SDEs and SDDEs in population dynamics

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Outline

1. Exponential model

$$\frac{dN}{dt} = rN.$$

Let $N(0) = N_0$, we have

$$N(t) = N_0 e^{rt}.$$

We see that the population remains constant when $r = 0$; the population grows exponentially when $r > 0$ and the population decrease exponentially when $r < 0$.

2. Logistic model

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N.$$

a) The stationary values: $dN/dt = 0$, that is: $N = 0$ or $N = K$.
b) Solution:

$$N(t) = \frac{KN_0e^{rt}}{K - N_0 + N_0e^{rt}}$$

3. Lotka-Volterra (predator-prey) model

$$\begin{cases} \frac{dN_1}{dt} = a_1 N_1 - b_1 N_1 N_2 \\ \frac{dN_2}{dt} = -a_2 N_2 + b_2 N_1 N_2 \end{cases}$$

The equilibrium point $dN_1/dt = 0$ and $dN_2/dt = 0$, that is $N_1 = a_2/b_2$ and $N_2 = a_1/b_1$. Letting $N = (N_1, N_2)^T$, and rewriting the equation above we have

$$dN = \mathrm{diag}(N_1, N_2)(a + bN)$$

where $a = (a_1, -a_2)^T$ and

$$b = \begin{pmatrix} 0 & -b_1 \\ b_2 & 0 \end{pmatrix}.$$

Consider the one-dimensional logistic equation

$$\dot{x}(t) = x(t)[b + ax(t)]$$

on $t \geq 0$ with $x(0) = x_0$.
(i) $a < 0$ and $b > 0$, global solution

$$x(t) = \frac{b}{-a + e^{-bt}(b + ax_0)/x_0}$$

(ii) $a > 0$ and $b > 0$, local solution

$$x(t) = \frac{b}{-a + e^{-bt}(b + ax_0)/x_0} \quad 0 \leq t < T,$$

where

$$T = -\frac{1}{b} \log\left(\frac{ax_0}{b + ax_0}\right).$$

Now consider a general Lotka-Volterra model for a system with n interacting components,

$$\dot{x}_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right), \ 1 \le i \le n.$$

This equation can be rewritten in the matrix form

$$\dot{x}(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[b + Ax(t)], \ t \ge 0,$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T, b = (b_1, \ldots, b_n)^T$ and $A = (a_{ij})_{n\times n}$

Stochastic population dynamics, X. Mao, et al 02 and X. Mao eta al 03.

Stochastically perturbing each parameter

$$a_{ij} \to a_{ij} + \sigma_{ij}\dot{w}(t)$$

results in a new stochastic new form

$$\dot{x}(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t))dt + \sigma x(t)dw(t)], \; t \geq 0.$$

Theorem

Let $\sigma_{ii} > 0, \sigma_{ij} \geq 0$, then for any $b \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, x_0 \in \mathbb{R}^n_+$, there is a unique solution $x(t)$. Moreover $X(t) \in \mathbb{R}^n_+$ with probability one.

$\mathbb{R}_+ := \{x \in \mathbb{R}^n, x_i > 0$ for all $1 \leq i \leq n\}$.

Another is to consider the stochastic perturbation of growth

$$b \to b + \sigma \dot{w}$$

results in a new stochastic new form

$$\dot{x}(t) = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t))dt + \sigma dw(t)],\ t \geq 0.$$

details see Hu and Wang, Jiang and Shi, Liu and Wang, Zhu and Yin.
For example: Jiang and Shi (05) investigated the equation

$$dN(t) = N(t)[(a(t) - b(t)N(t))dt + \alpha(t)dB(t)]$$

and showed there is a unique explicit solution when $a(t) > 0, b(t) > 0$.

