Population Systems: Stochastic versus Deterministic

Xuerong Mao

Department of Mathematics and Statistics
University of Strathclyde
Glasgow, G1 1XH
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
Outline

1. Stochastic Modelling

2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models

3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace

4. Summary
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on \( x(t) \)
     - Variance independent of \( x(t) \)
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behaviour of an underlying quantity. We here use the term *underlying quantity* to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are

- number of cancer cells,
- number of HIV infected individuals,
- numbers of interacting species,
- share price in a company,
- price of gold, oil or electricity.
One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behaviour of an underlying quantity. We here use the term *underlying quantity* to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are

- number of cancer cells,
- number of HIV infected individuals,
- numbers of interacting species,
- share price in a company,
- price of gold, oil or electricity.
One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behaviour of an underlying quantity. We here use the term *underlying quantity* to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are

- number of cancer cells,
- number of HIV infected individuals,
- numbers of interacting species,
- share price in a company,
- price of gold, oil or electricity.
One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behaviour of an underlying quantity. We here use the term *underlying quantity* to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are

- number of cancer cells,
- number of HIV infected individuals,
- numbers of interacting species,
- share price in a company,
- price of gold, oil or electricity.
One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behaviour of an underlying quantity. We here use the term *underlying quantity* to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are

- number of cancer cells,
- number of HIV infected individuals,
- numbers of interacting species,
- share price in a company,
- price of gold, oil or electricity.
Now suppose that at time $t$ the underlying quantity is $x(t)$. Let us consider a small subsequent time interval $dt$, during which $x(t)$ changes to $x(t) + dx(t)$. (We use the notation $d\cdot$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change.) By definition, the intrinsic growth rate at $t$ is $dx(t)/x(t)$. How might we model this rate?
If, given \( x(t) \) at time \( t \), the rate of change is *deterministic*, say \( R = R(x(t), t) \), then

\[
\frac{dx(t)}{x(t)} = R(x(t), t) \, dt.
\]

This gives the ordinary differential equation (ODE)

\[
\frac{dx(t)}{dt} = R(x(t), t)x(t).
\]
However the rate of change is in general not deterministic as it is often subjective to many factors and uncertainties e.g. system uncertainty, environmental disturbances. To model the uncertainty, we may decompose

\[
\frac{dx(t)}{x(t)} = \text{deterministic change} + \text{random change}.
\]
The deterministic change may be modeled by

\[ \bar{R} \, dt = \bar{R}(x(t), t) \, dt \]

where \( \bar{R} = \bar{r}(x(t), t) \) is the average rate of change given \( x(t) \) at time \( t \). So

\[ \frac{dx(t)}{x(t)} = \bar{R}(x(t), t) \, dt + \text{random change}. \]

How may we model the random change?
In general, the random change is affected by *many factors independently*. By the well-known central limit theorem this change can be represented by a normal distribution with mean zero and variance $V^2 dt$, namely

$$\text{random change} = N(0, V^2 dt) = V N(0, dt),$$

where $V = V(x(t), t)$ is the standard deviation of the rate of change given $x(t)$ at time $t$, and $N(0, dt)$ is a normal distribution with mean zero and variance $dt$. Hence

$$\frac{dx(t)}{x(t)} = \bar{R}(x(t), t) dt + V(x(t), t) N(0, dt).$$
A convenient way to model $N(0, dt)$ as a process is to use the Brownian motion $B(t)$ ($t \geq 0$) which has the following properties:

- $B(0) = 0$,
- $dB(t) = B(t + dt) - B(t)$ is independent of $B(t)$,
- $dB(t)$ follows $N(0, dt)$. 

The stochastic model can therefore be written as

$$\frac{dx(t)}{x(t)} = \tilde{R}(x(t), t)dt + V(x(t), t)dB(t),$$

or

$$dx(t) = \tilde{R}(x(t), t)x(t)dt + V(x(t), t)x(t)dB(t)$$

which is a stochastic differential equation (SDE).
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
If both $\bar{R}$ and $V$ are constants, say $\bar{R}(x(t), t) = \mu$, $V(x(t), t) = \sigma$, then the SDE becomes

$$dx(t) = \mu x(t) dt + \sigma x(t) dB(t).$$

This is a linear SDE. It is known as the geometric Brownian motion in finance and the exponential growth model in the population systems.
Mean-reverting process

