Sequential change-point detection for Hawkes processes

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A sequence of events:

User A re-tweeted the picture at 10:10am
User B liked the picture at 11:05am
User C commented on it at 3:01pm

People influence each other.
Acts of crimes come with follow-ups.
Discrete event data over networks

- Observe a sequence of discrete events over a network

\[(t_i, u_i), i = 1, 2, \ldots\]

- \(t_i\): time of \(i\)th event
- \(u_i\): node index of \(i\)th event

- Can be used to capture **spatial-temporal** dependent process

![Diagram](image)
Difference from other models

- Independent and identically distributed data (iid.): $P(x_1, x_2, x_3)$. 
- Time series data: 
- Asynchronous and interdependent data with red arrows indicating dependency.
(Incomplete) list of reference

- Modeling network of point processes, estimate parameters offline or online [Myers, Leskovec 14][Rodriguez et.al 11][Yang, Zha 13][Hall, Willett 14][Bertozzi, Short 2013]

- Non-parametric Hawkes process [Chen, Shojaie, Shea-Brown, Witten 2017]

- Detect change has been focusing on single data stream, usually assuming Poisson assumption [Zhang et.al 14]

- Granger causality based on network Hawkes processes [Xu et al. 2016]
Change-point detection

Use sequence of events $\{(t_i, u_i)\}$ to detect a change online.

Poisson to Hawkes

Hawkes, intensity increases, star topology
Outline

▶ Background

▶ Generalized likelihood ratio statistic

▶ Control false alarm

▶ Computation
Point processes

- A sequence of random events in time \( \{t_1, t_2, \ldots \} \)

\[ \mathcal{H}_t = \{t_1, \ldots, t_n : t_n < t\} \]

- Rate function \( \lambda(t) \) of an event happens at \( t \)

\[ \lambda_t dt = \mathbb{P}\{\text{event in } [t, t + dt] | \mathcal{H}_t\} \]
Hawkes process

Intensity depends on history

\[ \lambda(t) = \mu(t) + \alpha \sum_{i: t_i < t} \beta e^{-\beta(t-t_i)} \]

- \( \mu(t) \): background rate of event happening
- \( \phi(t) = \beta e^{-\beta t} \): kernel measures influence from the history
- \( \beta \): decay rate
- \( \alpha > 0 \): strength of self-exciting part
- \( \alpha = 0 \): no influence from past
Network Hawkes process

In the network setting

\[ \lambda_i(t) = \mu_i(t) + \sum_{j=1}^{d} \sum_{t_i < t} \alpha_{u_i,j} \beta e^{-\beta(t-t_i)} \]

- influence parameter \( \alpha_{ij} \): mutual excitation between nodes
  - \( \alpha_{ij} = 0 \): no mutual excitation between \( i \) and \( j \)
  - \( \alpha_{ii} = 0 \): no self excitation at \( i \)
Outline

- Background
- Sequential change-point detection
- Control false alarm
- Computation
Change-point detection

"... to detect abrupt changes in the statistical behavior of an observed signal or time series."

– Quickest detection by Poor and Hadjiliadis, 2008

I change-point represents an interesting event or anomaly such as a seismic event, solar flare, or human activity.

I wish to detect change-point online

▶ Shewhart chart, CUSUM, GLR, Shiryaev-Roberts procedures...
Sequential hypothesis test for 1-dimensional case

\[
\begin{cases}
H_0 : \lambda(s) = \mu, \quad 0 < s < t; \\
H_1 : \lambda(s) = \mu, \quad 0 < s < \kappa, \\
\lambda^*(s) = \mu + \alpha \int_{\kappa}^{s} \varphi(s - \tau) dN_\tau, \quad \kappa < s < t.
\end{cases}
\]