The delay Lotka- Volterra model for n interacting species is described by the n-dimensional delay differential equation

$$\frac{dx(t)}{dt} = \mathrm{diag}(x_1(t), \ldots, x_n(t))[b + Ax(t) + Bx(t - \tau)] \qquad (3.1)$$

Consider the stochastic perturbation of growth

$$b \to b + \beta \dot{w}$$

results in a new stochastic form

$$\frac{dx(t)}{dt} = \mathrm{diag}(x_1(t), \ldots, x_n(t))[(b + Ax(t) + Bx(t - \tau))dt + \beta dw(t)] \qquad (3.2)$$

Theorem

(Bahar and Mao) Assume that there are positive numbers $c_1, \ldots c_n$ and θ such that

$$\lambda^+_{\max}\left(\frac{1}{2}(\bar{C}A + A^T\bar{C}) + \frac{1}{4\theta}\bar{C}BB^T\bar{C} + \theta I\right) \leq 0$$

where $\bar{C} = \mathrm{diag}(c_1, \ldots, c_n)$ and I is the $n \times n$ identity matrix. Then given initial data $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}^n_+)$, there is a unique solution $x(t)$ and the solution will remain in $\mathbb{R}^n_+$ with probability one.

Bahar and Mao also studied
Boundedness: $\limsup_{t\to\infty} E|x(t)| \leq K$,
Extinction: $\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) < 0$.

Bahar and Mao investigated the case $A = 0$ and the stochastic perturbation of growth

$$b \to b + \sigma x(t)\dot{w}(t)$$

results in a new stochastic form

$$\frac{dx(t)}{dt} = \mathrm{diag}(x_1(t), \dots, x_n(t))[(b + Bx(t-\tau))dt + \sigma x(t)dw(t)] \tag{3.3}$$

Assume that equation (3.1) has an equilibrium state $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ in $\mathbb{R}^n$. That is

$$b + (A + B)\bar{x} = 0.$$

So (3.1) can be written as

$$\frac{dx(t)}{dt} = \mathrm{diag}(x_1(t), \cdots, x_n(t))[A(x(t) - \bar{x}) + B(x(t-\tau) - \bar{x})].$$

If $A \to A + \sigma \dot{w}(t)$

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))\Big([A(x(t) - \bar{x}) + B(x(t-\tau) - \bar{x})]dt + \sigma(x(t) - \bar{x})dw(t)\Big), \tag{3.4}$$

where $\sigma = (\sigma_{ij})_{n\times n}$.

Mao, Yuan & Zou investigated the following topics:

- will remain positive or never become negative,
- will not explode to infinity in a finite time,
- will be persistent (i.e. never become extinct),
- will tend to the equilibrium state $\bar{x}$.

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2. Global Positive Solution

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ and such that

$$\lambda_{\max}\Big(\frac{1}{2}[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma] + \frac{1}{4}\bar{C}BB^T\bar{C} + I\Big) \le 0, \qquad (3.5)$$

where $\bar{C} = \mathrm{diag}(c_1, \cdots, c_n)$, $\bar{X} = \mathrm{diag}(\bar{x}_1, \cdots, \bar{x}_n)$, and I is the $n \times n$ identity matrix. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, there is a unique solution $x(t)$ to equation (3.4) on $t \ge -\tau$ and the solution will remain in R^n_+ with probability 1, namely $x(t) \in R^n_+$ for all $t \ge -\tau$ almost surely.

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ and such that

$$\lambda_{\max}\Big(\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma + {}^{-1}\bar{C} + B^T\bar{C}B\Big) \le 0, \qquad (3.6)$$

where $\bar{C}$ and $\bar{X}$ are the same as defined in Theorem 2.1. Then for any given initial data $\{x(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, there is a unique solution $x(t)$ to equation (3.4) on $t \ge -\tau$ and the solution will remain in R^n_+ with probability 1, namely $x(t) \in R^n_+$ for all $t \ge -\tau$ almost surely.

Stochastic Persistence

Definition

The SDDE (3.4) is said to be persistent with probability 1 if, for every initial data $\xi = \{\xi(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, the solution $x(t; \xi)$ has the property that

$$\liminf_{t\to\infty} x_i(t;\xi) > 0 \quad a.s. \text{ for all } 1 \le i \le n. \tag{3.7}$$

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ *and such that either (4.25) or (3.6) holds. Then equation (3.4) is persistent with probability 1. Moreover, for any initial data* $\xi = \{\xi(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R_+^n)$*, the solution* $x(t;\xi)$ *has the property that*

$$\limsup_{t\to\infty} x_i(t;\xi) < \infty \quad a.s. \text{ for all } 1 \le i \le n. \tag{3.8}$$