If

$$\bar{R}(x(t), t) = \frac{\alpha(\mu - x(t))}{x(t)}, \quad V(x(t), t) = \sigma,$$

then the SDE becomes

$$dx(t) = \alpha(\mu - x(t))dt + \sigma x(t)dB(t).$$

This is known as the mean-reverting process.
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on \( x(t) \)
     - Variance independent of \( x(t) \)
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
If
\[ \bar{R}(x(t), t) = b + ax(t), \quad V(x(t), t) = \sigma x(t), \]
then the SDE becomes
\[ dx(t) = x(t) \left( [b + ax(t)]dt + \sigma x(t)dB(t) \right). \]

This is the well-known Logistic model in population.
If

\[ \bar{R}(x(t), t) = \mu, \quad V(x(t), t) = \frac{\sigma}{\sqrt{x(t)}}, \]

then the SDE becomes the well-known square root process

\[ dx(t) = \mu x(t) dt + \sigma \sqrt{x(t)} dB(t). \]

This is used widely in engineering and finance.
If

\[ \bar{R}(x(t), t) = \frac{\alpha(\mu - x(t))}{x(t)}, \quad V(x(t), t) = \frac{\sigma}{\sqrt{x(t)}}, \]

then the SDE becomes

\[ dx(t) = \alpha(\mu - x(t))dt + \sigma \sqrt{x(t)}dB(t). \]

This is the mean-reverting square root process used widely in finance and population.
If

\[ \tilde{R}(x(t), t) = \mu, \quad V(x(t), t) = \sigma(x(t))^{\theta-1}, \]

then the SDE becomes

\[ dx(t) = \mu x(t) dt + \sigma(x(t))^\theta dB(t), \]

which is known as the theta process.
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
Consider the Lotka-Volterra model for a system with \( n \) interacting species, namely

\[
\dot{x}(t) = \text{diag}(x_1(t), \cdots, x_n(t))[b + Ax(t)],
\]

(3.1)

where

\[
x = (x_1, \cdots, x_n)^T, \quad b = (b_1, \cdots, b_n)^T, \quad A = (a_{ij})_{n \times n}.
\]

We need some condition on \( A \), e.g., \( A + A^T \) is negative-definite, for the system not to explode to infinity at a finite time (namely, no explosion).
We first consider the corresponding SDE model where the variance dependent on the state, say

\[ dx(t) = x(t)\left( [b + Ax(t)]dt + \sigma x(t)dB(t) \right), \quad (3.2) \]

where \( \sigma = (\sigma_{ij})_{n \times n} \) and \( B(t) \) is a scalar Brownian motion.

*How is this SDE different from its corresponding ODE?*
Theorem 1

Assume that

\[ \sigma_{ii} > 0 \text{ for } 1 \leq i \leq n \text{ whilst } \sigma_{ij} \geq 0 \text{ for } i \neq j. \]

Then for any system parameters \( b \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n\times n} \), and any given initial data \( x(0) = x_0 \in \mathbb{R}^n \), there is a unique solution \( x(t) \) to equation (3.2) on \( t \geq 0 \) and the solution will remain in \( \mathbb{R}_+^n \) with probability 1, namely \( x(t) \in \mathbb{R}_+^n \) for all \( t \geq 0 \) almost surely.
**Significant Difference between ODE (3.1) and SDE (3.2)**

- **ODE (3.1):** The solution may explode to infinity at a finite time if, e.g., $A + A^T$ is positive definite.

- **SDE (3.2):** With probability one, the solution will no longer explode to infinity in a finite time, even in the case when $A + A^T$ is positive definite, as long as $\sigma_{ii} > 0$ for $1 \leq i \leq n$ whilst $\sigma_{ij} \geq 0$ for $i \neq j$. Moreover, the stochastic population system is persistent.
Significant Difference between ODE (3.1) and SDE (3.2)

