Asymptotic behaviour of a neural field lattice model with a Heaviside operator

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Neural field models are tissue level models that describe the spatio-temporal evolution of coarse grained variables such as synaptic or firing rate activity in populations of neurons.

They are often represented as evolution equations generated as continuum limits of computational models of neural field theory.

A seminal model was proposed by S. Amari:

\[
\frac{\partial}{\partial t} u(t, x) = -u(t, x) + \int_{\Omega} K(x-y)H(u(t, y) - \theta) \, dy, \quad x \in \Omega.
\]

Here \( \theta > 0 \) is a given threshold and \( H : \mathbb{R} \to \mathbb{R} \) is the Heaviside function defined by

\[
H(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0,
\end{cases} \quad x \in \mathbb{R}.
\]
Continuum neural models may lose their validity in capturing detailed dynamics at discrete sites when the discrete structures become dominant.

Neural field lattice models are also evolution equations, which avoid having to take the continuum limit. They describe the dynamics at each site of the neural field.

Han & Kloeden (2017) introduced and studied the following neural field lattice system based on a generalization of the Amari model

\[
\frac{d}{dt} u_i(t) = f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j} H(u_j(t) - \theta) + g_i(t), \quad i \in \mathbb{Z}^d, \quad \theta > 0.
\]

This lattice system is the topic of this talk.
Model reformulation: infinite dimensional differential inclusion

\[
\frac{du_i(t)}{dt} \in f_i(u_i(t)) + \sum_{j \in \mathbb{Z}^d} k_{i,j} \chi(u_j(t) - \theta) + g_i(t) \tag{1}
\]

with the set-valued mapping

\[
\chi(s) = \begin{cases} 
\{0\}, & s < 0, \\
[0,1], & s = 0, \\
\{1\}, & s > 0,
\end{cases} \quad s \in \mathbb{R}.
\]

Assumption A1: \( k_{i,j} \geq 0 \) and \( \sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa \) for all \( i, j \in \mathbb{Z}^d \) for some \( \kappa > 0 \).
**Function space:** we consider a space of bi-infinite real valued sequences with vectorial indices \( i = (i_1, \ldots, i_d) \in \mathbb{Z}^d \) and a positive sequence of weights \( (\rho_i)_{i \in \mathbb{Z}^d} \)

\[
\ell^2_\rho := \left\{ u = (u_i)_{i \in \mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} \rho_i u_i^2 < \infty \right\}
\]

which is a separable Hilbert space with the inner product and norm

\[
\langle u, v \rangle := \sum_{i \in \mathbb{Z}^d} \rho_i u_i v_i, \quad \| u \|_\rho := \sqrt{\sum_{i \in \mathbb{Z}^d} \rho_i u_i^2}
\]

for \( u = (u_i)_{i \in \mathbb{Z}^d}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2_\rho \).
Assumption A2: \( \rho_i > 0 \) for \( i \in \mathbb{Z}^d \) and \( \rho_\Sigma := \sum_{i \in \mathbb{Z}^d} \rho_i < \infty \).

\[ \ell^2 \subset \ell^\infty \subset \ell^2_{\rho}, \] where

\[ \ell^2 = \left\{ \mathbf{u} = (u_i)_{i \in \mathbb{Z}^d} : \sum_{i \in \mathbb{Z}^d} u_i^2 < \infty \right\}, \]

\[ \ell^\infty = \left\{ \mathbf{u} = (u_i)_{i \in \mathbb{Z}^d} : \sup_{i \in \mathbb{Z}^d} |u_i| < \infty \right\} \]

with norms \( \| \mathbf{u} \| := \sqrt{\sum_{i \in \mathbb{Z}^d} u_i^2} \) and \( \| \mathbf{u} \|_\infty := \sup_{i \in \mathbb{Z}^d} |u_i| \).

\[ \ell^2_{\rho} \] allows a wider range of solutions, e.g., with just bounded components or traveling waves, than the unweighted space \( \ell^2 \).
The reaction term

**Assumption F1:** the function \( f_i : \mathbb{R} \to \mathbb{R} \) is continuously differentiable with weighted equi-locally bounded derivatives, i.e., there exists a non-decreasing function \( L(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+) \) such that

\[
\max_{s \in [-r, r]} |f_i'(s)| \leq L(\rho_i r), \quad \forall r \in \mathbb{R}^+, \ i \in \mathbb{Z}^d.
\]