Theorem

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$$P\{\alpha_i < x_i(t;\xi) < \beta_i \text{ for all } t \ge -\tau,\ 1 \le i \le n\} \ge 1 - \varepsilon \qquad (3.9)$$

with

$$\alpha_i = \bar{x}_i h_l^{-1}\left[\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right] \quad \text{and} \quad \beta_i = \bar{x}_i h_r^{-1}\left[\frac{\varphi(\xi)}{\varepsilon c_i \bar{x}_i}\right], \qquad (3.10)$$

where we set

$$\varphi(\xi) = \sup_{-\tau \le s \le 0} V(\xi(s)) + \int_{-\tau}^{0} |\xi(s) - \bar{x}|^2 ds.$$

Asymptotic stability

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ and such that either (4.25) or (3.6) holds. Then for any initial data $\xi = \{\xi(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$, the solution of equation (3.4) has the property that

$$\lim_{t\to\infty} d(x(t;\xi), \mathcal{K}) = 0 \quad a.s. \tag{3.11}$$

with

$$\mathcal{K} = \{x \in \bar{R}^n_+ : (x - \bar{x})^T H(x - \bar{x}) = 0\} \tag{3.12}$$

where we set

$$H = \frac{1}{2}[\bar{C}A + A^T\bar{C} + \sigma^T\bar{C}\bar{X}\sigma] + \frac{1}{4}\bar{C}BB^T\bar{C} + I, \tag{3.13}$$

if condition (4.25) holds.

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ *and such that the symmetric matrix H defined by either (3.13) is negative-definite. Then for any initial data* $\xi = \{\xi(t) : -\tau \le t \le 0\} \in C([-\tau, 0]; R^n_+)$*, the solution of equation (3.4) has the property that*

$$\lim_{t \to \infty} x(t; \xi) = \bar{x} \quad a.s. \tag{3.14}$$

Theorem

(Mao, Yuan & Zou 05) Assume that there are positive numbers $c_1, \cdots, c_n$ *and such that either (4.25) or (3.6) holds and, moreover,*

$$\bar{C}\sigma + \sigma^T \bar{C} \tag{3.15}$$

is either positive-definite or negetive-definite. Then the conclusion (3.14) of Theorem 4.2 still holds.

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There are more literature

- Wu, Fuke; Hu, Shigeng, Stochastic Kolmogorov-type population dynamics with variable delay. Stoch. Models 25 (2009), 129-150.
- Wu, Fuke; Hu, Shigeng, Stochastic functional Kolmogorov-type population dynamics. J. Math. Anal. Appl. 347 (2008), 534-549.

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Population dynamics with jumps

The population may suffer sudden environmental shocks, e.g., earthquakes, hurricanes, epidemics, etc. However, stochastic Lotka-Volterra model cannot explain such phenomena. To explain these phenomena, introducing a jump process into underlying population dynamics provides a feasible and more realistic model. Consider a Lotka-Volterra model with jumps

$$
\begin{aligned}
dX(t) = \mathrm{diag}(X_1(t^-), \cdots, X_n(t^-))\Big[(a(t) - B(t)X(t^-))dt \\
+ \sigma(t)dW(t) + \int_{\mathbb{Y}} \gamma(t,u)\tilde{N}(dt,du)\Big].
\end{aligned}
\tag{4.1}
$$

Here $X(t^-)$ is the left limit of $X(t)$,

$$\sigma = (\sigma_1, \cdots, \sigma_n)^T, \gamma = (\gamma_1, \cdots, \gamma_n)^T,$$

W is a real-valued standard Brownian motion.

Let $T > 0$ and $\tilde{N}(dt, du) := N(dt, du) - dt\lambda(du)$ associated with a Poisson random measure $N : \mathcal{B}(\mathbb{R}_+ \times \mathbb{Z}) \times \Omega \to \mathbb{N} \cup \{0\}$ with the characteristic measure λ on the measurable space $(\mathbb{Z}, \mathcal{B}(\mathbb{Z}))$. For each $\mathbb{A} \in \mathcal{B}(\mathbb{Z})$, the Poisson random measure $N((0, t] \times \mathbb{A})$ can be represented by a point process p on $\mathbb{Z}$ with the domain D_p as a countable subset of $\mathbb{R}_+$, the collection of non-negative real numbers. That is, $N(t, \mathbb{A}) = \sum_{s \in D_p, s \leq t} I_{\mathbb{A}}(p(s))$. N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Y}$ of $[0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$, $\tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt$. We assume that W and N are independent.