- ODE (3.1): The solution may explode to infinity at a finite time if, e.g., \( A + A^T \) is positive definite.
- SDE (3.2): With probability one, the solution will no longer explode to infinity in a finite time, even in the case when \( A + A^T \) is positive definite, as long as \( \sigma_{ii} > 0 \) for \( 1 \leq i \leq n \) whilst \( \sigma_{ij} \geq 0 \) for \( i \neq j \). Moreover, the stochastic population system is persistent.
Example

\[
\frac{dy(t)}{dt} = y(t)[1 + y(t)], \quad t \geq 0, \quad x(0) = x_0 > 0
\]

has the solution

\[
y(t) = \frac{1}{-1 + e^{-t}(1 + x_0)/x_0} \quad (0 \leq t < T),
\]

which explodes to infinity at the finite time

\[
T = \log \left( \frac{1 + x_0}{x_0} \right).
\]

However, the SDE

\[
dx(t) = x(t)[(1 + x(t))dt + \sigma x(t)dw(t)]
\]

will never explode as long as \( \sigma \neq 0 \).
Lotka-Volterra model

Noise suppresses exponential growth
Noise expresses exponential growth

An example by R. Wallace

Left: $\sigma = 3$; right: $\sigma = 6$
Key Point:

When $a > 0$ and $\varepsilon = 0$ the solution explodes at the finite time $t = T$; whilst conversely, no matter how small $\varepsilon > 0$, the solution will not explode in a finite time. In other words,

noise may suppress explosion.
Consider the SDE model where the noise is independent of the state, namely

\[ dx(t) = x(t) \left( [b + Ax(t)] dt + \sigma dB(t) \right), \]  

(3.3)

where \( \sigma = (\sigma_{ij}) \in \mathbb{R}^{n \times n} \) and \( B(t) \) is an \( n \)-dimensional Brownian motion.

How is this SDE different from others?
For this SDE model, we require, e.g. $A + A^T$ is negative-definite, in order to have no explosion.

If $\lambda^+_{\text{max}}(Q) < 0$, where $Q = (q_{ij})_{n \times n}$ is defined by

$$q_{ij} = b_i + b_j - \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk},$$

then $\lim_{t \to \infty} x(t) = 0$ with probability 1, namely the noise makes the population extinct.

If both $\sigma$ and $-A$ are non-singular M-matrices while $\sum_{k=1}^{n} \sigma_{ik}^2 < 2b_i$ for $1 \leq i \leq n$, the population system is persistent. In this case

$$\lim_{t \to \infty} Ex(t) = (-A)^{-1}(\xi_1, \cdots, \xi_n)^T,$$

where $\xi_i = b_i - 0.5 \sum_{k=1}^{n} \sigma_{ik}^2$. 

Xuerong Mao  
SDE Models
For this SDE model, we require, e.g. $A + A^T$ is negative-definite, in order to have no explosion.

If $\lambda^+_{\text{max}}(Q) < 0$, where $Q = (q_{ij})_{n \times n}$ is defined by

$$q_{ij} = b_i + b_j - \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk},$$

then $\lim_{t \to \infty} x(t) = 0$ with probability 1, namely the noise makes the population extinct.

If both $\sigma$ and $-A$ are non-singular M-matrices while $\sum_{k=1}^{n} \sigma_{ik}^2 < 2b_i$ for $1 \leq i \leq n$, the population system is persistent. In this case

$$\lim_{t \to \infty} Ex(t) = (-A)^{-1}(\xi_1, \cdots, \xi_n)^T,$$

where $\xi_i = b_i - 0.5 \sum_{k=1}^{n} \sigma_{ik}^2$. 
For this SDE model, we require, e.g. $A + A^T$ is negative-definite, in order to have no explosion.

If $\lambda^+(Q) < 0$, where $Q = (q_{ij})_{n \times n}$ is defined by

$$q_{ij} = b_i + b_j - \sum_{k=1}^{n} \sigma_{ik} \sigma_{jk},$$

then $\lim_{t \to \infty} x(t) = 0$ with probability 1, namely the noise makes the population extinct.

If both $\sigma$ and $-A$ are non-singular M-matrices while $\sum_{k=1}^{n} \sigma_{ik}^2 < 2b_i$ for $1 \leq i \leq n$, the population system is persistent. In this case

$$\lim_{t \to \infty} Ex(t) = (-A)^{-1}(\xi_1, \cdots, \xi_n)^T,$$

where $\xi_i = b_i - 0.5 \sum_{k=1}^{n} \sigma_{ik}^2$. 
The SDE

\[ dx(t) = x(t)[(1 - x(t))dt + \sigma x(t)dw(t)] \]

will become extinct if \( \sigma > \sqrt{2} \) but will be persistent if \( \sigma < \sqrt{2} \).
Lotka-Volterra model
Noise suppresses exponential growth
Noise expresses exponential growth
An example by R. Wallace