**Assumption F2:** \( f_i(0) = 0 \) for all \( i \in \mathbb{Z}^d \).

**Assumption F3:** there exist constants \( \alpha > 0 \) and \( \beta_i \) with \( \beta := (\beta_i)_{i \in \mathbb{Z}^d} \in \ell^2_\rho \) such that

\[
sf_i(s) \leq -\alpha |s|^2 + \beta_i^2, \quad \forall s \in \mathbb{R}, \ \forall i \in \mathbb{Z}^d.
\]
Assumption (F1) implies that the $f_i$ are locally Lipschitz

$$|f_i(u_i) - f_i(v_i)| \leq L(\rho_i(|u_i| + |v_i|)) \cdot |u_i - v_i|$$

$$\leq L \left( \sqrt{\rho_\Sigma} (\|u\|_\rho + \|v\|_\rho) \right) \cdot |u_i - v_i|.$$ 

for all $u = (u_i)_{i \in \mathbb{Z}^d}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2$. 

Define $F(u) := (f_i(u_i))_{i \in \mathbb{Z}^d}$ for $u = (u_i)_{i \in \mathbb{Z}^d} \in \ell^2$.

**Lemma 1**  Assume that Assumptions (F1)–(F3) hold. Then $F : \ell^2 \rightarrow \ell^2$ is locally Lipschitz and satisfies the dissipativity condition

$$\langle F(u), u \rangle \leq -\alpha \|u\|_\rho^2 + \|\beta\|_\rho^2.$$
The interconnection term

Define the set-valued operator \( \mathcal{K}(u) := (\mathcal{K}_i(u))_{i \in \mathbb{Z}^d} \) for every \( u = (u_i)_{i \in \mathbb{Z}^d} \in \ell_\rho^2 \) by

\[
\mathcal{K}_i(u) = \sum_{j \in \mathbb{Z}^d} k_{i,j} \chi(u_j - \theta) = \sum_{j \in \mathbb{Z}^d} k_{i,j} \begin{cases} 
0, & u_j < \theta, \\
[0,1], & u_j = \theta, \\
1, & u_j > \theta.
\end{cases}
\]

**Lemma 2**  The set-valued operator \( \mathcal{K} \) maps \( \ell_\rho^2 \) to \( \ell_\rho^2 \).

To prove this we show that

\[
\|\mathcal{K}_i(u)\| \leq \sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa, \quad \forall i \in \mathbb{Z}^d
\]

Hence

\[
\|\mathcal{K}(u)\|_\rho^2 \leq \sum_{i \in \mathbb{Z}^d} \rho_i \|\mathcal{K}_i(u)\|^2 \leq \kappa^2 \rho_\Sigma < \infty.
\]
Here for any sets $B \subset \mathbb{R}$ and $U = (U_i)_{i \in \mathbb{Z}^d} \subset \ell^2_\rho$, we define

$$
\| B \| := \sup_{b \in B} |b|, \quad \| U \|_\rho := \left( \sup_{u = (u_i)_{i \in \mathbb{Z}^d} \in \ell^2_\rho} \| u \|_\rho^2 \right)^{1/2}.
$$

The forcing term

Define $G(t) := (g_i(t))_{i \in \mathbb{Z}^d}$ and assume that

**Assumption G1:** $G(\cdot) \in C_b(\mathbb{R}, \ell^2_\rho)$ and $\bar{g}(\cdot) \in L^1_{loc}(\mathbb{R}) \cap L^2(\mathbb{R}, \ell^2_\rho)$, where $\bar{g}(t) := \sup_{i \in \mathbb{Z}^d} |g_i(t)|$. 
Existence of solutions

The lattice differential inclusion (1) can be rewritten as a differential inclusion on $\ell^2_\rho$ as

$$
\dot{u}(t) \in G(u(t), t) := F(u(t)) + K(u(t)) + G(t). \quad (2)
$$

We define solution to the differential inclusion (2) componentwise.