- \( \kappa \): unknown change-point location
- \( \mu \) known, \( \alpha \) unknown
- count measure

\[
dN_t = \sum_{t_i \in H_t} \delta(t - t_i)dt, \quad H_t = \{t_1, \ldots, t_n : t_n < t\}.
\]
Network (multi-dimensional) case

\[
\begin{align*}
H_0: \quad & \lambda_i(s) = \mu_i, \quad 0 < s < t; \\
H_1: \quad & \lambda_i(s) = \mu_i, \quad 0 < s < \kappa, \\
& \lambda_i^*(s) = \mu_i + \sum_{j=1}^{d} \alpha_{ji} \int_{\kappa}^{s} \varphi(s - \tau) dN_{\tau,j}, \quad \kappa < s < t.
\end{align*}
\]

\(\alpha_{ij}:\) unknown
Sequential change-point detection

- Detect a change-point in the distribution of the process
- Classic formulation: sequential hypothesis testing
  a sequence of i.i.d. observations $y_1, y_2, \cdots \in \mathbb{R}$

$$H_0 : \quad y_t \sim f_0, \quad t = 1, 2, \ldots$$
$$H_1 : \quad y_t \sim f_0, \quad t = 1, \ldots, \kappa,$$
$$y_t \sim f_1, \quad t = \kappa + 1, \ldots$$

unknown change-point $\kappa > 0$

**goal**: detect change-point as quickly as possible
Stopping time procedure

- for a hypothesized $\kappa = k$:

$$\ell(t, k) = \log \frac{\prod_{i=1}^{k} f_0(y_i) \cdot \prod_{i=k+1}^{t} f_1(y_i)}{\prod_{i=1}^{t} f_0(y_i)} = \sum_{i=k+1}^{t} \log \frac{f_1(y_i)}{f_0(y_i)}$$

- stop the first time hitting a threshold $b$

$$T = \inf\{t : \max_{k < t} \ell(t, k) \geq b\}$$
Likelihood function

likelihood of \( \{0 < t_1 < \cdots < t_n < t\} \)

\[
\mathcal{L} = f(t_1, \ldots, t_n) = (1 - F^*(t)) \prod_{i=1}^{n} f(t_i \mid t_1, \ldots, t_{i-1})
\]

\[
= (1 - F^*(t)) \prod_{i=1}^{n} f^*(t_i) = \left( \prod_{i=1}^{n} \lambda_{t_i} \right) \exp \left\{ - \int_{0}^{t} \lambda_s ds \right\}.
\]
Generalized likelihood ratio (GLR) statistic

- GLR as a function of $t$, change-point location $\tau$, and unknown influence parameter

$$
\ell(t, \tau, \alpha) = \sum_{t_i \in (\tau, t)} \log \left[ \mu + \alpha \sum_{t_j \in (\tau, t_i)} \beta e^{-\beta(t_i - t_j)} \right] - \mu(t - \tau) - \alpha \sum_{t_i \in (\tau, t)} \left[ 1 - e^{-\beta(t - t_i)} \right].
$$

$$
T_{\text{one-dim}} = \inf \{ t : \max_{\tau < t} \max_{\alpha} \ell_{t, \tau, \alpha} > b \}.
$$
GLR statistic for network setting

$$\ell_{t, \tau, A^*} = \sum_{t_i \in (\tau, t)} \log \left[ \frac{\mu_{u_i} + \sum_{t_j \in (\tau, t_i)} \alpha^*_{u_i, u_j} \beta e^{-\beta(t_i - t_j)}}{\mu_{u_i} + \sum_{t_j \in (\tau, t_i)} \alpha_{u_i, u_j} \beta e^{-\beta(t_i - t_j)}} \right]$$

$$- \sum_{j=1}^{d} \sum_{t_i \in (\tau, t)} \left( \alpha^*_{j, u_i} - \alpha_{j, u_i} \right) \left[ 1 - e^{-\beta(t - t_i)} \right] ,$$
Sliding window procedure

