Geometric Statistics for High-Dimensional Data Analysis

Snigdhansu Chatterjee

School of Statistics, University of Minnesota

Joint work with Lindsey Dietz, Megan Heyman, Subhabrata (Subho) Majumdar, and Ujjal Mukherjee

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Major contributors




Outline

Quantiles: univariate, multivariate

Geometric quantiles for classification

The Indian Summer Monsoons: GSQ for feature selection

fMRI data: GSQ for spatio-temporal modeling

Univariate quantiles

- Suppose $X \in \mathbb{R}$ is a random variable.
- For any $\alpha \in (0, 1)$, the α^{th} quantile Q_α is the number below which X is observed with probability α, *i.e.* $Q_\alpha = \inf\{q : \mathbb{P}[X \leq q] \geq \alpha$.

Theorem

If X is (absolutely) continuous with cumulative distribution function $F(\cdot)$, then $F(X) \sim Uniform(0, 1)$, and there is a one-to-one relationship between α and Q_α.

Univariate quantiles: an alternative view

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- **(Alternative notation)** Define $u = 2\alpha - 1 \in (-1, 1)$. The u^{th} quantile Q_u is the (unique) minimizer of

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- $$= \mathbb{E}\{||X - q|| + < u, X - q >\}.$$
- *Define* quantiles in any inner-product space as minimizers of $\Psi_u(q) = \mathbb{E}\{||X - q|| + < u, X - q >\}$. (Haldane (1948), Chaudhuri (1996).)

Univariate to multivariate quantiles

Univariate quantiles:

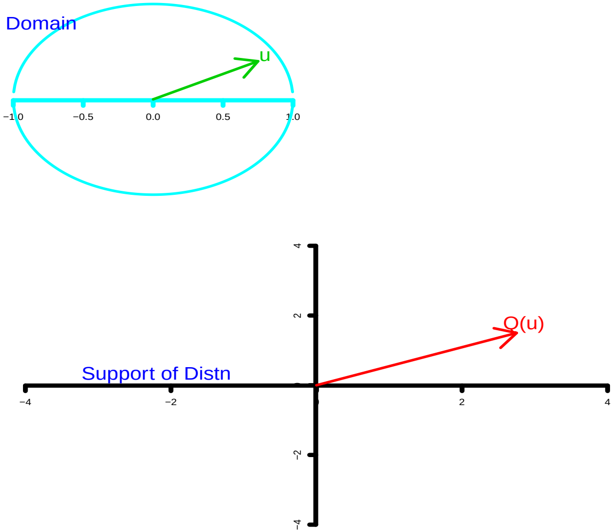
For every $u \in \{x : ||x|| < 1\} \subset \mathbb{R}$, $Q(u)$ minimizes
$\Psi_u(q) = \mathbb{E}\,[||X - q|| + < u, X - q >]$.

Write $x = x_u u/||u|| + x_{u^\perp}$.
Generally, for some $\lambda \geq 0$, the *generalized spatial quantile* (GSQ) are:

1. indexed by vectors in the unit ball $u \in \mathcal{B}_p = \{x : ||x|| < 1\}$, and
2. the u-th quantile $Q(u)$ is the minimizer of

$$\Psi_{u\lambda}(q) = \mathbb{E}\Big[|X_u - q_u|\{1 + \lambda(X_u - q_u)^{-2}||X_{u^\perp} - q_{u^\perp}||^2\}^{1/2} + ||u||(X_u - q_u)\Big].$$

Bivariate quantiles



Bahadur representation of generalized spatial quantiles

Theorem

The following asymptotic Bahadur-type representation holds with probability 1 for any u:

$$n^{1/2}(\hat{Q}(u) - Q(u)) = -n^{-1/2} H^{-1} S_n + O(n^{-(1+s)/4} (\log\ n)^{1/2} (\log\log\ n)^{(1+s)/4})$$

as $n \to \infty$.

(Apologies for not including the details.)

Projection quantiles

Generalized spatial quantiles minimize:

$$\Psi_{u\lambda}(q) = \mathbb{E}\Big[|X_u - q_u|\{1 + \lambda(X_u - q_u)^{-2}||X_{u^\perp} - q_{u^\perp}||^2\}^{1/2} + ||u||(X_u - q_u)\Big].$$

Set $\lambda = 0$ to get projection quantiles.

- Computationally extremely simple, no limitations from sample size and dimension (high p, low n allowed).
- Projection quantiles based confidence sets have exact coverage.
- Works on infinite-dimensional spaces.

Projection quantiles

Theorem

Projection quantiles have a one-to-one relationship with the unit ball, like univariate quantiles.

