Stock Loan Valuation Under Brownian-Motion Based and Markov Chain Stock Models

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What is a Stock Loan?

- Client (borrower) owns share of stock.
- Use as collateral to obtain, for a fee, loan from bank (lender).
- Upon loan maturity (or before, for American maturity), client may either:
  - Repay the loan (principal and interest).
  - Default (surrender the stock).

Stock Loan Problem

- For a given stock, maturity, principal, and loan interest rate, what is the fair value of the fee charged by bank?
- Notations:
  - q = Loan Principal
  - γ = Loan Interest Rate
  - r = Risk-Free Rate
  - c = Bank Service Fee
  - Amount borrower gets = $q - c$

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Two Examples: Borrower's Perspective

Stock	Closing Price 6/5/17	Repayment Amount 6/5/18*		
Apple (AAPL)	$ 155.99	$172.40		
Southern Co. (SO)	$ 50.81	$ 56.15		

*Assumes $\gamma = 0.1$ and $q =$ share price.

Two Examples: Borrower's Perspective

Stock	Closing Price 6/5/17	Repayment Amount 6/5/18*	Closing Price 6/5/18	Borrower's Decision
Apple (AAPL)	$ 155.99	$ 172.40	$ 191.83	Repay
Southern Co. (SO)	$ 50.81	$ 56.15	$ 43.93	Default

*Assumes $\gamma = 0.1$ and $q =$ share price.

Two Examples: Lender's Perspective

Stock	Closing Price 6/5/17	Cash Paid Out*	Borrower's Decision	Lender's Nominal Profit
Apple (AAPL)	$ 155.99	$ 153.99	Repay	172.40-153.99 =18.41
Southern Co. (SO)	$ 50.81	$ 48.81	Default	43.93-48.81 =(4.88)

*Assumes $c = \$2$ and $q =$ share price.

Perpetual Stock Loans (Xia and Zhou, 2007)

- Stock obeys Geometric Brownian Motion:

$$S_t = x \exp\left(\left(r - \delta - (\sigma^2/2)\right) t + \sigma W_t\right)$$

  where δ is the dividend yield and $x = S_0$.
- Bank collects dividends during loan period.
- The loan is perpetual.

Evaluating the Stock Loan: Preliminaries

- Let $V(x) \equiv$
$$\sup_{\tau \in T_0} \mathbb{E}\left[e^{-r\tau}\left(x \exp\left(\left(r - \delta - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right) - qe^{\gamma\tau}\right)_+\right].$$
- Assumption: $V(x) = x - q + c > 0$

Evaluating the Stock Loan: Preliminaries

For $a \in \mathbb{R}^+$, let:

- $\tau_a \equiv \inf \left[t \geq 0 : e^{-\gamma t} S_t = a \right]$
- $g(a) \equiv \mathbb{E} \left[e^{-r\tau_a} \left(S_{\tau_a} - q e^{\gamma \tau_a} \right)_+ \right]$

$$= (a - q) \, \mathbb{E} \left[e^{(\gamma - r)\tau_a} I_{\tau_a < \infty} \right].$$

Solving for the Value Function: Case 1

Case 1: If $\delta = 0$ and $\gamma - r \leq \frac{\sigma^2}{2}$, then

1. $g(a) = \frac{(a-q)x}{a}$ and
2. $V(x) = x$.

Solving for the Value Function: Case 2

Let $a_0 \equiv \frac{\left(q\left[\sqrt{\left(\frac{\sigma}{2}-\frac{\gamma-r+\delta}{\sigma}\right)^2+2\delta}+\frac{\sigma}{2}+\frac{\gamma-r+\delta}{\sigma}\right]\right)}{\left(\sqrt{\left(\frac{\sigma}{2}-\frac{\gamma-r+\delta}{\sigma}\right)^2+2\delta}-\frac{\sigma}{2}+\frac{\gamma-r+\delta}{\sigma}\right)}$.

Case 2: If $\delta > 0$, or $\delta = 0$ and $\gamma - r > \frac{\sigma^2}{2}$, and $q < a_0 \leq x$, then

1. $g(a)$ attains its maximum at $a = x$ and
2. $V(x) = x - q$.

