Valuing Equity-Linked Insurance Products

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Motivation

- To value guarantees and options in Variable Annuities

- Variable Annuities
  - = Investment Funds (Mutual Funds)
  - + Rider(s) : Guaranteed Minimum Benefits
Equity-linked death benefit

- $x$ age at issue of policy (time 0)
- $T_x$ time of death
- payment at time $T_x$
- depends on $S(T_x)$,
- or more generally on $S(t), 0 \leq t \leq T_x$
Goal:

- Calculate $E[e^{-\delta T_x} \times \text{payment}]$
- the expectation of the discounted value of the payment

$\delta$ valuation force of interest
Equity-linked products are very popular in the market nowadays.

Example: Guaranteed Minimum Death Benefits

Payoff:

\[ \max(S(T_x), K) = S(T_x) + [K - S(T_x)]_+ = K + [S(T_x) - K]_+ , \]

where \( T_x \) is the time-until-death random variable for a life age \( x \), \( S(t) \) is the price of equity-index at time \( t \), and \( K \) is the guaranteed amount.
Mathematical Problem:

- \( \{X(t); \ t \in [0, \infty), \text{ or } t = 0, 1, 2, \ldots\} \) a random process
- Running minimum: \( m(t) = \min_{0 \leq s \leq t} X(s) \)
- Running maximum: \( M(t) = \max_{0 \leq s \leq t} X(s) \)
- \( \tau \) a random variable
- We are interested in the distributions of \( X(\tau) \) and \((X(\tau), M(\tau))\) or \((X(\tau), m(\tau))\).
Exponential stopping of Brownian motion

- $X(t) = \mu t + \sigma W(t)$
- $\{W(t)\}$: standard Wiener process

$$f_{X(\tau)}(x) = \begin{cases} \kappa e^{-\alpha x}, & \text{if } x < 0, \\ \kappa e^{-\beta x}, & \text{if } x > 0. \end{cases}$$

with $\kappa = \frac{2\lambda}{\sigma^2(\beta - \alpha)}$, $\alpha < 0$ and $\beta > 0$ solutions of the quadratic equation $\frac{\sigma^2}{2} \xi^2 + \mu \xi - \lambda = 0$

$$f_{X(\tau), M(\tau)}(x, m) = \frac{2\lambda}{\sigma^2} e^{\alpha(m-x)-\beta m}, \quad -\infty < x \leq m, \ m \geq 0$$
Erlang stopping of Brownian motion

\[ f_\tau(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad t > 0, \]

Let \( \hat{f}_{X(\tau)}(z) \) denote the two-sided Laplace transform of \( f_{X(\tau)}(x) \).

\[ \hat{f}_{X(\tau)}(z) = E[E[e^{-zX(\tau)}|\tau]] = E[e^{Dz^2\tau - \mu z \tau}] = \hat{f}_\tau(-Dz^2 + \mu z) \]

where

\[ \hat{f}_\tau(z) = \left( \frac{\lambda}{z + \lambda} \right)^n. \]
\[ f_{X(\tau)}(z) = \left( \frac{\lambda}{-Dz^2 + \mu z + \lambda} \right)^n \]

\[ = \left( \frac{\lambda}{-D(z + \beta)(z + \alpha)} \right)^n \]

\[ = \kappa^n \left( \frac{1}{z + \beta} - \frac{1}{z + \alpha} \right)^n, \]

for \(-\beta < z < -\alpha\).
Lemma

Let $a$ and $b$ be elements with the property that $ab = a + b$. We have

$$(a + b)^n = P_n(a) + P_n(b),$$

where

$$P_n(x) = \sum_{k=0}^{n-1} \binom{n-1 + k}{k} x^{n-k}.$$ 

If $a$ and $b$ have the property that $ab = \nu(a + b)$ for some number $\nu \neq 0$, then

$$(a + b)^n = \nu^n P_n\left(\frac{a}{\nu}\right) + \nu^n P_n\left(\frac{b}{\nu}\right).$$
Let 
\[ a = \frac{-1}{\alpha + z}, \quad b = \frac{1}{\beta + z}, \quad \nu = \frac{1}{\beta - \alpha}. \]

