Optimal harvesting of populations in random environments

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The Model

- Many species of animals like whales, elephant seals, bison, and rhinoceroses, are at risk of being harvested to extinction.

- Excessive harvesting has already led to both local and global extinctions of species.

- A significant percentage of the endangered birds and mammals of the world are threatened by harvesting, hunting or other types of overexploitation.
The Model

- Harvesting strategies have to be carefully chosen.
- After significant harvests, it takes time for the harvested population to get back to the pre-existing level.
- The harvested population fluctuates randomly in time due to environmental stochasticity.
The Model

We present a stochastic model of population harvesting and find the optimal harvesting strategy that maximizes the asymptotic yield of harvested individuals.

In most stochastic models that exist in the literature the population is either assumed to become extinct in finite time, or it can end up being harvested to extinction.

In our framework, if the population goes extinct under some harvesting strategy, the asymptotic yield is 0 and therefore this strategy cannot be optimal.
The Model

We consider a population whose density $\tilde{X}(t)$ at time $t \geq 0$, in the absence of harvesting, follows the stochastic differential equation (SDE)

$$d\tilde{X}(t) = \tilde{X}(t)\mu(\tilde{X}(t)) \, dt + \sigma \tilde{X}(t) \, dB(t), \quad \tilde{X}(0) = x > 0,$$

(0.1)

where $(B(t))_{t \geq 0}$ is a standard one dimensional Brownian motion. This describes a population $\tilde{X}$ with per-capita growth rate given by $\mu(x) > 0$ when the density is $\tilde{X} = x$. The infinitesimal variance of fluctuations in the per-capita growth rate is given by $\sigma^2$. 
Assumptions

The function $\mu : [0, \infty) \to \mathbb{R}$ satisfies:

- $\mu$ is locally Lipschitz.
- $\mu$ is decreasing.
- As $x \to \infty$ we have $\mu(x) \to -\infty$.
- The function $p(x) := x\mu(x)$ has a unique maximum.
The Model

The standard example we are interested in is the stochastic Verhulst-Pearl diffusion

\[ d\tilde{X}(t) = \tilde{X}(t)(\bar{\mu} - \kappa \tilde{X}(t)) \, dt + \sigma \tilde{X}(t) \, dB(t), \quad \tilde{X}(0) = x > 0, \]

where \( \bar{\mu} > 0 \) is the per-capita growth rate and \( \kappa > 0 \) is the intracompetition strength.
The Model

The process $\tilde{X}$ does not reach 0 or $\infty$ in finite time and the stochastic growth rate $\mu(0) - \frac{\sigma^2}{2}$ determines the long-term behavior in the following way:

- (Persistence) If $\mu(0) - \frac{\sigma^2}{2} > 0$ and $\tilde{X}(0) = x > 0$, then $(\tilde{X}(t))_{t \geq 0}$ converges weakly to its unique invariant probability measure $\nu$ on $(0, \infty)$.

- (Extinction) If $\mu(0) - \frac{\sigma^2}{2} < 0$ and $\tilde{X}(0) = x > 0$, then $\lim_{t \to \infty} \tilde{X}(t) = 0$ almost surely.
The Model

Assume that the population is harvested at time $t \geq 0$ at the stochastic rate $h(t) \in U := [0, M]$ for some fixed $M > 0$. Adding the harvesting to the dynamics yields the SDE

$$dX(t) = X(t)(\mu(X(t)) - h(t)) \, dt + \sigma X(t) \, dB(t), \quad X(0) = x > 0.$$ 

A stochastic process $(h(t))_{t \geq 0}$ taking values in $U$ is said to be an admissible strategy if $(h(t))_{t \geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the Brownian motion $(B(t))_{t \geq 0}$. 
The Model

One can show that the optimal harvesting strategy is a stationary Markov strategy. These are the admissible strategies of the form

$$h(t) = v(X(t))$$

where $v : \mathbb{R}^{++}_+ \mapsto U$ is a measurable function. Using a stationary Markov strategy $v(\cdot)$, the dynamics becomes

$$dX(t) = X(t)(\mu(X(t)) - v(X(t))) \, dt + \sigma X(t) \, dB(t), \quad X(0) = x > 0.$$
The Model