Solving for the Value Function: Case 3

Let $a_0 \equiv \frac{\left(q\left[\sqrt{\left(\frac{\sigma}{2}-\frac{\gamma-r+\delta}{\sigma}\right)^2+2\delta}+\frac{\sigma}{2}+\frac{\gamma-r+\delta}{\sigma}\right]\right)}{\left(\sqrt{\left(\frac{\sigma}{2}-\frac{\gamma-r+\delta}{\sigma}\right)^2+2\delta}-\frac{\sigma}{2}+\frac{\gamma-r+\delta}{\sigma}\right)}$.

Case 3: If $\delta > 0$, or $\delta = 0$ and $\gamma - r > \frac{\sigma^2}{2}$, and $a_0 > x$, then

1. $g(a)$ attains its maximum on $[q \vee x, \infty)$ at $a = a_0$ and
2. $V(x) = g(a_0)$.

Finite Maturity Stock Loans, Mean-Reverting Model

- Assume the stock loan matures at time $T < \infty$ and maturity is European.
- Assume the stock price obeys the mean-reverting model.

$$S_t = e^{X_t}$$

$$dX_t = a(L - X_t)dt + \sigma dW_t$$

where $a > 0$ is the rate of reversion and L is the equilibrium level.

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Key Idea: Change of time

- Let $\phi_t \equiv \int_0^t \left(1/\alpha\left(\phi_s, \Gamma^{\phi_s}\right)\right) ds$ and $\alpha(t,\omega) = \alpha(t) \equiv \sigma e^{at}$.
- Then the mean-reverting model can be written explicitly:

$$X_t = e^{-at}(\log x - L) + L + e^{-at} W\left(\frac{1}{\phi_t}\right).$$

Solving for the Value Function (P. and Zhang, 2010)

Let $u \equiv T - s$. Under the mean-reverting model with European maturity,

$$V(u,x) = \frac{e^{(\gamma-r)u+\frac{B^2}{4A}+C}}{\sqrt{(\phi_{u+s})^{-1}}}\frac{1}{\sqrt{A}}\left[1-\Phi\left(\sqrt{A}\left(P-\frac{B}{2A}\right)\right)\right]$$

$$-\frac{qe^{(\gamma-r)u}}{\sqrt{(\phi_{u+s})^{-1}}}\frac{1}{\sqrt{2A}}\left[1-\Phi\left(P\sqrt{2A}\right)\right]$$

where

$C \equiv e^{-a(u+s)}(\log x - L) + L$

$B \equiv e^{-a(u+s)}$

$A \equiv \dfrac{1}{2(\phi_{u+s})^{-1}}$

$P \equiv e^{a(u+s)}(\log q - L) + L - \log x,$

Markov Chain Model for Perpetual Case

- α_t denote a Markov chain with state space $\{1, 2\}$ and generator $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$.
- Stock obeys $\frac{dS_t}{S_t} = \mu(\alpha_t)dt,\ S_0 = x \geq 0,\ t \geq 0$
- $\mu_1 = \mu(1) > 0$ and $\mu_2 = \mu(2) < 0$ are given return rates.

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- Stopping Time: τ (perpetual case)
- Payoff Function:

$$J(x, i, \tau) \equiv E\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)_+ I_{\tau<\infty} | S_0 = x, \alpha_0 = i\right],$$

where $x_+ = \max\{0, x\}$.
- Value Function: $V(x, i) = \sup_\tau J(x, i, \tau)$, where the sup is taken over all stopping times τ.

Sufficient Conditions for a Closed-Form Solution

- $\mu_2 < r < \gamma < \mu_1$.
- $r > \rho_0$ where $\rho_0 \equiv$
  $\frac{1}{2}\left(\mu_1 - \lambda_1 + \mu_2 - \lambda_2 + \sqrt{((\mu_1 - \lambda_1) - (\mu_2 - \lambda_2))^2 + 4\lambda_1\lambda_2}\right)$
  is the larger root of the equation
  $\Phi(x) = (x + \lambda_1 - \mu_1)(x + \lambda_2 - \mu_2) - \lambda_1\lambda_2$.
- $\lambda_i > \gamma - r$, for $i = 1, 2$.