(**A**) For any $t \geq 0$ and $i, j = 1, 2, \cdots, n$ with $i \neq j$, $a_i(t) > 0, b_{ii}(t) > 0, b_{ij}(t) \geq 0, \sigma_i(t)$ and $\gamma_i(t, u)$ are bounded functions, $\hat{b}_{ii} := \inf_{t \in \mathbb{R}_+} b_{ii}(t) > 0$ and $\gamma_i(t, u) > -1, u \in \mathbb{Y}$.

Theorem

Under assumption (**A**), for any initial condition $X(0) = x_0 \in \mathbb{R}^n_+$, Eq. (4.1) has a unique global solution $X(t) \in \mathbb{R}^n_+$ for any $t \geq 0$ almost surely.

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Sketch of Proof Since the drift coefficient does not fulfill the linear growth condition, the general theorems of existence and uniqueness cannot be implemented to this equation. However, it is locally Lipschitz continuous, therefore for any given initial condition $X(0) \in \mathbb{R}_+^n$ there is a unique local solution $X(t)$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. By Eq. (4.1) the ith component $X_i(t)$ of $X(t)$ admits the form for $i = 1, \cdots, n$

$$dX_i(t) = X_i(t^-)\Big[\Big(a_i(t) - \sum_{j=1}^{n} b_{ij}(t)X_j(t^-)\Big)dt + \sigma_i(t)dW(t) + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big].$$

Noting that for any $t \in [0, \tau_e)$

$$X_i(t) = X_i(0) \exp\Big\{ \int_0^t \Big(a_i(s) - \sum_{j=1}^n b_{ij}(s) X_j(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du) \Big) ds + \int_0^t \sigma_i(s) dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u)) \tilde{N}(ds,du) \Big\},$$

together with $X_i(0) > 0$, we can conclude $X_i(t) \geq 0$ for any $t \in [0, \tau_e)$.

Now consider the following two auxiliary SDEs with jumps

$$\begin{aligned} dY_i(t) &= Y_i(t^-)\Big[\Big(a_i(t) - b_{ii}(t)Y_i(t^-)\Big)dt + \sigma_i(t)dW(t) \\ &\quad + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big], \\ Y_i(0) &= X_i(0), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} dZ_i(t) &= Z_i(t^-)\Big[\Big(a_i(t) - \sum_{i\neq j} b_{ij}(t)Y_j(t) - b_{ii}(t)Z_i(t^-)\Big)dt + \sigma_i(t)dW(t) \\ &\quad + \int_{\mathbb{Y}} \gamma_i(t,u)\tilde{N}(dt,du)\Big], \\ Z_i(0) &= X_i(0). \end{aligned} \tag{4.3}$$

Then by the comparison theorem we can conclude that

$$Z_i(t) \leq X_i(t) \leq Y_i(t), t \in [0, \tau_e). \tag{4.4}$$

If we can show that $Y_i(t)$ will not be exploded in any finite time and

$$\mathbb{P}(Z_i(t) > 0 \text{ on } t \in [0, \tau_e)) = 1.$$

Hence $\tau_e = \infty$ and $X_i(t) > 0$ almost surely for any $t \in [0, \infty)$. The proof is therefore complete.

Theorem

Let assumption (**A**) hold.

(1) For any $p \in [0, 1,]$ there is a constant K such that

$$\sup_{t \in \mathbb{R}_+} \mathbb{E}|X(t)|^p \leq K. \tag{4.5}$$

(2) Assume further that there exists a constant $\bar{K}(p) > 0$ such that for some $p > 1, t \geq 0, i = 1, \cdots, n$

$$\int_{\mathbb{Y}} |\gamma_i(t, u)|^p \lambda(du) \leq \bar{K}(p). \tag{4.6}$$

Then there exists a constant $K(p) > 0$ such that

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Then there exists a constant $K(p) > 0$ such that

$$\sup_{t \in \mathbb{R}_+} \mathbb{E}|X(t)|^p \leq K(p). \tag{4.7}$$

Sketch of the proof: We shall prove (4.7) firstly. Define a Lyapunov function for $p > 1$

$$V(x) := \sum_{i=1}^{n} x_i^p, x \in \mathbb{R}_+^n. \tag{4.8}$$

Applying the Itô formula, we obtain

$$\mathbb{E}(e^t V(X(t))) = V(x_0) + \mathbb{E}\int_0^t e^s[V(X(s)) + \mathcal{L}V(X(s), s)]ds.$$