**Definition**  A function $u(t) = (u_i(t))_{i \in \mathbb{Z}^d} : [t_0, t_0 + T) \rightarrow \ell^2_\rho$ is called a solution to the differential inclusion (2) if it is an absolutely continuous function $u(t) : [t_0, t_0 + T) \rightarrow \ell^2_\rho$ such that

$$
\dot{u}_i(t) \in f_i(u_i(t)) + K_i(u(t)) + g_i(t), \quad \forall \, i \in \mathbb{Z}^d, \text{ a.e.}
$$
Theorem Let $T > 0$ and suppose that Assumptions $(A1)$–$(A2)$, $(F1)$–$(F3)$ and $(G1)$ hold.

Then for any $t_0 \in \mathbb{R}$ and initial data $u_o = (u_{o,i})_{i \in \mathbb{Z}^d} \in \ell^2_{\rho}$, the differential inclusion (2) admits a solution $u(t; t_0, u_o)$ with $u(t_0; t_0, u_o) = u_o$ which exists on $[t_0, t_0 + T]$.

Proof There are many existence theorems for abstract differential inclusions on Banach spaces.

They require the upper semi-continuity of the set-valued mapping $\mathcal{K}$ from $\ell^2_{\rho}$ to $\ell^2_{\rho}$.

The weighted norm makes life difficult

Instead we will use finite dimensional approximations and a compactness argument to construct a solution
Approximation of the set-valued operator

Let \( \mathbb{Z}_N^d := \{i = (i_1, \ldots, i_d) \in \mathbb{Z}^d : |i_1|, \ldots, |i_d| \leq N\} \) and define the truncated operator

\[
\mathcal{K}_i^N(u) = \sum_{j \in \mathbb{Z}_N^d} k_{i,j} \chi(u_j - \theta), \quad u = (u_i)_{i \in \mathbb{Z}^d} \in \ell^2_{\rho}.
\]

We need the inequality for nonempty compact subsets of \( \mathbb{R}^d \)

\[
\text{dist}_{\mathbb{R}^d}(A_1 + B_1, A_2 + B_2) \leq \text{dist}_{\mathbb{R}^d}(A_1, A_2) + \text{dist}_{\mathbb{R}^d}(B_1, B_2)
\]

**Lemma 3** For every \( i \in \mathbb{Z}^d \), the set-valued mapping \( u \mapsto \mathcal{K}_i^N(u) \) is upper semi continuous from \( \ell^2_{\rho} \) into the nonempty compact convex subsets of \( \mathbb{R}^1 \), i.e.,

\[
\text{dist}_{\mathbb{R}^1}(\mathcal{K}_i^N(u^m), \mathcal{K}_i^N(\hat{u})) \to 0 \text{ as } u^m \to \hat{u} \text{ in } \ell^2_{\rho}.
\]
Since $\mathbf{u}^m \to \hat{\mathbf{u}}$ in $\ell^2_\rho$, for every $\varepsilon > 0$ there exists $M(\varepsilon)$ such that

$$\|\mathbf{u}^m - \hat{\mathbf{u}}\|_\rho^2 = \sum_{j \in \mathbb{Z}^d} \rho_i |u_j^m - \hat{u}_j|^2 < \varepsilon^2, \quad \forall \ m \geq M(\varepsilon).$$

Considering only $j \in \mathbb{Z}_N^d$ as appearing in $\mathbb{K}_i^N$, then

$$|u_j^m - \hat{u}_j| < \varepsilon / \sqrt{\rho_N}, \quad \forall \ m \geq M_{\varepsilon}, \ j \in \mathbb{Z}_N^d.$$

where $\rho_N := \min_{j \in \mathbb{Z}_N^d} \rho_j$.

The set-valued mapping $s \mapsto \chi(s - \theta)$ is upper semi continuous for $s \in \mathbb{R}$.