- $X_t \equiv e^{-\gamma t} S_t$, so that

$$dX_t = X_t \left[-\gamma + \mu(\alpha_t)\right] dt.$$

- Letting $\xi \equiv \gamma - r > 0$, the value function becomes

$$V(x, i) = \sup_{\tau} E\left[e^{\xi\tau} \left(X_\tau - q\right)_+ I_{\tau<\infty} | X_0 = x, \alpha_0 = i\right]$$

HJB Equation and Variational Inequalities

With $f_i \equiv \mu_i - \gamma$, the generator for this value function is

$$\mathcal{A}h(x,i) = xf_i h'(x,i) + Qh(x,\cdot)(i),$$

where $Qh(x,\cdot)(1) = \lambda_1(h(x,2) - h(x,1))$, and $Qh(x,\cdot)(2) = \lambda_2(h(x,1) - h(x,2))$.

The associated variational inequalities are

$$\max\{\xi h(x,1) + \mathcal{A}h(x,1), (x-q)_+ - h(x,1)\} = 0,$$

$$\max\{\xi h(x,2) + \mathcal{A}h(x,2), (x-q)_+ - h(x,2)\} = 0.$$

Solution via Smooth-Fit Substitution

- Start on the region $(0, x^*)$ with free boundary x^*, i.e. the case in which $(\xi + \mathcal{A})h(x, i) = 0$, $i = 1, 2$.
- Solve the case in which $(\xi + \mathcal{A})h(x, 1) = 0$ for $h(x, 2)$ and substitute into $(\xi + \mathcal{A})h(x, 2) = 0$.

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Characteristic Equation

Substitution gives a 2nd order ODE with characteristic equation

$$\phi(\beta) = f_1 f_2 \beta^2 + [f_1(\xi - \lambda_2) + f_2(\xi - \lambda_1)]\,\beta + [(\xi - \lambda_1)(\xi - \lambda_2) - \lambda_1\lambda_2]\,.$$

Characteristic Equation: Solutions (P. and Zhang, 2014)

$$\beta_1 = \frac{-D_1 + \sqrt{D_1^2 - 4f_1 f_2 D_2}}{2f_1 f_2},$$

$$\beta_2 = \frac{-D_1 - \sqrt{D_1^2 - 4f_1 f_2 D_2}}{2f_1 f_2},$$

where

$$D_1 = f_1(\xi - \lambda_2) + f_2(\xi - \lambda_1),$$
$$D_2 = (\xi - \lambda_1)(\xi - \lambda_2) - \lambda_1 \lambda_2.$$

$$x^* = \left(\frac{\xi - \lambda_1 + f_1}{\xi - \lambda_1}\right)\left(\frac{q\beta_2}{\beta_2 - 1}\right).$$

Stopping Time Solution

$$h(x,1) = \begin{cases} A_2 x^{\beta_2} & \text{if } 0 \le x \le x^*, \\ A_0 x + B_0 & \text{if } x > x^*, \end{cases}$$

$$h(x,2) = \begin{cases} \kappa_2 A_2 x^{\beta_2} & \text{if } 0 \le x \le x^*, \\ x - q & \text{if } x > x^*. \end{cases}$$

$$A_0 = -\frac{\lambda_1}{\xi - \lambda_1 + f_1}$$

$$B_0 = \frac{\lambda_1 q}{\xi - \lambda_1}$$

$$\kappa_2 = \frac{1}{\lambda_1}\left[-(\xi - \lambda_1) - f_1 \beta_2\right]$$

$$A_2 = \frac{A_0 x^* + B_0}{(x^*)^{\beta_2}}$$

$h(x, i) = V(x, i)$, $i = 1, 2$. Moreover, let

$$D = (0, \infty) \times \{1\} \cup (0, x^*) \times \{2\}$$

denote the continuation region. Then

$$\tau^* = \inf\{t \geq 0;\ (X_t, \alpha_t) \notin D\}$$

is an optimal exercising time.