Detection procedure is a stopping time

\[ T = \inf \{ t : \max_{\tau < t} \max_{A} \ell_{t,\tau, A} > b \} , \]
Example

\[ T = \inf \{ t : \max_{\tau < t} \max_A \ell_{t, \tau, A} > b \}, \]

ARL = 10000 unit time. Window length = 200 unit time.
Outline

► Background

► Sequential change-point detection

► Control false alarm

► Computation
Performance metrics

- **No change:**
  Average-Run-Length (ARL): $\mathbb{E}^\infty[T]$

- **After change:**
  Expected detection delay: $\sup_k \mathbb{E}^k[T - k | T > k]$
To find ARL approximation, we find the following probability

\[ P_{H_0}\left\{ \max_{t<t_0} \max_{\mathbf{A}} A_{t,t-L,A} > b \right\}, \quad b \to \infty \]

using the change-of-measure technique [Yakir, Siegmund, 2010]

**Theorem (Average-Run-Length, Li, X. 2017)**

When \( b \to \infty \)

\[ \mathbb{E}^\infty[T] = e^b \left[ \int \cdots \int_{a_{ij}} \nu \left( \frac{2\xi}{\eta^2} \right) \frac{\phi \left( \frac{LI-b}{\sqrt{L\sigma^2}} \right)}{\sqrt{L\sigma^2}} da_{ij} \right]^{-1} \cdot (1 + o(1)). \]
TABLE I

<table>
<thead>
<tr>
<th>Setting</th>
<th>$I$</th>
<th>$I_0$</th>
<th>$\sigma^2$</th>
<th>$\sigma_0^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poi. → Haw.</td>
<td>$\frac{\mu}{1-\alpha} \log \left( \frac{1}{1-\alpha} \right) - \frac{\alpha}{1-\alpha} \mu$</td>
<td>$\mu \log \left( \frac{1}{1-\alpha} \right) - \frac{\alpha}{1-\alpha} \mu$</td>
<td>$\left[ \log \left( \frac{1}{1-\alpha} \right) \right]^2 \cdot \left[ \frac{\mu}{1-\alpha} + \frac{\alpha(2-\alpha)\mu}{(1-\alpha)^3} \right]$</td>
<td>$\mu \left[ \log \left( \frac{1}{1-\alpha} \right) \right]^2$</td>
</tr>
<tr>
<td>high dim.</td>
<td>$\lambda^* \left( \log(\lambda^*) - \log(\mu) \right)$</td>
<td>$-e^T(\lambda^* - \mu)$</td>
<td>$e^T \left( H \circ C \right) e$</td>
<td>$\mu^T \left( \log(\lambda^*) - \log(\mu) \right)^{(2)}$</td>
</tr>
<tr>
<td>one dim.</td>
<td>$\frac{\mu}{1-\alpha} \log \left( \frac{1}{1-\alpha} \right) - \frac{\mu}{1-\alpha} + \frac{\mu}{1-\alpha}$</td>
<td>$\frac{\mu}{1-\alpha} \cos \left( 1 - \frac{\mu}{1-\alpha} \right) - \frac{\mu}{1-\alpha} + \frac{\mu}{1-\alpha}$</td>
<td>$\left[ \log \left( \frac{1}{1-\alpha} \right) \right]^2 \cdot \left[ \frac{\mu}{1-\alpha} + \frac{\alpha(2-\alpha)\mu}{(1-\alpha)^3} \right] + \left( 1 - \frac{\lambda^*}{1-\alpha} \right)^2 \cdot \left[ \frac{\mu}{1-\alpha} + \frac{\alpha(2-\alpha)\mu}{(1-\alpha)^3} \right]$</td>
<td>$\left[ \log \left( \frac{1}{1-\alpha} \right) \right]^2 \cdot \left[ \frac{\mu}{1-\alpha} + \frac{\alpha(2-\alpha)\mu}{(1-\alpha)^3} \right]$</td>
</tr>
<tr>
<td>multi dim.</td>
<td>$\lambda^* \left[ \log(\lambda^*) - \log(\lambda) \right]$</td>
<td>$-e^T(\lambda^* - \lambda)$</td>
<td>$e^T \left( G \circ C^* + F \circ C \right) e$</td>
<td>$e^T \left( R \circ C^* + G \circ C \right) e$</td>
</tr>
</tbody>
</table>