Since there is only a finite number of terms in the sum defining $\mathbb{K}_i^N$ it follows by the above inequality that the set-valued mapping $\mathbf{u} \mapsto \mathbb{K}_i^N(\mathbf{u})$ is also upper semi continuous.
Lemma 4  For each $i \in \mathbb{Z}^d$ and every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\text{dist}_{R^1} (\mathcal{K}_i(u), \mathcal{K}^N_i(u)) \leq \varepsilon \quad \text{for all} \quad N \geq N(\varepsilon, i), \quad u \in \ell^2_\rho.$$ 

Write $\mathcal{K}_i(u) := \mathcal{K}^N_i(u) + \mathcal{E}_i^N(u)$, where

$$\mathcal{E}_i^N(u) := \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}^d_N} k_{i,j} \chi(u_j - \theta).$$

Then for each $i \in \mathbb{Z}^d$ and all $u \in \ell^2_\rho$

$$\text{dist}_{R^1} (\mathcal{K}_i(u), \mathcal{K}^N_i(u)) = \text{dist}_{R^1} (\mathcal{K}^N_i(u) + \mathcal{E}_i^N(u), \mathcal{K}^N_i(u) + \{0\}) \leq \text{dist}_{R^1} (\mathcal{E}_i^N(u), \{0\}) \leq \|\mathcal{E}_i^N(u)\|$$

$$\leq \sum_{j \in \mathbb{Z}^d \setminus \mathbb{Z}^d_N} k_{i,j} \leq \varepsilon \quad \forall \quad N \geq N(\varepsilon),$$

because $\|\chi(u_j - \theta)\| \leq 1$ for all $u \in \ell^2_\rho$. 
Finite dimensional lattice inclusion

For each $i \in \mathbb{Z}_N^d$ and $u^N(t) = (u^N_i(t))_{i \in \mathbb{Z}_N^d} \equiv \mathbb{R}^{(2N+1)d}$ define

$$\frac{d}{dt} u^N_i(t) \in G^N_i(u^N(t), t) := f_i(u^N_i(t)) + R^N_i(u^N(t)) + g_i(t)$$

The set-valued mapping $G^N_i(u^N, t)$ is nonempty, compact, convex valued as well as upper semicontinuous in $u^N$ and measurable in $t$.

Moreover, for any $u^N \in \mathbb{R}^{(2N+1)d}$,

$$\|G^N_i(u^N, t)\| \leq |f_i(u^N_i)| + \|R^N_i(u^N)\| + |g_i(t)|$$

$$\leq L(\rho_i |u_i^N|) |u_i^N| + \sum_{j \in \mathbb{Z}_N^d} k_{i,j} + \bar{g}(t)$$

$$\leq L(\rho_\Sigma |u^N|) |u^N| + \kappa + \bar{g}(t),$$

where $|\cdot|$ is the Euclidean norm.
Therefore by standard existence theorems for finite dimensional inclusions, e.g., see Aubin & Cellina, there exists a solution

\[ u^N(t; t_0, u^N_o) = (u^N_i(t; t_0, u^N_o))_{i \in \mathbb{Z}_N^d}. \]

\[ \implies \text{ for each } i \in \mathbb{Z}_N^d \text{ there exists a selection } \sigma^N_i(t) \in \mathcal{K}_i^N(u^N(t; t_0, u^N_o)), \text{ a.e.,} \]

with a.e.

\[
\frac{d}{dt} u^N_i(t; t_0, u^N_o) = f_i(u^N_i(t; t_0, u^N_o)) + \sigma^N_i(t) + g_i(t), \quad i \in \mathbb{Z}_N^d.
\]
Componentwise convergent subsequence

Extend the solution \( u^N(t; t_0, u^N_0) = (u^N_i(t; t_0, u^N_0))_{i \in \mathbb{Z}^d} \) to \( v^N(t) = (v^N_i(t))_{i \in \mathbb{Z}^d} \) in \( \ell^2_{\rho} \) with zero elements and modify \( \sigma^N_i, g_i \) similarly. Then \( v^N(t) \) satisfies the infinite dimensional lattice ODE, a.e.,

\[
\frac{d}{dt} v^N_i(t) = f_i(v^N_i(t)) + \tilde{\sigma}^N_i(t) + g^N_i(t), \quad i \in \mathbb{Z}^d.
\]

Multiplying by \( v^N_i(t) \) gives

\[
\frac{d}{dt} |v^N_i(t)|^2 = 2f_i(v^N_i(t)) v^N_i(t) + 2\tilde{\sigma}^N_i(t) v^N_i(t) + 2g^N_i(t) v^N_i(t) \leq -\alpha |v^N_i(t)|^2 + 2\beta^2_i + \frac{2}{\alpha} (\kappa^2 + \bar{g}^2(t))
\]

as \( \sigma^N_i(t) \in \mathcal{K}^N_i(u^N(t; t_0, u^N_0)) \Rightarrow |\sigma^N_i(t)| \leq \sum_{j \in \mathbb{Z}^d} k_{i,j} \leq \kappa \quad \forall \ i \in \mathbb{Z}^d. \)
for each $i \in \mathbb{Z}^d$, $T \in \mathbb{R}^+$ exists $\mu_i, T$ independent of $N$ such that

$$|v_i^N(t)| \leq \mu_i, T \quad \text{for all} \quad t \in [t_0, t_0 + T], \quad N \in \mathbb{N}.$$ 

