Multiparameter Persistence via Geometric Topology

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Outline

Motivation

Persistent homology and Morse theory

Multiparameter persistence and Cerf theory
Kernel Density Estimation

- Suppose we’re given experimental data $x_1, \ldots, x_N$, lying in $\mathbb{R}^n$.
- Mathematically, we assume these are sampled from an underlying distribution $f$, which we’d like to understand as best we can.
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• Mathematically, we assume these are sampled from an underlying distribution $f$, which we’d like to understand as best we can.
• One such method is (Gaussian) Kernel density estimation:
Kernel Density Estimation

- How do we select the bandwidth $\alpha$?
- How do we select the threshold $y$?
Kernel Density Estimation

- How do we select the bandwidth $\alpha$?
- How do we select the threshold $y$?
- Want an understanding of the connected components of $\{\hat{f}_\alpha \geq y\}$. How they bifurcate, come together, etc.
- Which modes persist through many values of $\alpha$, $y$? This leads to a global understanding of the parameter space, not just a local one.
Persistent homology and Morse theory

- Persistent homology is the homology of a filtered space.
- Let $M$ be a smooth manifold equipped with a Morse function $f : M \to \mathbb{R}$. We study the sub-level sets of $f$. 
• Persistent homology is the homology of a filtered space.
• Let $M$ be a smooth manifold equipped with a Morse function $f : M \to \mathbb{R}$. We study the sub-level sets of $f$. 
“The critical points determine the topology of the sub-level sets.”
The output is a *persistence module*, and can be thought of as a functor $\mathbb{R} \to \text{vect}_k$. That is, a one-parameter family of vector spaces with linear maps between them.

**Theorem (Crawley-Boevey, Gabriel)**

*Every (tame) persistence module splits into a sum of indecomposables. The indecomposables are given by connected intervals, and hence, either of finite length or infinite length.*
We’d like to vary two (or more) parameters simultaneously.

A *multiparameter persistence module* is a family of vector spaces and linear maps depending on a variety of parameters.

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Idea: Restrict to modules arising in geometric settings.
• J. Cerf studied families of smooth functions on smooth manifolds in his work on pseudo-isotopy theory.
• Cerf showed this function space admits a stratification for which
  1. Morse functions are codimension 0 (open and dense), and
  2. ‘generalized’ Morse functions are codimension 1 (generic families include finitely many of them).
• Morse functions allow quadratic singularities, whereas generalized Morse functions allow for cubic singularities.
Given a one-parameter family of smooth functions \( \tilde{f} : M \times I \to \mathbb{R} \), form a fibered version

\[
f : M \times I \to \mathbb{R} \times I,
\]

\[
(m, t) \mapsto (\tilde{f}(m, t), t).
\]

We’re interested in modules of the form

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H_i(f^{-1}((-\infty; a], [t_1, t_2]); k).
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Think of as a composite

$$\mathbb{R} \times \text{Int}_I \xrightarrow{F} \text{Top} \xrightarrow{H_i} \text{vect}_k$$

where

$$F(a, [t_1, t_2]) = f^{-1}((-\infty, a], [t_1, t_2]).$$
The Cerf diagram

**Definition**

The *Cerf diagram* of a generic family of Morse functions \( \tilde{f} : M \times I \to \mathbb{R} \) is given by

\[
\bigcup_{(m,t) \in \Sigma(\tilde{f})} \tilde{f}(m, t), t) \subset \mathbb{R} \times I.
\]

The Cerf diagram tracks the critical values as the function \( \tilde{f}(\_-, t) \) varies.
Simple example

\[ H^* F \sim = 0 \]

\[ H_0 \]

\[ H_1 \]

\[ 1 \]

\[ 0 \]
Simple example

\[ H_* F \cong H_0 \oplus H_1 \]
Definition

A point $p$ is a fold singularity of index $j$, if near $p$, there exist coordinates so that $f$ looks like

$$(x, t) \mapsto \left( -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{n} x_i^2, t \right).$$
Cusp singularities

Definition

A point $p$ is a cusp singularity of index $j + 1/2$, if near $p$, there exist coordinates so that $f$ looks like

$$(z, x, t) \mapsto \left( z^3 + 3tz - \sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{n-q} x_i^2, t \right),$$
### Definition

For an open subset $U$, a *wrinkle of index* $j + 1/2$ is a map equivalent to

$$w(x, z, t) = \left( z^3 + 3 \left( |t|^2 - 1 \right) z - \sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{n-q} x_i^2, t \right).$$
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A map $f$ is *wrinkled* if there exist disjoint open subsets $U_i$ such that

- $f|_{U_i}$ is a wrinkle, and
- $f|_{M \setminus \bigcup U_i}$ is a submersion.
Wrinkling of smooth mappings and its applications

- The singular set of a wrinkle is a sphere, with equator given by cusps of index $s + 1/2$. The upper hemisphere consists of folds of index $s$ and the lower hemisphere consists of folds of index $s + 1$.

\textbf{Fig. 4.} A fibered wrinkle, \textit{a}) in the source, \textit{b}) in the image
Dented cylinder

\[ F : \mathbb{R} \times \text{Int}_I \rightarrow \text{Top} \]
Dented cylinder
Dented cylinder
Dented cylinder
Dented cylinder
Dented cylinder

\[ H_* F \cong \bigoplus \bigoplus \]

(intersect both)
Theorem (Bubenik-C.)

Let $\tilde{f} : M \times I \to \mathbb{R}$ be a map with wrinkles. Each wrinkle gives rise to an indecomposable summand of $H_*(f)$. A wrinkle of index $j + 1/2$ contributes to $H_j$ and $H_{j+1}$.

- View each semi-infinite rectangle as a cobordism.
- Proof uses left/right half handle attachments from Morse theory for manifolds with boundary.
Future directions

- Representations of quivers.
- Level set persistence, leading to a cosheaf.
- Simplifications: Reidemeister moves, or more ambitiously, the h-principle.