Left: $\sigma = 2$; right: $\sigma = 0.5$
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary

Xuerong Mao  SDE Models
Consider the linear scalar ordinary differential equation

$$\dot{y}(t) = a + by(t) \quad \text{on } t \geq 0$$

with initial value $y(0) = y_0 > 0$, where $a, b > 0$. This equation has its explicit solution

$$y(t) = \left(y_0 + \frac{a}{b}\right)e^{bt} - \frac{a}{b}.$$ 

Hence

$$\lim_{t \to \infty} \frac{1}{t} \log(y(t)) = b,$$

that is, the solution tends to infinity exponentially.
On the other hand, consider its corresponding SDE

$$dx(t) = [a + bx(t)]dt + \sigma x(t)dB(t) \quad \text{on } t \geq 0,$$  \hspace{1cm} (3.5)

where $\sigma > 0$ and $B(t)$ is a scalar Brownian motion. Given initial value $x(0) = x_0 > 0$, this SDE has its explicit solution

$$x(t) = \exp \left[ (b - \frac{1}{2} \sigma^2) t + \sigma B(t) \right] \left( x_0 + a \int_0^t \exp \left[ (b - \frac{1}{2} \sigma^2) s + \sigma B(s) \right] ds \right).$$

If $\sigma^2 > 2b$ then the solution of equation (3.5) obeys

$$\limsup_{t \to \infty} \frac{\log(x(t))}{\log t} \leq \frac{\sigma^2}{\sigma^2 - 2b} \quad \text{a.s.}$$  \hspace{1cm} (3.6)
This shows that for any $\varepsilon > 0$, there is a positive random variable $T_\varepsilon$ such that, with probability one,

$$x(t) \leq t^{\varepsilon+\sigma^2/(\sigma^2-2b)} \quad \forall t \geq T_\varepsilon.$$  

In other words, the solution will grow at most polynomially with order $\varepsilon + \sigma^2/(\sigma^2 - 2b)$. In particular, by increasing the noise intensity $\sigma$, we can make the order as close to 1 as we like. Comparing this polynomial growth with the exponential growth of the solution (3.4), we see the important fact that the noise suppresses the exponential growth.
Generally, consider a nonlinear system described by an \( n \)-dimensional ordinary differential equation

\[
\dot{y}(t) = f(y(t), t),
\]

whose coefficient obeys

\[
\langle y, f(y, t) \rangle \leq \alpha + \beta |y|^2, \quad \forall (y, t) \in \mathbb{R}^n \times \mathbb{R}_+.
\]

Clearly, the solution of this equation may grow exponentially.
However, its stochastically linearly perturbed system

\[
dx(t) = f(x(t), t)dt + \sum_{i=1}^{m} A_i x(t) dB_i(t) \tag{3.7}
\]

may grow at most polynomially with probability one, where \( B_i(t) \) are \( m \) independent Brownian motions and \( A_i \in \mathbb{R}^{n \times n} \).
In fact, if there are two positive constants $\gamma$ and $\delta$ such that

$$\sum_{i=1}^{m} |A_i x|^2 \leq \gamma |x|^2, \quad \sum_{i=1}^{m} |x^T A_i x|^2 \geq \delta |x|^4$$

(3.8)

for all $x \in \mathbb{R}^n$, and

$$\delta > \beta + \frac{1}{2} \gamma,$$

(3.9)

then, for any initial value $x(0) = x_0 \in \mathbb{R}^n$, the solution of the stochastically controlled system (3.7) obeys

$$\limsup_{t \to \infty} \frac{\log(|x(t)|)}{\log t} \leq \frac{\delta}{2\delta - 2\beta - \gamma} \quad \text{a.s.}$$

(3.10)
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
Consider the $n$-dimensional linear ordinary differential equation

$$\frac{dy(t)}{dt} = q + Qy(t), \quad t \geq 0$$  \hspace{1cm} (3.11)

with initial value $y(0) = y_0 \in \mathbb{R}^n$, where $q \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$. Assume that

$$\lambda_{\max}(Q + Q^T) < 0.$$  \hspace{1cm} (3.12)