Null: 2D Poisson process $\mu_i = 0.5$; $\beta = 1$; $L = 300$ and 400.
Challenge in analysis: overlapping windows cause dependent statistics.

\[ P_\infty \left\{ \sup_{t < m, \alpha \in \Theta} \ell_{t, t-L, \alpha} > x \right\} \]

\( \ell_{t, t-L, \alpha} \) and \( \ell_{t', t-L, \alpha'} \) are correlated!
Proof technique: Change-of-measure

- Technique $\ell_{t,\tau,A}$ can be approximated as Gaussian random field (Siegmund, Yakir, Zhang, 2010)

\[ \mathbb{P}_\infty \left\{ \sup_{t < m, \alpha \in \Theta} \ell_{t, \tau, \alpha} > x \right\} \]
Proof technique: Change-of-measure

- Technique $\ell_{t,\tau,A}$ can be approximated as Gaussian random field

$$
\mathbb{P}_\infty \left\{ \sup_{t<m, \alpha \in \Theta} \ell_{t,\tau,\alpha} > x \right\} = \mathbb{E} \left[ \frac{\int_t \int_{\alpha \in \Theta} e^{\ell_{t,\tau,\alpha}} dt d\alpha}{\int_{t'} \int_{\alpha' \in \Theta} e^{\ell_{t',\tau',\alpha'}} dt' d\alpha'} ; \sup_{t<m, \alpha \in \Theta} \ell_{t,\tau,\alpha} > x \right]_{=1}
$$
$$\mathbb{P}_\infty \left\{ \sup_{t<m, \alpha \in \Theta} \ell_{t, \tau, \alpha} > x \right\}$$

$$= \mathbb{E} \left[ \frac{\int_t \int_{\alpha \in \Theta} e^{\ell_{t, \tau, \alpha}} dt d\alpha}{\int_{t'} \int_{\alpha' \in \Theta} e^{\ell_{t', \tau', \alpha'}} dt' d\alpha'} ; \sup_{t<m, \alpha \in \Theta} \ell_{t, \tau, \alpha} > x \right]$$

$$= \int_t \int_{\alpha \in \Theta} \mathbb{E} \left[ \frac{e^{\ell_{t, \tau, \alpha}}}{\int_{t'} \int_{\alpha' \in \Theta} e^{\ell_{t', \tau', \alpha'}} dt' d\alpha'} ; \sup_{t<m, \alpha \in \Theta} \ell_{t, \tau, \alpha} > x \right] dt d\alpha$$

$$= e^{-x} \int_t \int_{\alpha \in \Theta} \mathbb{E}_{t, \tau, \alpha} \left[ \frac{\mathcal{M}_{t, \tau, \alpha}}{S_{t, \tau, \alpha}} e^{-[\tilde{\ell}_{t, \tau, \alpha} + \log \mathcal{M}_{t, \tau, \alpha}] ; \tilde{\ell}_{t, \tau, \alpha} + \log \mathcal{M}_{t, \tau, \alpha} > 0} \right] dt d\alpha$$
Asymptotic for several important quantities

\[ \mathcal{M}_{t,\tau,\alpha} = \sup_{t'} e^{\ell_{t',\tau',\alpha} - \ell_{t,\tau,\alpha}}, \]

\[ S_{t,\tau,\alpha} = \int_{t'} e^{\ell_{t',\tau',\alpha} - \ell_{t,\tau,\alpha}} dt', \]

\[ \tilde{l}_{t,\tau,\alpha} = \ell_{t,\tau,\alpha} - x, \]