We will call $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a yield function if the following properties hold:

- $\Phi$ is continuous.
- $\Phi(0) = 0$.
- $\Phi$ has subpolynomial growth that is, there is $n \in \mathbb{N}$ such that $\frac{\Phi(x)}{x^n} \to 0$ for $x \to \infty$. 
The Model

Our aim is to find the optimal strategy $h(t)$ that almost surely maximizes the asymptotic yield

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \Phi(X(t)h(t)) \, dt.$$
The Model

In other words we want to find $v$ such that, for any initial population size $X(0) = x > 0$, we have with probability 1 that

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \Phi\left(X(t)v(X(t))\right) dt = \sup_{h} \liminf_{T \to \infty} \frac{1}{T} \int_0^T \Phi\left(X(t)h(t)\right) dt =: \rho^*$$

where the supremum is taken over all admissible strategies $h$. 
Optimal harvesting of populations in random environments

Results

Assume that $\Phi(x) = x, x \in [0, \infty)$ and that $\mu(0) - \frac{\sigma^2}{2} > 0$. The optimal control $v$ has the bang-bang form

$$v(x) = \begin{cases} 
0 & \text{if } 0 < x \leq x^* \\
M & \text{if } x > x^*
\end{cases}$$

for a unique $x^* \in (0, \infty)$. The optimal asymptotic yield satisfies

$$\rho^* \leq \sup_{x \in \mathbb{R}_+} x \mu(x).$$
Idea of Proof

The general theory of ergodic control shows that the HJB equation

$$\rho = \begin{cases} x\mu(x)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} & \text{if } f_x > 1 \\ x(\mu(x) - M)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} + Mx & \text{if } f_x \leq 1. \end{cases}$$

has a smooth enough solution $V^*$ that is increasing i.e. $V_x^* \geq 0$.  

Idea of Proof

One can show that the optimal control is Markov and satisfies

\[ v(x) = \begin{cases} 
0 & \text{if } V_x^* > 1 \\
M & \text{if } V_x^* < 1.
\end{cases} \]

A careful analysis shows that \( V_x \) crosses the line \( y = 1 \) only once and that the crossing happens from above.
Generalizations

We note that many of the existing models that look at the optimal harvesting of a population in a stochastic environment assume that the yield function $\Phi$ is the identity i.e. $\Phi(x) = x, x \geq 0$. This assumption is not always justifiable.
Generalizations

Assume that $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is weakly convex, $\Phi$ grows at most polynomially, $\Phi \in C^1(\mathbb{R}_+)$ and the population survives in the absence of harvesting, that is $\mu(0) - \frac{\sigma^2}{2} > 0$. The optimal control $v$ has the bang-bang form

$$v(x) = \begin{cases} 0 & \text{if } 0 < x \leq x^* \\ M & \text{if } x > x^* \end{cases}$$

for a unique $x^* \in (0, \infty)$. 
Generalizations

One can associate with the unharvested population process $X$ the family of generators $(\mathcal{L}_u)_{u \in [0,M]}$ defined by their action on $C^2$ functions with compact support in $\mathbb{R}_{++}$ as

$$\mathcal{L}_u f(x) := x[\mu(x) - u]f_x + \frac{1}{2}\sigma^2 x^2 f_{xx}.$$ 

Results from ergodic optimal control show that the HJB

$$\max_{u \in [0,M]} [\mathcal{L}_u V(x) + \Phi(xu)] = \rho$$

admits a classical solution $V^* \in C^2(\mathbb{R}_+)$ satisfying $V^*(1) = 0$ and $\rho = \rho^* > 0$.
Continuous optimal harvesting strategies

We have seen that when $\Phi$ is the identity or $\Phi$ is weakly convex, the optimal harvesting strategies are bang-bang. However, if $\Phi$ satisfies