Moreover,

$$\left| \frac{d}{dt} v_i^N(t) \right| \leq \|G_i^N(v^N(t), t)\| \leq L(\rho_i |v_i^N(t)|) |v_i^N(t)| + \kappa + \bar{g}(t),$$

for each $i \in \mathbb{Z}^d$, $T \in \mathbb{R}^+$ exists $\tilde{\mu}_i, T$ independent of $N$ such that

$$\left| \frac{d}{dt} v_i^N(t) \right| \leq \tilde{\mu}_i, T \quad \text{for all} \quad t \in [t_0, t_0 + T], \quad N \in \mathbb{N}.$$ 

i.e., $\{v_i^N(\cdot)\}_{N \in \mathbb{N}}$ is uniformly bounded and equi-Lipschitz continuous on $[t_0, t_0 + T]$ for each $i \in \mathbb{Z}^d$. 
By the Ascoli-Arzelà Theorem for each \( i \in \mathbb{Z}^d \), there is a \( v_i^*(\cdot) \in C([t_0, t_0 + T], \mathbb{R}_+) \) and a convergent subsequence \( \{v_i^{Nm}(\cdot)\}_{m \in \mathbb{N}} \) and such that

\[
v_i^{Nm}(\cdot) \rightarrow v_i^*(\cdot) \quad \text{in} \quad C([t_0, t_0 + T], \mathbb{R}_+).
\]

Moreover, for the subsequence of derivatives

\[
\frac{d}{dt} v_i^{Nm}(\cdot) \rightarrow \frac{d}{dt} v_i^*(\cdot) \quad \text{in} \quad L^1([t_0, t_0 + T], \mathbb{R}).
\]

The limit function \( v_i^*(\cdot) \) shares the equi-Lipschitz continuity of the subsequence \( \{v_i^{Nm}(\cdot)\}_{m \in \mathbb{N}} \) and hence is absolutely continuous on \([t_0, t_0 + T]\).

The argument can be strengthened to obtain a common diagonal subsequence that converges for all \( i \in \mathbb{Z}^d \).
Convergent subsequence in $\ell^2_\rho$

Recall that we extended $u^N(\cdot; t_0, u^N_0)$ to $v^N(\cdot) = (v^N_i(\cdot))_{i \in \mathbb{Z}^d}$ in $\ell^2_\rho$. By the dissipativity condition we obtain

$$\|v^N(t)\|_\rho \leq \nu_T \quad \forall t \in [t_0, t_0 + T], \quad N \in \mathbb{N}.$$ 

and with more work

$$\left\| \frac{d}{dt} v^N(t) \right\|_\rho^2 \leq \tilde{\nu}_T, \quad t \in [t_0, t_0 + T], \quad N \in \mathbb{N}.$$ 

By the Ascoli-Arzelà Theorem in $C([t_0, t_0 + T], \ell^2_\rho)$ there is a $\hat{v}(\cdot) \in C([t_0, t_0 + T], \ell^2_\rho)$ and convergent subsequence $\{v^{Nm}(\cdot)\}_{m \in \mathbb{N}}$ such that

$$\sup_{t \in [t_0, t_0 + T]} \|v^{Nm}(t) - \hat{v}(t)\|_\rho \to 0 \quad \text{as} \quad m \to \infty.$$
Equivalence of limit points

It can be assumed WLOG that the two convergent subsequences above for the componentwise limit are the same.