Example: simulated data plots

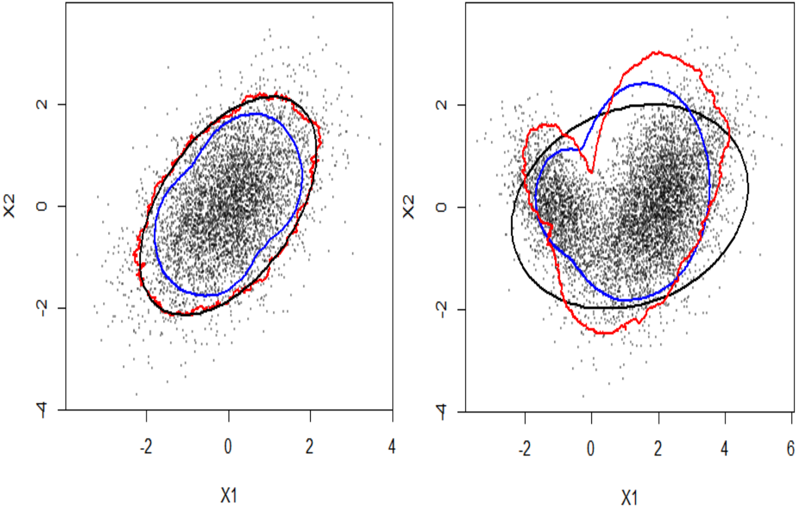


Figure: Simulated data with a few GSQ (covered areas are deliberately different)

Outline

GSQ-depths are great for classification

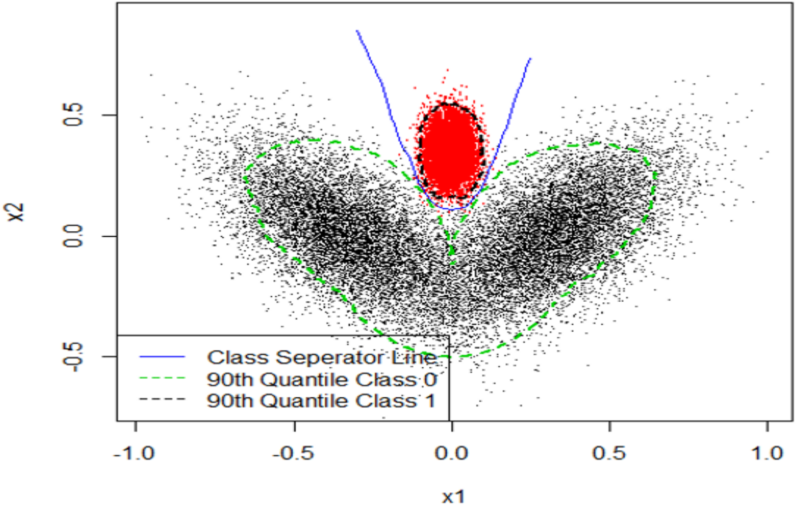


Figure: A simulated 2-class classification problem with GSQ-depth classifier

GSQ-depth based classification: some results

Method	CPU Time	Accuracy
GSQ	3.67	0.925
Random Forest	16714.20	0.895
SVM	966.86	0.842
LDA	0.28	0.74
Logit	0.35	0.69

Table: Arcene classification without feature selection (neural nets did not converge)

Outline

The data on monsoons

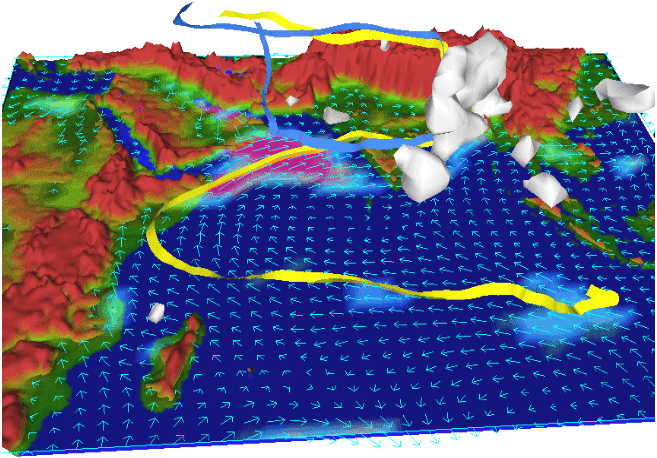
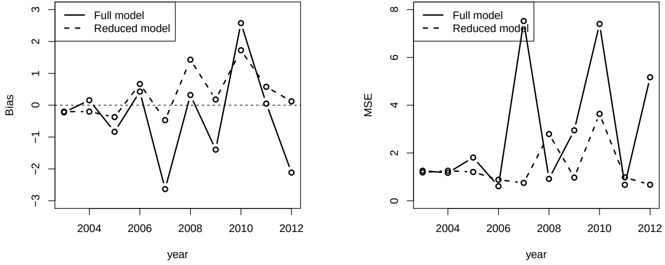