1. $\Phi \in C^2(\mathbb{R}_+)$,

2. $\Phi$ is strictly concave,

then the optimal harvesting strategy is \textit{continuous} and given by

$$v = \begin{cases} 
0 & \text{if } [\Phi']^{-1}(V_x^*(x)) \leq 0, \\
\frac{[\Phi']^{-1}(V_x^*(x))}{x} & \text{if } 0 < [\Phi']^{-1}(V_x^*(x)) < xM, \\
M & \text{if } [\Phi']^{-1}(V_x^*(x)) \geq xM.
\end{cases}$$
Application: Logistic equation

The most interesting example is when the population follows the Verhulst-Pearl model

\[ dX(t) = X(t)(\bar{\mu} - \kappa X(t))\ dt + \sigma X(t)\ dB(t). \]
Application: Logistic equation

If we harvest according to a constant strategy $\ell > 0$ then the SDE becomes

$$dX(t) = X(t)(\bar{\mu} - \kappa X(t) - \ell) \, dt + \sigma X(t) \, dB(t).$$

It is then easy to see that, as long as $\bar{\mu} - \ell - \frac{\sigma^2}{2} > 0$, the asymptotic yield is

$$L(\ell) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \ell X(t) \, dt = \ell \frac{\bar{\mu} - \ell - \frac{\sigma^2}{2}}{\kappa}.$$
**Application: Logistic equation**

We can maximize this yield $L(\ell)$, which is quadratic in $\ell$.

The maximum will be at

$$\ell^* = \frac{1}{2} \left( \frac{\mu}{\mu} - \frac{\sigma^2}{2} \right)$$

and the maximal asymptotic yield (among constant harvesting strategies) is

$$L(\ell^*) = \frac{\left( \mu - \frac{\sigma^2}{2} \right)^2}{4\kappa}.$$
Note that $L(\ell^*)$ is also called *maximum sustainable yield (MSY)* in the literature. Since $x \mu(x) = \overline{\mu}x - \kappa x^2$ we note that

$$\sup_{x \in \mathbb{R}^+} x \mu(x) = \frac{\overline{\mu}^2}{4\kappa}.$$ 

One sees that the optimal asymptotic yield $\rho^*$ satisfies

$$\frac{\left(\mu - \frac{\sigma^2}{2}\right)^2}{4\kappa} \leq \rho^* \leq \frac{\mu^2}{4\kappa}.$$
Optimal boundary

Our main result does not give us information about the optimal boundary \( x^* \), the point at which one starts harvesting. We can show, in the Verhulst-Pearl setting that \( x^* \) is the solution to

\[
H'(\eta) = 0.
\]

for an \( H \) which can be found explicitly in terms of incomplete gamma functions.
Unbounded harvesting

In the first part of the talk the harvesting in time $dt$ looked like $h(t)X(t)\,dt$ for some adapted process $(h(t))_{t\geq 0}$ that was always bounded above by a fixed constant $M > 0$.

However, in certain instances one can assume that there is absolute control over the harvested population. As such, it makes sense to study the following problem.
Unbounded harvesting

We consider a population whose density $X(t)$ at time $t \geq 0$, in the absence of harvesting, follows the stochastic differential equation (SDE)

$$dX(t) = X(t)\mu(X(t))\,dt + \sigma(X(t))\,dB(t).$$

We make some natural assumptions, that imply that the above SDE has solutions, and that the boundary points 0 and $\infty$ are unattainable and non-attracting. These assumptions then make sure that the SDE has unique weak solutions and that the population does not go extinct, nor blow up.
Unbounded harvesting

We denote the density of the scale function of $X$ by

$$S'(x) = \exp \left( - \int_c^x \frac{2\mu(y)y}{\sigma^2(y)} \, dy \right),$$

where $c \in \mathbb{R}_+$ is an arbitrary constant. The density of the speed measure $m$ is, in turn, denoted by

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}.$$
Assumptions

(A1) The function $\mu$ is nonincreasing and fulfills the limiting conditions
\[ \lim_{x \to 0^+} \mu(x) > \eta \quad \text{and} \quad \lim_{x \to \infty} \mu(x) < -\eta \quad \text{for some} \quad \eta > 0. \]

(A2) The function $\mu(x)x$ has a unique maximum point $\hat{x} = \arg\max \{ \mu(x)x \}$
so that $\mu(x)x$ is increasing on $(0, \hat{x})$ and decreasing on $(\hat{x}, \infty)$.