By assumption (**A**) and (4.6), we can deduce that there exists constant $K > 0$ such that

$$V(x) + \mathcal{L}V(x, t) \leq K.$$

Hence

$$\mathbb{E}(e^t V(X(t))) \leq V(x_0) + \int_0^t Ke^s ds = V(x_0) + K(e^t - 1),$$

which yields the desired assertion (4.7).

Corollary

Under assumption (**A**), there exists an invariant probability measure for the solution $X(t)$ of Eq. (4.1).

Corollary

Under assumption (**A**), the solution $X(t)$ of Eq. (4.1) is stochastically bounded, i.e. for any $\epsilon \in (0, 1)$, there is a constant $H := H(\epsilon)$ such that for any $x_0 \in \mathbb{R}^n_+$

$$\limsup_{t \to \infty} \mathbb{P}\{|X(t)| \leq H\} \geq 1 - \epsilon.$$

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Theorem

Let assumption (**A**) hold. Assume further that for some constant $\delta > -1$ and any $t \geq 0$

$$\gamma_i(t, u) \geq \delta, u \in \mathbb{Y}, i = 1, \cdots, n, \tag{4.9}$$

and there exists constant $K > 0$ such that

$$\int_0^t \int_{\mathbb{Y}} |\gamma(s, u)|^2 \lambda(du) ds \leq Kt. \tag{4.10}$$

Then the solution $X(t), t \geq 0$, of Eq. (4.1) has the property

$$\limsup_{t \to \infty} \frac{1}{t} \left[\ln(|X(t)|) + \frac{\min_{1 \leq i \leq n} \hat{b}_{ii}}{\sqrt{n}} \int_0^t |X(s)| ds \right] \leq \max_{1 \leq i \leq n} \check{a}_i, \quad \text{a.s.} \tag{4.11}$$

some properties of processes $Y_i(t)$ defined by (4.2)

Lemma

Under assumption (**A**), Eq. (4.2) admits a unique positive solution $Y_i(t), t \geq 0$, which admits the explicit formula

$$Y_i(t) = \frac{\Phi_i(t)}{\frac{1}{X_i(0)} + \int_0^t \Phi_i(s) b_{ii}(s) ds}, \tag{4.12}$$

where

$$\Phi_i(t) := \exp\Big(\int_0^t \Big[a_i(s) - \frac{1}{2}\sigma_i^2(s) + \int_{\mathbb{Y}} (\ln(1+\gamma_i(s,u)) - \gamma_i(s,u))\lambda(du) \Big] ds$$
$$+ \int_0^t \sigma_i(s) dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(s,u)) \tilde{N}(ds,du) \Big).$$

In order to get this lemma, actually we have developed the general variation-of-constant formula

Lemma

Let $F, G, f, g : \mathbb{R}_+ \to \mathbb{R}$ and $H, h : \mathbb{R}_+ \times \mathbb{Y} \to \mathbb{R}$ be Borel-measurable and bounded functions with property $H > -1$, and $Y(t)$ satisfy

$$\begin{aligned} dY(t) &= [F(t)Y(t) + f(t)]dt + [G(t)Y(t) + g(t)]dW(t) \\ &\quad + \int_{\mathbb{Y}} [Y(t^-)H(t,u) + h(t,u)]\tilde{N}(dt,du), \\ Y(0) &= Y_0. \end{aligned} \tag{4.13}$$

Then the solution can be explicitly expressed as:

$$\begin{aligned} Y(t) = \Phi(t)\Big(Y_0 + \int_0^t \Phi^{-1}(s)\Big[\Big(f(s) - G(s)g(s) - \int_{\mathbb{Y}} \frac{H(s,u)h(s,u)}{1+H(s,u)}\lambda(du)\Big)ds \\ + g(s)dW(s) + \int_{\mathbb{Y}} \frac{h(s,u)}{1+H(s,u)}\tilde{N}(ds,du)\Big]\Big), \end{aligned}$$