When \( x \to \infty \)

\[ \mathbb{E}_{t,\tau,\alpha} \left[ \frac{\mathcal{M}_{t,\tau,\alpha} e^{-[\tilde{l}_{t,\tau,\alpha} + m_{t,\tau,\alpha}]; \tilde{l}_{t,\tau,\alpha} + \mathcal{M}_{t,\tau,\alpha} > 0}}{S_{t,\tau,\alpha}} \right] \approx \mathbb{E}_{t,\tau,\alpha} \left[ \frac{\hat{\mathcal{M}}_{t,\tau,\alpha}}{\hat{S}_{t,\tau,\alpha}} \right] \frac{1}{\sqrt{(t - \tau)\sigma^2}} \phi \left( \frac{(t - \tau)I - x}{\sqrt{(t - \tau)\sigma^2}} \right). \]

Mill’s ratio

\[ \mathbb{E}_{t,\tau,\alpha} \left[ \frac{\hat{\mathcal{M}}_{t,\tau,\alpha}}{\hat{S}_{t,\tau,\alpha}} \right] \approx \nu \left( \frac{2\xi}{\eta^2} \right), \]
Expected detection delay

When \( b \to \infty \), expected detection delay when change happens at time 0

\[
\mathbb{E}_1[T] = \frac{b}{I} (1 + o(1)).
\]

\( I \): Kulback-Leibler divergence between \( f_0 \) and \( f_1 \)

One-dimensional

\[
I = \frac{\mu}{1 - \alpha} \log \left( \frac{1}{1 - \alpha} \right) - \frac{\alpha}{1 - \alpha} \mu
\]

Multi-dimensional

\[
I = \bar{\lambda}^* \left( \log(\bar{\lambda}^*) - \log(\mu) \right) - \mu^T (\bar{\lambda}^* - \mu)
\]

\[
\bar{\lambda}^* = (I - A^*)^{-1} \mu
\]
Outline

- Background
- Sequential change-point detection
- Control false alarm
- Online computation
Online parameter estimation

- When forming the detection statistic, have to estimate parameters \( \{a_{ij}\} \)
- Estimate the influence parameters online
  - E-M type algorithm (tuning parameter free)
  - Online convex optimization

\[
T = \inf \{t : \max_{\tau < t} \max_A \ell_{t,\tau,A} > b\},
\]

Sliding window
EDD Bound

- Performance measure for online estimator:

\[ \mathcal{R}^a_t = \sum_{i=1}^{t} \{- \log f_{\hat{\theta}_{i-1}}(x_i)\} - \inf_{\theta \in \Gamma} \sum_{i=1}^{t} \{- \log f_{\theta}(x_i)\} \]

best estimator in hindsight

Upper bound for EDD (Cao, Li, X., Xu, 2018)

\[ \mathbb{E}_{\theta,0}[T] \leq \frac{b}{I(\theta, \theta_0)} + \frac{\mathbb{E}_{\theta,0}[\mathcal{R}^a_T]}{I(\theta, \theta_0)} + O(1) \]

EDD \leq \text{optimal EDD} + \text{online estimator regret bound}
Numerical Examples
Twitter data

Figure 6: Exploratory results on Twitter for the detected change points.

a) Mr. Robot wins the Golden Globe
b) First Lady’s dress getting attention
c) Suresh Raina makes his team won

d) Court hearing on Martin Shkreli
e) Rihanna listens to ANTI
f) Suresh Raina makes his team won

Figure 7: Exploratory results on Memetracker for the detected change points.

Figure 8: Illustration of network topology we detect.

Account associated to a TV series named Mr. Robot. We identified that the statistic increases around late January 10th and early 11th. This, surprisingly corresponds to the winning of the 2016 Golden Globe Award.

Figure 6-b shows the statistic computed based on the events of the First lady of the USA and 30 of her randomly selected followers. This statistic revealed a sudden increase in 13th of January. We found a related event - Michelle Obama stole the show during the president’s final State of the Union address by wearing a marigold dress which sold out even before the president finished the speech.