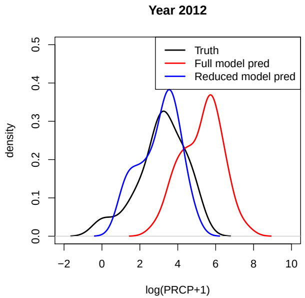
Figure: Air from the eastern Indian Ocean (yellow) and air descending over Arabia (blue) converge in the Somali jet. Low pressure at 30S. {*Courtesy: UMn Climate Expeditions team.*}

Variable dropped	$\hat{e}_n(\mathcal{S}_{-j})$
- Tmax	0.1490772
- X120W	0.2190159
- ELEVATION	0.2288938
- X120E	0.2290021
- ΔTT_Deg_Celsius	0.2371846
- X80E	0.2449195
- LATITUDE	0.2468698
- TNH	0.2538924
- Nino34	0.2541503
- X10W	0.2558397
- LONGITUDE	0.2563105
- X100E	0.2565388
- EAWR	0.2565687
- X70E	0.2596766
- v_wind_850	0.2604214
- X140E	0.2609039
- X40W	0.261159
- SolarFlux	0.2624313
- X160E	0.2626321
- EPNP	0.2630901
- TempAnomaly	0.2633658
- u_wind_850	0.2649837
- WP	0.2660394
$<$none$>$	0.2663496
- POL	0.2677756
- Tmin	0.268231
- X20E	0.2687891

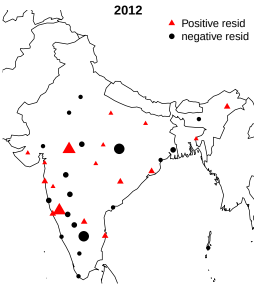


(a) (b)

Figure: Comparing full model rolling predictions with reduced models: (a) Bias across years, (b) MSE across years.



(c)



(d)

Figure: Comparing full model rolling predictions with reduced models: (c) density plots for 2012, (d) stationwise residuals for 2012

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A brief outline

- We consider 19 tests subjects, with 2 kinds of visuals tasks.
- Each subject went through 9 runs, where they saw faces or scrambled images, and had to react.
- We fit a spatio-temporal model. Temporally, we fit a AR(5) with quadratic drift. Spatially, we consider different layers nearest neighbor voxels.
- We measure the degree of spatial dependency in different regions of the brain.
- The figures below are for one subject in one run.

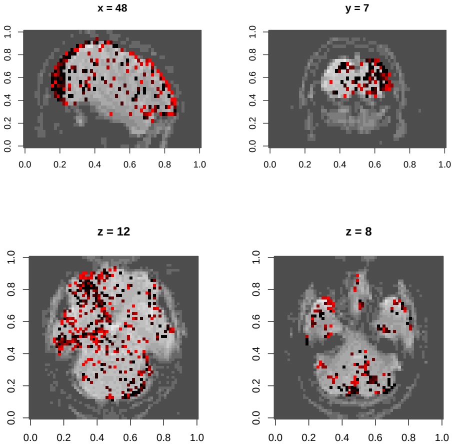

Figure: Plot of significant p-values at 95% confidence level at the specified cross-sections.

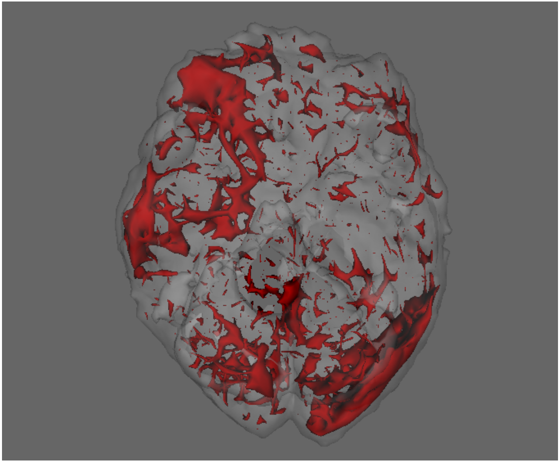

Figure: A smoothed surface obtained from the *p*-values clearly shows high spatial dependence in right optic nerve, auditory nerves, auditory cortex and left visual cortex areas

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Thank you