Given $\epsilon > 0$, take

$$\begin{aligned} \mu_1 &= r - \frac{\sigma^2}{2} + \frac{\sigma}{\sqrt{\epsilon}}, \\ \mu_2 &= r - \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{\epsilon}}, \\ \lambda_1 &= \lambda_2 = \frac{1}{\epsilon}. \end{aligned}$$

As $\epsilon \to 0$,

- $S_t = S_t^\epsilon$ converges weakly to

$$S_t^0 = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

- $x^* = x^{\epsilon,*} \to x^0 \equiv \beta_0 q/(\beta_0 - 1)$
- $V(x,1)$ and $V(x,2)$ both converge to

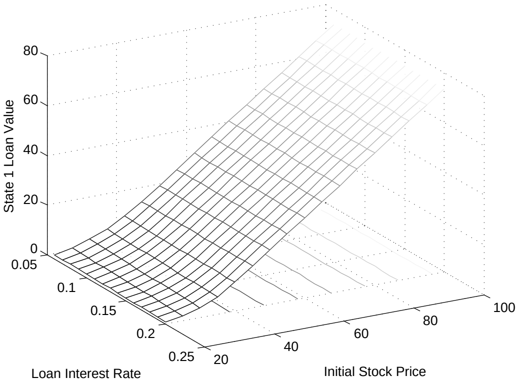
$$V^0(x) = \begin{cases} A^0 x^{\beta_0} & \text{if } x < x^0, \\ x - q & \text{if } x \geq x^0, \end{cases}$$

where $A^0 \equiv \dfrac{(\beta_0 - 1)^{\beta_0 - 1} q^{1-\beta_0}}{(\beta_0)^{\beta_0}}$.

Numerical Examples: Default Parameters and Initial Conditions

Parameter	Value
r	0.05
q	30
S_0	33
γ	0.1
λ_1	135.25
λ_2	130.95
μ_1	4.89
μ_2	-5.13

Numerical Examples: γ versus S_0



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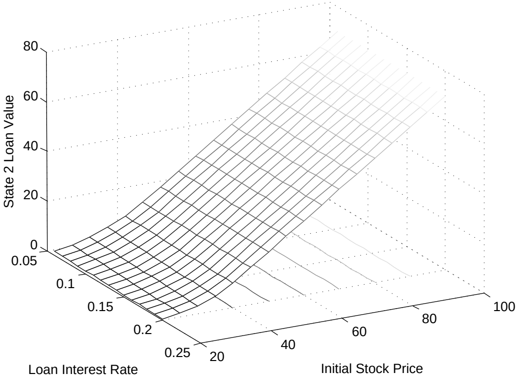
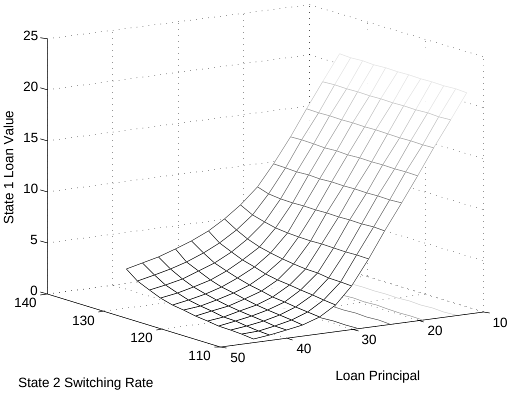