Equation (3.12) can be solved explicitly, which implies easily that

$$\limsup_{t \to \infty} |y(t)| \leq |q|.$$  \hspace{1cm} (3.13)

That is, under condition (3.12), the solution of equation (3.11) is asymptotically bounded and the bound is independent of the initial value.
Generally, consider an $n$-dimensional ordinary differential equation

$$\frac{dy(t)}{dt} = f(y(t), t), \quad t \geq 0. \tag{3.14}$$

Assume that $f$ is sufficiently smooth and, in particular, it obeys

$$-2\langle x, f(x, t) \rangle \leq c_1 + c_2|x|^2, \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R} \tag{3.15}$$

for some non-negative numbers $c_1$ and $c_2$. Our aim here is to show that the solution of its corresponding SDE

$$dx(t) = f(x(t), t)dt + g(x(t))dB(t) \tag{3.16}$$

may grow exponentially with probability one.
First, let the dimension of the state space $n$ be an even number and choose the dimension of the Brownian motion $m$ to be 2. Let $g : \mathbb{R}^n \to \mathbb{R}^{n\times 2}$ by

$$g(x) = (\xi, Ax),$$

where $\xi = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n$ and, of course, $\xi \neq 0$, while

$$A = \begin{pmatrix}
0 & \sigma \\
-\sigma & 0 \\
0 & \sigma \\
-\sigma & 0
\end{pmatrix}$$

with $\sigma > 0$. So the SDE (3.16) becomes
Lotka-Volterra model
Noise suppresses exponential growth
Noise expresses exponential growth
An example by R. Wallace

\[ dx(t) = f(x(t), t)dt + \xi dB_1 + \sigma \begin{pmatrix} x_2(t) \\ -x_1(t) \\ \vdots \\ x_n(t) \\ -x_{n-1}(t) \end{pmatrix} dB_2(t). \] (3.17)
If we have

\[ |\xi|^2 > c_1 \quad \text{and} \quad \sigma^2 = c_1 + c_2 + |\xi|^2 \quad (3.18) \]

then the solution of equation (3.17) obeys

\[ \liminf_{t \to \infty} \frac{1}{t} \log(|x(t)|) \geq \frac{|\xi|^2 - c_1^2}{4|\xi|^2} \quad \text{a.s.} \quad (3.19) \]

Alternatively, if we choose

\[ |\xi|^2 > c_1 \quad \text{and} \quad \sigma^2 = 3|\xi|^2 + c_2 - c_1 \quad (3.20) \]

then

\[ \liminf_{t \to \infty} \frac{1}{t} \log(|x(t)|) \geq \frac{1}{2}(|\xi|^2 - c_1) \quad \text{a.s.} \quad (3.21) \]
We next consider the dimension of the state space $n$ be an odd number but $n \geq 3$. Choose the dimension of the Brownian motion $m$ to be 3 and design $g : \mathbb{R}^n \to \mathbb{R}^{n \times 3}$ by

$$g(x) = (\xi, A_2 x, A_3 x),$$

where $\xi = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n$ and, of course, $\xi \neq 0$, while

$$A_2 = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \\ \vdots & \ddots & \ddots & \ddots \\ -\sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 \\ -\sigma & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & -\sigma \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & \cdots & \ddots \\ -\sigma & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

with $\sigma > 0$. 
So the stochastically perturbed system (3.16) becomes

\[ dx(t) = f(x(t), t)dt + \xi dB_1 + \sigma \begin{pmatrix} x_2(t) \\
-x_1(t) \\
\vdots \\
x_{n-1}(t) \\
-x_{n-2}(t) \\
0 \end{pmatrix} dB_2 + \sigma \begin{pmatrix} 0 \\
x_2(t) \\
-x_3(t) \\
\vdots \\
x_n(t) \\
-x_{n-1}(t) \end{pmatrix} dB_3. \] (3.22)
If we choose $\xi$ and $\sigma$ as (3.18) then the solution of equation (3.22) obeys (3.19), while if we choose $\xi$ and $\sigma$ as (3.20) then the solution of equation (3.22) obeys (3.21).
Outline