Lemma

where

$$\Phi(t) := \exp\Big[\int_0^t \Big(F(s) - \frac{1}{2}G^2(s) + \int_{\mathbb{Y}} [\ln(1 + H(s,u)) - H(s,u)]\lambda(du)\Big) ds + \int_0^t G(s)dW(s) + \int_0^t \int_{\mathbb{Y}} \ln(1 + H(s,u))\tilde{N}(ds,du)\Big]$$

is the fundamental solution of corresponding homogeneous linear equation

$$dZ(t) = F(t)Z(t)dt + G(t)Z(t)dW(t) + Z(t^-)\int_{\mathbb{Y}} H(t,u)\tilde{N}(dt,du). \tag{4.14}$$

Theorem

Let assumption (**A**) hold. Assume further that there exists constant $c_1 > 0$ such that, for any $t \geq 0$ and $i = 1, \cdots, n$,

$$a_i(t) - \sigma_i^2(t) - \int_{\mathbb{Y}} \frac{\gamma_i^2(t,u)}{1+\gamma_i(t,u)} \lambda(du) \geq c_1, \tag{4.15}$$

then the solution $Y_i(t), t \geq 0$ of Eq. (4.2) is stochastically permanent, i.e. for any $\epsilon \in (0,1)$ there exit positive constants $H_1 := H_1(\epsilon)$ and $H_2 := H_2(\epsilon)$ such that

$$\liminf_{t\to\infty} \mathbb{P}\{Y_i(t) \leq H_1\} \geq 1-\epsilon \text{ and } \liminf_{t\to\infty} \mathbb{P}\{Y_i(t) \geq H_2\} \geq 1-\epsilon.$$

Theorem

Let assumption (**A**) *hold. Assume further that there exists constant* $c_1 > 0$ *such that, for any* $t \geq 0$ *and* $i = 1, \cdots, n$,

$$a_i(t) - \sigma_i^2(t) - \int_{\mathbb{Y}} \frac{\gamma_i^2(t,u)}{1+\gamma_i(t,u)} \lambda(du) \geq c_1, \tag{4.16}$$

Then Eq. (4.2) *has the property*

$$\lim_{t\to\infty} \mathbb{E}|Y_i(t,x) - Y_i(t,y)|^{\frac{1}{2}} = 0 \text{ uniformly in } (x,y) \in \mathbb{K} \times \mathbb{K}, \tag{4.17}$$

where $\mathbb{K}$ *is any compact subset of* $(0,\infty)$.

Theorem

Let the condition of Theorem 20 If $a_i, b_{ii}, \sigma_i, \gamma_i$ are time-independent, Eq. (4.2) reduces to

$$dY_i(t) = Y_i(t^-)\left[(a_i - b_{ii}Y_i(t))dt + \sigma_i dW(t) + \int_{\mathbb{Y}} \gamma_i(u)\tilde{N}(dt, du)\right], \tag{4.18}$$

the solution $Y_i(t, x)$ of Eq. (4.18) has a unique invariant measure.

If a stochastic population dynamic system is perturbed by Lévy noise as the following:

$$dX(t) = \mathrm{diag}(X_1(t), \cdots, X_n(t))\Big[(b + AX(t)dt + \sigma X(t)dW(t) + \int_{\mathbb{Y}} H(X(t^-), u)\tilde{N}(dt, du)\Big]. \tag{4.19}$$

Here $W(t)$ is a scalar Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, $X(t^-) := \lim_{s\uparrow t} X(s)$, $N(dt, du)$ is a real-valued Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Y}$ of $[0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$, $\tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt$, and $H : \mathbb{R}^n \times \mathbb{Y} \to \mathbb{R}^n$. We further assume that W is independent of N.