Numerical Examples: q versus λ_2



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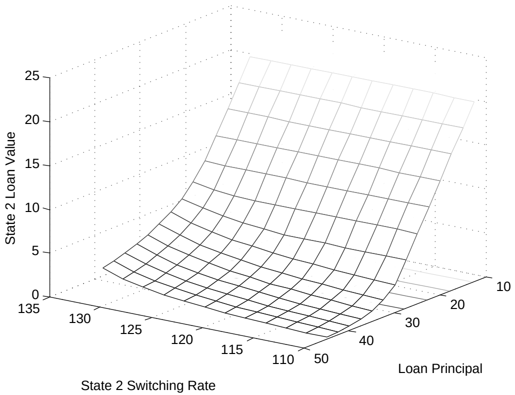
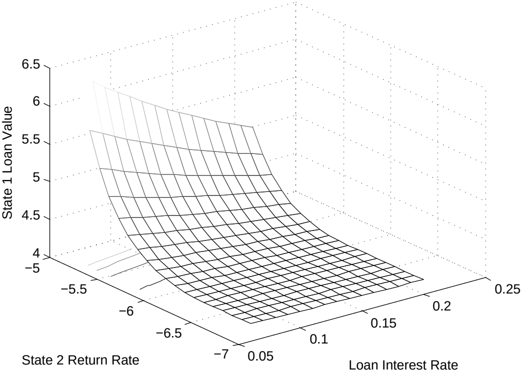
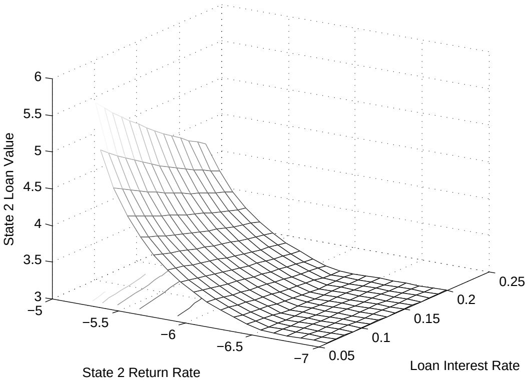


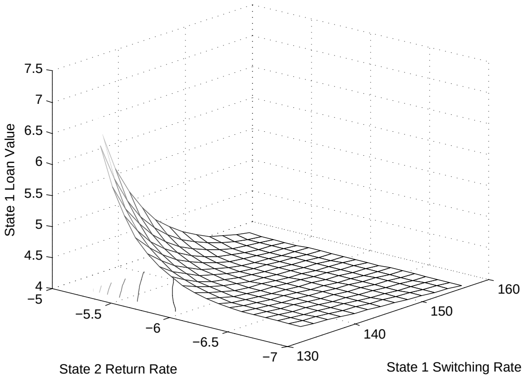
Numerical Examples: γ versus μ_2



Numerical Examples: γ versus μ_2



Numerical Examples: λ_1 versus μ_2



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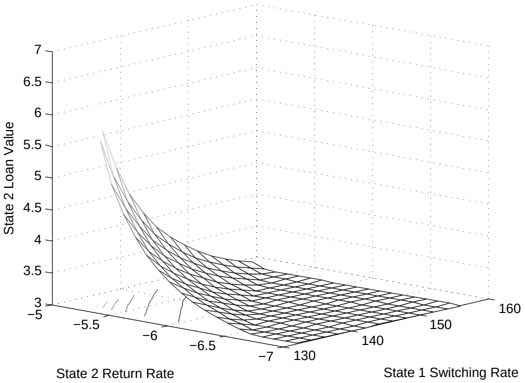

Conclusions

- Combined continuous and discrete properties
- Closed-form formulas for optimal stopping time and value function
- The stock loan valuation can be determined by the corresponding exercise time, which is given in terms of a single threshold level.

Directions for Further Study

- Model calibration
- Time variables

- Xia, Jianming and Xun Yu Zhou, Stock Loans, *Mathematical Finance*, April 2007, 307-317.
- Norberg R, The Markov chain market, *ASTIN Bulletin*, 2003, **33**: 265–287.
- Prager D and Zhang Q, Stock loan valuation under a regime-switching model with mean-reverting and finite maturity, *Journal of Systems Science and Complexity*, 2010: 572–583.
- Prager D and Zhang Q, Valuation of Stock Loans under a Markov Chain Model, *Journal of Systems Science and Complexity*, 2014: 1–17.

- All stock market data is from Yahoo! Finance.
- All figures were produced using MATLAB.