Figure 6-c is related to Suresh Raina, an Indian professional cricketer. We selected a small social circle around him as the center of a star-shaped influence network. We noticed that he led his team to win an important game on Jan. 20, which corresponds to a sharp increase of the computed statistic.

The scenario for Figure 6-d is also interesting as it reflects the activity on the network surrounding Mr. Shkreli, the former chief executive of Turing Pharmaceuticals, who is facing federal securities fraud charges. At Feb. 4th he was invited to congress for a hearing to be questioned about drug price hikes.

The last example, in Figure 6-f, demonstrates an increase in the statistic related to the network of Daughter around 25th of January who is attributed to releasing his new album at Jan. 25th.

We also analyzed the well-known dataset Memetracker for a specific question. First we selected the news websites which mentioned the word “Obama” in their memes during 2nd to 5th of November 2008. We extracted top 70 web domains based on the frequency and construct an underlying influence network based on the reported linking information available within the dataset. This subset always includes the credible news agencies such as BBC, CNN, Huffington, Guardian, etc.

By tracking the event data we successfully pinpoint a change right at the time that Barak Obama was elected as the 44th President of the United States.

First lady and her 30 randomly selected followers

First Lady’s dress got attention

Star topology with randomly select 30 followers
Mr. Robot wins Golden Globe

Suresh Raina makes his team won

Court hearing on Martin Shkreli

Rihanna releases ANTI
Memetracker data

- Network based on reported linking information
- 70 web domains
- Node examples: BBC, CNN, Guardian...
- Tracking “Obama” as meme of the website

Obama wins in 2008
Michelle Obama stole the show during the president's final State of the Union address on Jan. 13. We found a related event – Michelle Obama's dress getting attention – which corresponded to a sharp increase in network statistic involving her followers.

The scenario for Fig. 6-d is interesting as it reflects the activity on the network surrounding Mr. Shkreli, the former chief executive of Turing Pharmaceuticals, who is facing federal securities fraud charges. At Feb. 4th he was invited to congress for a hearing to be questioned about his involvement in the drug price hikes.

The 5th example is about Rihanna who listened to her ANTI album and announced the release of her new album in a tweet on Jan. 25th. That post was retweeted 170K times and received 280K likes and creates a sudden change in network statistic related to the network of Daughter around that time.

By tracking the event data we successfully pinpoint a change point at Jan. 25th.

The scenario for Fig. 6-b is the statistic computed based on the Memetracker dataset. This subset always includes the credible news websites such as The New York Times, The Washington Post, The Guardian, etc. We also analyzed the well-known Twitter network dataset.

First we selected the news websites which mentioned the word “Obama” in their memes during 2nd to 5th of November and 28th of December 2008. We extracted top 70 web domains based on the reported linking information available within the dataset. This subset always includes the credible news websites such as The New York Times, The Washington Post, The Guardian, etc. We also analyzed the well-known Twitter network dataset.

The scenario for Fig. 6-c is related to Suresh Raina, an Indian professional cricketer. We selected a small social circle around him as the center of a star-shaped influence network. We noticed that he led his team to win an important game on Jan. 20.

The diagram shows the network topology we detected. The 4th example is about the Beijing Olympic opening in 2008, which corresponds to a sharp increase of the connectedness statistic for a friend of a Twitter person.

Israel announces ceasefire in Gaza War in 2009.

More comparisons: Twitter data

- Select 116 important real-life events: release of a movie/album, winning an award, and the Pulse Nightclub shooting.
- Baseline 1: binning data into discrete time intervals
- Baseline 2: ignore network topology, sum all 1-D statistics
Summary

Change-point detection for network Hawkes process

- Change-point detection: Sequential hypothesis test framework
- Generalized likelihood ratio statistic
- Efficient online algorithms
- Theoretical results for controlling false-alarm

References:


2. Sequential change-point detection via online convex optimization. Y. Cao, L. Xie, Y. Xie and H. Xu. Accepted by Entropy, Special Issue on ”Information Theory in Machine Learning and Data Science”. Conference version appeared at AISTATS 2018.