Since $\mathbf{v}^{Nm}(t) \to \hat{\mathbf{v}}(t)$ in $\ell^2_{\rho}$, $\forall \varepsilon > 0 \exists M(\varepsilon)$ such that

$$
\|\mathbf{v}^{Nm}(t) - \hat{\mathbf{v}}(t)\|_{\rho}^2 = \sum_{i \in \mathbb{Z}^d} \rho_i \left| \mathbf{v}^{Nm}_i(t) - \hat{\mathbf{v}}_i(t) \right|^2 < \varepsilon^2, \quad m \geq M(\varepsilon).
$$

$$
\implies \left| \mathbf{v}^{Nm}_i(t) - \hat{\mathbf{v}}_i(t) \right| < \varepsilon/\sqrt{\rho_i}, \quad m \geq M(\varepsilon), \ i \in \mathbb{Z}^d.
$$

Thus, for every fixed $i \in \mathbb{Z}^d$,

$$
\left| \hat{\mathbf{v}}_i(t) - \mathbf{v}^*_i(t) \right| \leq \left| \mathbf{v}^{Nm}_i(t) - \hat{\mathbf{v}}_i(t) \right| + \left| \mathbf{v}^{Nm}_i(t) - \mathbf{v}^*_i(t) \right|
\leq \varepsilon/\sqrt{\rho_i} + \varepsilon.
$$

$$
\implies \hat{\mathbf{v}}_i(t) = \mathbf{v}^*_i(t) \text{ for every } i \in \mathbb{Z}^d \text{ and } t \in [t_0, t_0 + T].
$$
Limit as solution of the lattice inclusion

Rearrange the ODE for the convergent subsequence \( \{v^{Nm}(\cdot)\}_{m \in \mathbb{N}} \) to obtain

\[
\sigma_{im}^{Nm}(t) = \frac{d}{dt} v_{im}^{Nm}(t) - f_i(v_{im}^{Nm}(t)) - g_i(t), \quad i \in \mathbb{Z}^d, \text{ a.e.}
\]

With the limits \( v_i^*(\cdot) \) constructed above define

\[
\sigma_i^*(t) := \frac{d}{dt} v_i^*(t) - f_i(v_i^*(t)) - g_i(t), \quad i \in \mathbb{Z}^d, \text{ a.e.} \quad (3)
\]

The terms on the RHS converge in \( L^1([t_0, t_0 + T], \mathbb{R}) \) for each \( i \in \mathbb{Z}^d \). Hence

\[
\sigma_{im}^{Nm}(\cdot) \to \sigma_i^*(\cdot) \quad \text{in} \quad L^1([t_0, t_0 + T], \mathbb{R}) \quad \forall i \in \mathbb{Z}^d.
\]
We show that $\sigma_i^*(t) \in \mathcal{K}_i(\bm{v}^*(t)) \ \forall \ i \in \mathbb{Z}^d$ and a.e. $t \in [t_0, t_0 + T]$.

For each $N \in \mathbb{N}$ consider

\[
\begin{align*}
\int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\sigma_i^*(t), \mathcal{K}_i(\bm{v}^*(t))) \, dt \\
\leq \int_{t_0}^{t_0+T} |\sigma_i^*(t) - \sigma_i^{Nm}(t)| \, dt + \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\sigma_i^{Nm}(t), \mathcal{K}_i(\bm{v}^{Nm}(t))) \, dt \\
+ \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\mathcal{K}_i^N(\bm{v}^{Nm}(t)), \mathcal{K}_i^N(\bm{v}^*(t))) \, dt \\
+ \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\mathcal{K}_i^N(\bm{v}^*(t)), \mathcal{K}_i(\bm{v}^*(t))) \, dt.
\end{align*}
\]
the term (iv) ≡ 0 since $K_i^N(v^*(t)) \subset K_i(v^*(t)) \ \forall \ t \in [t_0, t_0 + T]$.

the term (i) converges to 0 as $m \to \infty$ since $\sigma_i^{Nm}(\cdot)$ converge to $\sigma_i^*(\cdot)$ in $L^1([t_0, t_0 + T], \mathbb{R})$. Thus, $\forall \ \varepsilon > 0, \ \exists \ M_1(\varepsilon)$ such that

$$(i) < \varepsilon \ \ \forall \ m \geq M_1(\varepsilon).$$

the term (ii) can be estimated by

$$(ii) \leq \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\sigma_i^{Nm}(t), K_i(v^{Nm}(t))) \ dt$$

$$+ \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (K_i(v^{Nm}(t)), K_i^N(v^{Nm}(t))) \ dt.$$ 