Thus
\[
\hat{f}_{X(\tau)}(z) = \kappa^n \nu^n \left\{ P_n \left( \frac{a}{\nu} \right) + P_n \left( \frac{b}{\nu} \right) \right\} \\
= \kappa^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} \left( \frac{1}{\beta - \alpha} \right)^k \left( \frac{-1}{\alpha + z} \right)^{n-k} \\
+ \kappa^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} \left( \frac{1}{\beta - \alpha} \right)^k \left( \frac{1}{\beta + z} \right)^{n-k}.
\]
Distribution of \( X(\tau) \)

Note that \( (\frac{-1}{\alpha+z})^j \) is the two-sided Laplace transform of the function \( \frac{(-x)^{j-1}}{(j-1)!} e^{-\alpha x} I(x<0) \), and \( (\frac{1}{\beta+z})^j \) is that of \( \frac{x^{j-1}}{(j-1)!} e^{-\beta x} I(x>0) \). Hence

\[
f_{X(\tau)}(x) = \begin{cases} 
\kappa^n e^{-\alpha x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)! (\beta-\alpha)^{n-j}} (-x)^{j-1}, & \text{if } x < 0, \\
\kappa^n e^{-\beta x} \sum_{j=1}^{n} \frac{\binom{2n-j-1}{n-j}}{(j-1)! (\beta-\alpha)^{n-j}} x^{j-1}, & \text{if } x > 0,
\end{cases}
\]
If the drift $\mu$ of the Brownian motion $X(t)$ is zero, it follows from the reflection principle that

$$\Pr(X(t) \leq x, M(t) > y) = \Pr(X(t) \leq x - 2y), \quad y \geq \max(x, 0).$$
By changing the probability measure, we can change the drift. If the drift $\mu$ is an arbitrary constant, then

$$\Pr(X(t) \leq x, M(t) > y) = e^{y\mu/D} \Pr(X(t) \leq x - 2y),$$

$$y \geq \max(x, 0).$$

The joint density function of $X(t)$ and $M(t)$ is

$$f_{X(t),M(t)}(x, y) = -\frac{\partial^2}{\partial y \partial x} \Pr(X(t) \leq x, M(t) > y)$$

$$= -\frac{\partial}{\partial y} [e^{y\mu/D} f_{X(t)}(x - 2y)], \quad y \geq \max(x, 0).$$
Distribution of $(X(\tau), M(\tau))$

- We can replace $t$ by $\tau$, then

$$f_{X(\tau), M(\tau)}(x, y) = -\frac{\partial}{\partial y} [e^{y\mu/D} f_{X(\tau)}(x - 2y)], \quad y \geq \max(x, 0).$$

From this we obtain the distribution of $(X(\tau), M(\tau))$. 
Exponential stopping of Lévy process

- moment generating function of $X(t)$ is
  \[ E[e^{zX(t)}] = e^{t\Psi(z)}, \]
  where $\Psi(z)$ denotes the Lévy exponent of the process
- the mgf of $X(\tau)$:
  \[ E[e^{zX(\tau)}] = E[E[e^{zX(\tau)}|\tau]] = E[e^{\tau\Psi(z)}] = \frac{\lambda}{\lambda - \Psi(z)}. \]
Suppose that $\Psi(z)$ is a rational function and that the roots of

$$\Psi(z) = \lambda$$

are distinct. Let $\{\alpha_j\}$ and $\{\beta_k\}$ be the roots with negative and positive real part, respectively.