Joint work with Shuang Li, Mehrdad Farajtabar, Apurv Verma, Le Song.
Crime correlation detection system

- Given one incident, identify the most related crime incidences based on **correlations in police report narratives**.
- Display found related cases on the map.

Collaboration with Atlanta Police Department.
Partially funded by Atlanta Police Foundation.
E-M algorithm

Introduce the auxiliary variables $p_{ij}$ for all pairs of events $(i, j)$ within the window such that $t_j < t_i$

Lower bound

$$
\ell_{t, \tau, \alpha} \geq \sum_{t_i \in (\tau, t)} \left( p_{ii} \log(\mu) + \sum_{t_j \in (\tau, t_j)} p_{ij} \log [\alpha \beta e^{-\beta(t_i-t_j)}] \right) - \sum_{t_j \in (\tau, t)} p_{ij} \log p_{ij} - \mu(t - \tau) - \alpha \sum_{t_i \in (\tau, t)} [1 - e^{-\beta(t-t_i)}]
$$

E-M step

$$
p^{(k)}_{ij} = \frac{\alpha^{(k)} \beta e^{-\beta(t_j-t_i)}}{\mu + \alpha^{(k)} \beta \sum_{t_m \in (\tau, t)} e^{-\beta(t_j-t_m)}} \quad \{\text{E-step}\}
$$

$$
\alpha^{(k+1)} = \frac{\sum_{i < j} p^{(k)}_{ij}}{\sum_{t_i \in (\tau, t)} [1 - e^{-\beta(t-t_i)}]} \quad \{\text{M-step}\}
$$
Online convex optimization

\[ H_0 : \quad x_t \sim f_{\theta_0}, \quad t = 1, 2, \ldots \]
\[ H_1 : \quad x_t \sim f_{\theta_0}, \quad t = 1, 2, \ldots, \kappa \]
\[ x_t \sim f_{\theta_1}, \quad t = \kappa + 1, \ldots \]

\( \theta_0 \): known,
\( \theta_1 \): unknown

- GLR: estimate \( \theta_1 \) using all data in \((0, t]\)

\[ \hat{\theta}_1(x_1, \ldots, x_t) \]

have to store all data

- Robust GLR: estimate unknown \( \theta_1 \) via **online learning** recursively

\[ \hat{\theta}_1 \leftarrow \hat{\theta}_1 + \text{update}(x_t) \]
Online learning

- Online learning for parameter estimation:
  stochastic gradient descent

\[
\hat{\theta}_t = \hat{\theta}_{t-1} + \text{step-size} \cdot \nabla_{\theta_1} \log \frac{f_{\theta_1}(x_t)}{f_{\theta_0}(x_t)} \bigg|_{\theta_1 = \hat{\theta}_t}
\]

online mirror descent...

- Commonly used in machine learning (Hazan 2014)
Define $S(\gamma) = \{ T : \mathbb{E}_\infty[T] \geq \gamma \}$.

For $b = \log \gamma$, $T(b)$ belongs to $S(\gamma)$.

**Nearly optimality of Robust GLR**

When online estimator satisfying **logarithm regret**

$$
\sup_{\nu \geq 1} \mathbb{E}_{\theta,\nu}[T(b) - \nu + 1 | T(b) \geq \nu] 
- \inf_{T'(b) \in S(\gamma)} \sup_{\nu \geq 1} \mathbb{E}_{\theta,\nu}[T'(b) - \nu + 1 | T(b) \geq \nu]
= O(\log \log \gamma).
$$

**logarithm regret:** $\mathbb{E}_{\theta,0}[\mathcal{R}_t^a] \leq C \log t$