The first term here vanishes since $\sigma_i^{Nm}(t) \in K_i(v^{Nm}(t)) \ \forall \ t$, so

$$(ii) \leq \int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (K_i(v^{Nm}(t)), K_i^N(v^{Nm}(t))) \ dt.$$
For the second term, since $v^{Nm}(t) \in \ell^2_\rho$ for all $t \in [t_0, t_0 + T]$, so by Lemma 4 there exists $N(\varepsilon)$ such that

$$\text{dist}_\mathbb{R} \left( K_i(v^{Nm}(t)), K_i^N(v^{Nm}(t)) \right) \leq \varepsilon \quad \text{for } N \geq N(\varepsilon), \ t \in [t_0, t_0 + T],$$

$$\implies (\text{ii}) \leq T \varepsilon \quad \text{for } N \geq N(\varepsilon).$$

- By the upper semi continuity of $K_i^N$ (Lemma 3), $\forall \varepsilon > 0$ there exists $M_2(\varepsilon, N)$ such that

$$\text{dist}_\mathbb{R} \left( K_i^N(v^{Nm}(t)), K_i^N(v^*(t)) \right) < \varepsilon \quad \forall \ m \geq M_2(\varepsilon, N).$$

$$\implies (\text{iii}) \leq T \varepsilon \quad \text{for } m \geq M_2(\varepsilon, N).$$

In summary, fixing $N \geq N(\varepsilon)$, for any $m \geq \max\{M_1(\varepsilon), M_2(\varepsilon, N)\}$

$$\int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} \left( \sigma_i^*(t), K_i(v^*(t)) \right) \, dt < (2T + 1)\varepsilon.$$
The LHS is independent of \( m \) and \( N \), so

\[
\int_{t_0}^{t_0+T} \text{dist}_\mathbb{R} (\sigma_i^*(t), \mathcal{K}_i(v^*(t))) \, dt = 0.
\]

\[
\Rightarrow \text{dist}_\mathbb{R} (\sigma_i^*(t), \mathcal{K}_i(v^*(t))) = 0, \quad t \in [t_0, t_0 + T], \text{ a.e.},
\]

\[
\Rightarrow \sigma_i^*(t) \in \mathcal{K}_i(v^*(t)), \quad t \in [t_0, t_0 + T], \text{ a.e.}
\]

Finally, equation for \( \sigma_i^* \) can be rewritten as

\[
\frac{d}{dt} v_i^*(t) = f_i(v_i^*(t)) + \sigma_i^*(t) + g_i(t), \quad i \in \mathbb{Z}^d, \text{ a.e.},
\]

with \( \sigma_i^*(t) \in \mathcal{K}_i(v^*(t)) \implies v^*(t) = (v_i^*(t)))_{i \in \mathbb{Z}^d} \) is a solution of the lattice differential inclusion.

The existence proof is complete.
Nonautonomous set-valued dynamical system

The attainability set of the lattice inclusion

\[ \Phi(t, t_0, u_o) := \left\{ v \in \ell^2_\rho : \exists \text{ a solution } u(\cdot; t_0, u_o) \text{ with } \right. \]
\[ \left. \quad u(t_0; t_0, u_o) = u_o \text{ such that } v = u(t; t_0, u_o) \right\} . \]

generates a two-parameter set-valued semi-group.

**Lemma 5** Suppose that Assumptions (A1)–(A2), (F1)–(F2) and (G1) hold. Then for any \( t > t_0 \) and \( u_o = (u_i, o)_{i \in \mathbb{Z}^d} \in \ell^2_\rho \), the attainability set \( \Phi(t, t_0, u_o) \) is a nonempty compact subset of \( \ell^2_\rho \). Moreover, the set-valued mapping \( u_o \mapsto \Phi(t, t_0, u_o) \) is upper semi continuous in \( u_o \) in \( \ell^2_\rho \) for any \( t \geq t_0 \).
Lemma 6  Suppose that \((A1)-(A2), (F1)-(F2)\) and \((G1)\) hold. Then the set-valued dynamical system \(\Phi(t, t_0, u_0)\) has a nonautonomous pullback absorbing set with closed and bounded component sets

\[
\Lambda(t) := \{ u \in \ell^2 : \|u\|_{\rho}^2 \leq R(t) \}, \quad \forall t \in \mathbb{R},
\]

where

\[
R(t) := \frac{2}{\alpha} \left( \|\beta\|_{\rho}^2 + \frac{\kappa^2}{\alpha} \rho_{\Sigma} + \rho_{\Sigma} \int_{-\infty}^{t} \bar{g}^2(s)e^{-\alpha(t-s)}ds \right) + 1
\]

This follows easily from the dissipativity condition.