$$\frac{\lambda}{\lambda - \Psi(z)} = \sum_j \frac{\lambda}{\Psi'(\alpha_j)} \frac{1}{\alpha_j - z} + \sum_k \frac{\lambda}{\Psi'(\beta_k)} \frac{1}{\beta_k - z},$$
The pdf of $X(\tau)$ is

$$f_{X(\tau)}(x) = \begin{cases} 
\sum_j a_j e^{-\alpha_j x}, & \text{if } x < 0, \\
\sum_k b_k e^{-\beta_k x}, & \text{if } x \geq 0,
\end{cases}$$

where

$$a_j = \frac{-\lambda}{\Psi'(\alpha_j)}$$

and

$$b_k = \frac{\lambda}{\Psi'(\beta_k)}.$$
Let $\{X(t); \ t \geq 0\}$ be a Brownian motion (with drift and diffusion parameters $\mu$ and $\sigma$) extended by independent jumps in both directions. The downward jumps form an independent compound Poisson process; the frequency of these jumps is $\nu$. Similarly, the upward jumps forms another independent compound Poisson process with Poisson parameter $\omega$. 
Assume that the pdf of each downward jump is

$$\sum_{j=1}^{m} A_j v_j e^{-v_j x}, \quad x > 0,$$

with $\sum_{j=1}^{m} A_j = 1$ and $0 < v_1 < v_2 < ... < v_m$, and that the pdf of each upward jump is

$$\sum_{k=1}^{n} B_k w_k e^{-w_k x}, \quad x > 0,$$

with $\sum_{k=1}^{n} B_k = 1$ and $0 < w_1 < w_2 < ... < w_n$. 
Then

\[ \Psi(z) = Dz^2 + \mu z - \nu \sum_{j=1}^{m} A_j \frac{z}{v_j + z} + \omega \sum_{k=1}^{n} B_k \frac{z}{w_k - z}, \]

where

\[ D = \sigma^2 / 2. \]

Under the assumption that the \( m + n + 2 \) solutions of the equation \( \Psi(z) = \lambda \) are distinct, the density function of \( X(\tau) \) is given above. In the case of mixtures (all \( A'_i \)'s and \( B'_i \)'s positive), the solutions are distinct and real as we have the following interlacing relationship:

\[-\infty < \alpha_{m+1} < -\nu_m < ... < -\nu_1 < \alpha_1 < 0 < \beta_1 < w_1 < ... < w_n < \beta_{n+1} < \infty.\]
For Lévy process, we have

\[ M(\tau) \text{ and } [M(\tau) - X(\tau)] \text{ are independent} \]

\[ [X(\tau) - M(\tau)] \text{ and } m(\tau) \text{ have the same distribution} \]

Therefore

\[ \mathbb{E}[e^{zX(\tau)}] = \mathbb{E}[e^{zM(\tau)}]\mathbb{E}[e^{zm(\tau)}], \]

a version of the celebrated \textit{Wiener-Hopf factorization}. 
\[ M_{X(\tau)}(z) = \frac{\lambda}{\lambda - \Psi(z)}; \]

\( M_{X(\tau)}(z) \) is the mgf \( E[e^{zX(\tau)}] \) when the expectation exists. The zeros of the denominator are the poles of \( M_{X(\tau)}(z) \).

Because \( 0 \leq M(\tau) < \infty \), the mgf \( E[e^{zM(\tau)}] \) is an analytic function of \( z \) with negative real part and it has no negative zeros. Similarly, the mgf \( E[e^{zm(\tau)}] \) is an analytic function of \( z \) with positive real part and it has no positive zeros.
Thus

\[ E[e^{zm(\tau)}] \propto \left[ \prod_{j=1}^{m} (z + v_j) \right] \left[ \prod_{j=1}^{m+1} \frac{1}{z - \alpha_j} \right], \]

\[ E[e^{zM(\tau)}] \propto \left[ \prod_{k=1}^{n} (z - w_k) \right] \left[ \prod_{k=1}^{n+1} \frac{1}{z - \beta_k} \right]. \]

