow{d_{j+1}} C_j \xrightarrow{d_j} C_{j-1} \xrightarrow{d_{j-1}} \cdots \xrightarrow{d_1} C_0 \to 0$$

and

$$H_j(B) \cong \ker d_j / \operatorname{im} d_{j+1}.$$

RIVET requires the C_i to be free or one-critical.

Minimal Presentation

Let $M[P]$ be the indecomposable module with $M[P]_Q = \begin{cases} k & \text{if } Q \geq P \\ 0 & \text{otherwise} \end{cases}$.

For any set of points Σ, let $F(\Sigma) = \bigoplus_{P \in \Sigma} M[P]$.

Proposition

For a bipersistence module N of rank 1 with grades of appearance given by the set G, let R be the set of points which are least upper bounds of pairs of distinct points in G. Then there exists a surjective map $F[G] \to N$ and an injection $F[R] \hookrightarrow F[G]$ whose image is the kernel of the former map.

Example

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k^2 & \longleftarrow & k^2 & \longleftarrow & k \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k^2 & \longleftarrow & k & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
\quad \to \quad
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & k & \longleftarrow & k \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & k & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
$$

Example

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k^2 & \longleftarrow & k & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & 0 & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k^2 & \longleftarrow & k^2 & \longleftarrow & k \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k^2 & \longleftarrow & k & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longleftarrow & k & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
$$

Resolving d_{j+1}

In general, $C_i = \bigoplus_{i\text{-simplices } \sigma} C[\sigma]$ where $C[\sigma]_P$ is k if σ exists at P and 0 otherwise.

Let G_i be the set of all the grades of appearance of all the i simplices, counted with multiplicity, and let R_i be the set of least upper bounds of pairs of distinct points coming from the same simplex in G_i.

Then we have two maps

$$F[G_{j+1}] \to C_{j+1} \to C_j \quad \text{and} \quad F[R_j] \to F[G_j].$$

Resolving d_{j+1}

In general, $C_i = \bigoplus_{i\text{-simplices } \sigma} C[\sigma]$ where $C[\sigma]_P$ is k if σ exists at P and 0 otherwise.

Let G_i be the set of all the grades of appearance of all the i simplices, counted with multiplicity, and let R_i be the set of least upper bounds of pairs of distinct points coming from the same simplex in G_i.

Then we have two maps

$$F[G_{j+1}] \to C_{j+1} \to C_j \quad \text{and} \quad F[R_j] \to F[G_j].$$

Claim: We can lift the first map to a map $F[G_{j+1}] \to F[G_j]$.

Resolving d_{j+1}

In general, $C_i = \bigoplus_{i\text{-simplices } \sigma} C[\sigma]$ where $C[\sigma]_P$ is k if σ exists at P and 0 otherwise.

Let G_i be the set of all the grades of appearance of all the i simplices, counted with multiplicity, and let R_i be the set of least upper bounds of pairs of distinct points coming from the same simplex in G_i.

Then we have two maps

$$F[G_{j+1}] \to C_{j+1} \to C_j \quad \text{and} \quad F[R_j] \to F[G_j].$$

Claim: We can lift the first map to a map $F[G_{j+1}] \to F[G_j]$. This gives a map:

$$f : F[G_{j+1}] \oplus F[R_j] \to F[G_j].$$

Resolving d_j

For each $j-1$ simplex τ, let $L[\tau]$ be the greatest upper bound of the grades of appearance of τ and let L be the set consisting of $L[\tau]$ for all such τ. Then, there exists a canonical injection $C_{j-1} \hookrightarrow F[L]$.

Resolving d_j

For each $j-1$ simplex τ, let $L[\tau]$ be the greatest upper bound of the grades of appearance of τ and let L be the set consisting of $L[\tau]$ for all such τ. Then, there exists a canonical injection $C_{j-1} \hookrightarrow F[L]$. Composing with C_j gives us a map $C_j \to C_{j-1} \to F[L]$ and thus a map

$$g : F[G_j] \to F[L].$$

Resolving d_j

For each $j-1$ simplex τ, let $L[\tau]$ be the greatest upper bound of the grades of appearance of τ and let L be the set consisting of $L[\tau]$ for all such τ. Then, there exists a canonical injection $C_{j-1} \hookrightarrow F[L]$. Composing with C_j gives us a map $C_j \to C_{j-1} \to F[L]$ and thus a map

$$g : F[G_j] \to F[L].$$

Proposition

In the presentation $F[G_{j+1}] \oplus F[R_j] \xrightarrow{f} F[G_j] \xrightarrow{g} F[L]$, $H_j(B) \cong \ker d_j / \operatorname{im} d_{j+1} \cong \ker g / \operatorname{im} f$.

Table of Contents

1 Background
- Degree-Rips Bifiltration
- Difficulties

2 Computing Bifiltrations
- Grades of Appearance
- Resolving Relations

3 Comments

Future Work

Performance speedups

- Clearing
- Dualization