The sets \(\Lambda(t), t \in \mathbb{R}\), are positively invariant in the sense that

\[
\Phi(t, t_0, \Lambda(t_0)) \subset \Lambda(t), \quad \forall t \geq t_0.
\]
Lemma 7  Suppose that (A1)–(A2), (F1)–(F2) and (G1) hold. Then the set-valued dynamical system $\Phi(t, t_0, u_0)$ is asymptotically upper semi compact.

The proof follows the paper


• First we obtain the tail estimate of for a sequence of solutions $\{u^n(t)\}$ inside the absorbing set

$$\sum_{|i| \geq M_1(\varepsilon)} \rho_i u^n_i(t)^2 \leq \varepsilon, \quad n \geq N_1(\varepsilon), \ t_0 \leq t - T(\varepsilon).$$

• Then we use the Ascoli-Arzelà Theorem to obtain a weakly convergent subsequence

• Finally we use the tail estimate to show that the subsequence is in fact strongly convergent
**Definition** \( \mathcal{A} = \{A(t)\}_{t \in \mathbb{R}} \) is a pullback attractor for \( \Phi \) if

\[ \begin{align*}
\star & \text{ } A(t) \subset X \text{ is compact for each } t \in \mathbb{R}; \\
\star & \text{ } \mathcal{A} \text{ is invariant, i.e., } \Phi(t, t_0, A(t_0)) = A(t), \; \forall t \geq t_0; \\
\star & \text{ } \mathcal{A} \text{ is pullback attracts the absorbing set,} \\
& \lim_{\tau \to \infty} \text{dist}_X(\Phi(t, t-\tau, \Lambda(t-\tau)), A(t)) = 0.
\end{align*} \]

**Theorem** Suppose that \((A1)-(A2), (F1)-(F2)\) and \((G1)\) hold. Then the set-valued dynamical system \( \Phi(t, t_0, u_o) \) systems generated by the neural lattice model possesses a unique pullback attractor \( \mathcal{A} = \{A(t)\}_{t \in \mathbb{R}} \) with components given by

\[ A(t) = \bigcap_{t_0 \geq 0} \bigcup_{s \geq t_0} \Phi(t, t-\tau, \Lambda(t-\tau)). \]
**Forward omega limit sets**

Pullback attractors involve information about the dynamics of the system in the past. They need not be asymptotically stable.

Nonautonomous omega limit sets involve information about the dynamics in the future.

The lattice inclusion system $\Phi(t, t_0, u_o)$ has a positively invariant forward absorbing set

$$\Lambda_0 := \left\{ u \in \ell^2_\rho : \|u\|_\rho^2 \leq R_0 \right\}$$

where

$$R_0 := \frac{2}{\alpha} \left( \|\beta\|_\rho^2 + \frac{\kappa^2}{\alpha} \rho_\Sigma + \overline{G} \right) + 1,$$

$$\overline{G} := \sup_{t \geq 0} e^{-\alpha t} \int_{t_0}^t \|G(s)\|_\rho^2 e^{\alpha s} ds < \infty.$$
Similarly to the pullback case it can be shown that

**Proposition**  The lattice inclusion system $\Phi(t, t_0, u_0)$ is forward asymptotic compact in $\Lambda_0$.

Hence for each $t_0 \in \mathbb{R}$ the nonautonomous omega limit set

$$\omega_{t_0, \Lambda_0} = \{ x \in \ell^2_{\rho} : \exists \ t_n \to \infty, x_n \in \Phi(t_n, t_0, \Lambda_0), x_n \to x \text{ as } n \to \infty \}$$

is a nonempty and compact subset of $\Lambda_0$.

$$\text{dist}_{\ell^2_{\rho}} (\Phi(t_n, t_0, \Lambda_0), \omega_{t_0, \Lambda_0}) \to 0 \text{ as } t \to \infty.$$