1. Stochastic Modelling
2. Well-known Models
   - Linear SDE models
   - Non-linear SDE models
3. Stochastic vs Deterministic
   - Lotka-Volterra model
     - Variance dependent on $x(t)$
     - Variance independent of $x(t)$
   - Noise suppresses exponential growth
   - Noise expresses exponential growth
   - An example by R. Wallace
4. Summary
Following Bailey (The Mathematical Theory of Infectious Diseases, Second Edition, Griffin, London, 1975), let $X$ be the proportion of the human population that is infective and $Y$ be the proportion of infective (female) mosquitoes. The deterministic epidemic equations are then

\[
\begin{align*}
dX/dt &= abmY(t)[1 - X(t)] - rX(t), \\
dY/dt &= aX(t)[1 - Y(t)] - \mu Y(t),
\end{align*}
\]

where $m$ is the number of female mosquitoes per human host, $a$ the number of bites per unit time on man by a single mosquito, $b$ the proportion of infected bites on man that produce an infection, $r$ the per capita rate of recovery in humans and $\mu$ the per capita mortality rate for mosquitoes.
The deterministic steady state endemic levels, when $dX/dt = dY/dt = 0$, are

$$X_\infty = \frac{a^2 bm - \mu r}{a(abm + r)}$$

and

$$Y_\infty = \frac{a^2 bm - \mu r}{abm(a + \mu)}$$

If

$$a^2 bm \leq \mu r$$

the endemic level is zero, and initial outbreaks will collapse.
If \( a = b = m = r = 1 \) and \( \mu = 0.9 \), then the equations are

\[
\begin{align*}
\frac{dX}{dt} &= Y(t)(1 - X(t)) - X(t), \\
\frac{dY}{dt} &= X(t)(1 - Y(t)) - 0.9Y(t).
\end{align*}
\]

(3.24)

The endemic state \((X_\infty = 0.05, Y_\infty = 0.0526)\) is a stable state.
Consider the corresponding SDE model

\[
\begin{align*}
\text{d}X(t) &= (Y(t)[1 - X(t)] - X(t))\,dt + 0.4\,Y(t)\,dB_1(t), \\
\text{d}Y(t) &= (X(t)[1 - Y(t)] - 0.9\,Y(t))\,dt + 0.4\,X(t)\,dB_2(t).
\end{align*}
\]

The following simulation shows that the system diverges markedly from the low endemic level (initially (0.05,0.05)), generating repeated and increasing peaks of infection.
Stochastic Modelling
Well-known Models
Stochastic vs Deterministic
Summary

Lotka-Volterra model
Noise suppresses exponential growth
Noise expresses exponential growth
An example by R. Wallace

Xuerong Mao
SDE Models
On the other hand, consider another SDE model

\[
\begin{align*}
  dX(t) &= \left( Y(t)[1 - X(t)] - X(t) \right) dt \\
        &\quad + 0.4X(t)dB_1(t) + 0.4Y(t)dB_2(t), \\
  dY(t) &= \left( X(t)[1 - Y(t)] - 0.9Y(t) \right) dt \\
        &\quad + 0.4X(t)dB_1(t) + 0.4Y(t)dB_2(t).
\end{align*}
\]

The following simulation shows that Individual outbreaks do not converge on the endemic level, but fall to zero, in the presence of sufficient *symmetric* stochasticity.
Stochastic Modelling
Well-known Models
Stochastic vs Deterministic
Summary

Lotka-Volterra model
Noise suppresses exponential growth
Noise expresses exponential growth
An example by R. Wallace

Xuerong Mao
SDE Models
Noise changes the behaviour of population systems significantly.

- Noise may suppress the potential population explosion.
- Noise may make the population extinct.
- Noise may make the population persistent.
- Noise may suppress or express the exponential growth.
Noise changes the behaviour of population systems significantly.

Noise may suppress the potential population explosion.

Noise may make the population extinct.

Noise may make the population persistent.

Noise may suppress or express the exponential growth.
Noise changes the behaviour of population systems significantly.
Noise may suppress the potential population explosion.
Noise may make the population extinct.
Noise may make the population persistent.
Noise may suppress or express the exponential growth.
Noise changes the behaviour of population systems significantly.

- Noise may suppress the potential population explosion.
- Noise may make the population extinct.
- Noise may make the population persistent.
- Noise may suppress or express the exponential growth.
Noise changes the behaviour of population systems significantly.

- Noise may suppress the potential population explosion.
- Noise may make the population extinct.
- Noise may make the population persistent.
- Noise may suppress or express the exponential growth.