(A3) $\lim_{x \to 0^+} m((x, y)) < \infty$ for $x < y$. 
Assumptions

It is worth pointing out that assumption (A1) guarantees that the per-capita growth rate $\mu$ vanishes at some given point $x_0 = \mu^{-1}(0)$. In typical population models this point coincides with the carrying capacity of the population. We naturally have that $\hat{x} < x_0$. 
Assumptions

It is also worth mentioning that condition (A3) is needed for the existence of a stationary distribution for the process $X$. Under our boundary assumptions, it guarantees that 0 is repelling for $X$ and the condition $\lim_{x \to 0^+} S((x, y)) = +\infty$, for $x < y$, is satisfied.
Model

A stochastic process \((Z_t)_{t \geq 0}\) is said to be an admissible strategy if \((Z_t)_{t \geq 0}\) is non-negative, nondecreasing, right continuous and adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\) generated by the Brownian motion \((B(t))_{t \geq 0}\).
Model

Assume that \((Z_t)_{t \geq 0}\) is admissible and that at time \(t\) we harvest in the infinitesimal period \(dt\) an amount \(dZ_t\). Then our harvested population’s dynamics is given by

\[
dX^Z(t) = X^Z(t)\mu(X^Z(t))\,dt + \sigma(X^Z(t))\,dB(t) - dZ_t, \quad X^Z(0) = x > 0.\]
Model

We consider the following ergodic singular control problem:

$$\sup_{Z} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_{x} \int_{0}^{T} dZ_{s}.$$ 

where we take the supremum over all admissible strategies $Z$. 
The optimal harvesting strategy is

\[ Z_t^{b^*} = \begin{cases} 
(x - b^*)^+ & \text{if } t = 0, \\
L(t, b^*) & \text{if } t > 0 
\end{cases} \]

where \( L(t, b^*) \) is the local time push of the process \( X^Z \) at the boundary \( b^* \).
Results

The optimal harvesting boundary $b^*$ as well as the maximal expected average asymptotic yield $\ell^*$ are the solutions of the optimality conditions

$$\ell^* = \mu(b^*)b^* = \frac{1}{S'(b^*)m((0, b^*))}. $$

Moreover,

$$\sup_{Z \in \Lambda} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \int_0^T dZ_s = \lim_{T \to \infty} \frac{\mathbb{E}_x[Z_{T}^{b^*}]}{T} = \ell^* = \mu(b^*)b^*. $$
Almost sure results

One may wonder whether our findings could be extended further to a setting focusing on the almost sure maximization problem

$$\sup_{Z \in \Lambda} \liminf_{T \to \infty} \frac{1}{T} \int_0^T dZ_t = \sup_{Z \in \Lambda} \liminf_{T \to \infty} \frac{Z_T}{T}.$$
Almost sure results

Under some further conditions, we can show the following

For any admissible strategy $Z \in \Lambda$ and any $X_0^Z = x \in (0, \infty)$

$$
P_x \left\{ \liminf_{T \to \infty} \frac{Z_T}{T} \leq \liminf_{T \to \infty} \frac{Z_T^{b^*}}{T} = \ell^* = \mu(b^*)b^* \right\} = 1.
$$
Almost sure results

Our main result on the sign of the relationship between volatility and the optimal harvesting strategy is summarized in the following.

Increased volatility increases the optimal harvesting threshold $b^*$ and decreases the long run average cumulative yield $\ell^* = \mu(b^*)b^*$. 
Conclusions

We have analysed the optimal harvesting strategies which maximize the asymptotic yield of harvested individuals.

The optimal strategies never drive the harvested species extinct.

We do not need discount factors, which are usually hard to estimate.
Thank you for your attention!