For the jump-diffusion coefficient we assume that
(**H1**) For any $m \geq 1, x \in \mathbb{R}^n, u \in \mathbb{Y}$ and $i = 1, \cdots, n$

$$H_i(x,u) > -1, \qquad H_i(0,u) = 0, \tag{4.20}$$

$$\sup_{0<|x|\leq m} \int_{\mathbb{Y}} (\ln|1 + H_i(x,u)|)^2 \lambda(du) < \infty, \tag{4.21}$$

and for each $k > 0$ there exists $L_k > 0$ such that

$$\int_{\mathbb{Y}} |H(x,u) - H(y,u)|^2 \lambda(du) \leq L_k |x-y|^2 \tag{4.22}$$

whenever $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq k$.

Theorem

Let the assumption (**H1**) *hold. Assume further that for* $p \in (0, 1)$ *there exist constants* $\delta > 0, \alpha > 2$ *such that for* $x \in \mathbb{R}^n, i = 1, \dots n,$

$$J_i(x, p) := \int_{\mathbb{Y}} [(1 + H_i(x, u))^p - 1 - pH_i(x, u)] \, \lambda(du) \leq -\delta |x|^\alpha + o(|x|^\alpha) \tag{4.23}$$

where $o(|x|^\alpha)/|x|^\alpha \to 0$ *as* $|x| \to \infty$*. Then, for any initial condition* $\bar{x} \in \mathbb{R}^n_+$*, Eq.* (4.19) *has a unique global solution* $X(t) \in \mathbb{R}^n_+$ *for any* $t \geq 0$ *almost surely.*

Under the condition
$\sigma_{ii} > 0$ if $1 \leq i \leq n$ while $\sigma_{ij} \geq 0$ if $i \neq j$.
Mao, Marion and Renshaw reveal the important fact that Brownian motion noise can suppress a potential population explosion. The Theorem above shows that Lévy noise can also play the same role, without any conditions imposed on the diffusion coefficient σ.
We also can have the moment properties and pathwise estimation.

Population models under regime-switching

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j \end{cases} \qquad (4.24)$$

for $\Delta > 0$. As usual $\gamma_{ij} \geq 0$ is transition rate from i to j if $i \neq j$ and

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$.

Zhu & Yin, On competitive Lotka-Volterra model in random environments

Authors used regime-switching diffusions to model the dynamics of the population sizes of n different species in an ecosystem subject to the random changes of the external environment, i.e.

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))\Big([b(r(t)) + A(r(t))x(t)]dt \\ + \sigma(r(t))dW(t)\Big).$$

Many properties have been studied.

Yuan, Mao & Lygeros, Stochastic hybrid delay population dynamics: well-posed models and extinction

$$dx(t) = \mathrm{diag}(x_1(t), \cdots, x_n(t))\Big(\big[(b(r(t)) + A(r(t))x(t)$$
$$+ B(r(t))x(t-\tau)\big] dt + \sigma(r(t))dW(t) \Big).$$

Theorem

Assume that there exist $nN+1$ positive numbers $c_1(i), \dots, c_n(i)$ for $i \in S$ and θ such that

$$\max\left(\frac{1}{2}[C(i)A(i) + A^T(i)C(i)] + \frac{1}{4\theta}C(i)B(i)B^T(i)C(i) + \theta I\right) \le 0, \tag{4.25}$$

where $C(i) = \mathrm{diag}(c_1(i), \dots, c_n(i))$ and I is the $n \times n$ identity matrix. Then for any given initial data $i_0 \in S$ and $\{\xi(t) : -\tau \le t \le 0\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n_+)$, there is a unique solution $(r(t), x(t))$ to equations on $t \ge -\tau$ such that $x(t) \in \mathbb{R}^n_+$ for all $t \ge -\tau$ almost surely.

Theorem

Assume that there exist $nN+1$ *positive numbers* $c_1(i), \cdots, c_n(i)$ *for* $i \in S$ *and such that*

$$\sup_{i\in S} \max \Big(\frac{1}{2}[C(i)A(i) + A^T(i)C(i)] + \frac{1}{4} C(i)B(i)B^T(i)C(i) + I \Big) < 0 \tag{4.26}$$

where again $C(i) = \mathrm{diag}(c_1(i), \ldots, c_n(i))$. *Then the SHDPD is ultimately bounded in mean, i.e. there exists a positive constant* K *such that*

$$\limsup_{t\to\infty} E|x(t)| \le K.$$

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