As each mgf takes the value 1 when \( z = 0 \), we have

\[ E[e^{zm(\tau)}] = \left[ \prod_{j=1}^{m} \frac{z + v_j}{v_j} \right] \left[ \prod_{j=1}^{m+1} \frac{-\alpha_j}{z - \alpha_j} \right], \]

\[ E[e^{zM(\tau)}] = \left[ \prod_{k=1}^{n} \frac{w_k - z}{w_k} \right] \left[ \prod_{k=1}^{n+1} \frac{\beta_k}{\beta_k - z} \right]. \]
Hence, the pdf of $M(\tau)$ is

$$f_{M(\tau)}(x) = \sum_{k=1}^{n+1} b^*_k e^{-\beta_k x}, \quad x > 0.$$ 

where

$$b^*_k = \left[ \prod_{i=1}^{n} \frac{w_i - \beta_k}{w_i} \right] \left[ \prod_{i=1, i \neq k}^{n+1} \beta_i \right] \beta_k.$$
Similarly, we obtain the pdf of $m(\tau)$,

$$f_{m(\tau)}(x) = \sum_{j=1}^{m+1} a_j^* e^{-\alpha_j x}, \quad x < 0,$$

where

$$a_j^* = \left[ \prod_{i=1}^{m} \frac{\alpha_j + v_i}{v_i} \right] \left[ \prod_{i=1, i \neq j}^{m+1} \frac{-\alpha_i}{\alpha_j - \alpha_i} \right] (-\alpha_j).$$
Now

$$f_{X(\tau), M(\tau)}(x, y) = f_{M(\tau), X(\tau) - M(\tau)}(y, x - y)$$

$$= f_{M(\tau)}(y)f_{X(\tau) - M(\tau)}(x - y)$$

$$= f_{M(\tau)}(y)f_{m(\tau)}(x - y)$$

$$= \left[ \sum_{k=1}^{n+1} b_k^* e^{-\beta_k y} \right] \left[ \sum_{j=1}^{m+1} a_j^* e^{-\alpha_j(x-y)} \right]$$

$$= \sum_{j=1}^{m+1} \sum_{k=1}^{n+1} a_j^* b_k^* e^{-(\beta_k - \alpha_j)y - \alpha_j x}.$$
For $t \geq 0$, let the *surplus* of a company at time $t$ be

$$U(t) = u - X(t),$$

where $u$ is a positive number representing the initial surplus. Let

$$T = \inf\{t : U(t) \leq 0\}$$

be the *time of ruin*. 
Then,

\[ \Pr(M(\tau) \geq u) = \Pr(\tau > T) = E[e^{-\lambda T}]. \]

Thus, knowing the Laplace transform, with respect to \( \lambda \), of the time of ruin random variable is equivalent to knowing the distribution of the \( M(\tau) \).
In particular, if \( \{X(t)\} \) is a Lévy process with an upward jump pdf of the linear combination of exponential, we can use the results in Section 9 of Albrecher, Gerber and Yang (2010), with \( w(x) \equiv 1 \) and \( w_0 = 1 \), to obtain a closed-form expression for \( \Pr(M(\tau) \geq u) \).
Geometric stopping of a random walk

Random walk

- $X(t) = X_1 + \cdots + X_t$, \hspace{1cm} X(0) = 0

- $X_1, X_2, \ldots$ are i.i.d. r.v.'s, with $\Pr\{X_t = 1\} = p_1$, $\Pr\{X_t = 0\} = p_0$, $\Pr\{X_t = -1\} = p_{-1}$, $p_1 + p_0 + p_{-1} = 1$. 

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Valuing Equity-Linked Insurance Products
Geometric distribution

Let \( \tau \) be an independent r.v. with a geometric distribution (say parameter \( \pi \)), such that

\[
\Pr\{\tau = t\} = (1 - \pi)\pi^t, \quad t = 0, 1, 2, \ldots .
\]

Its probability generating function (pgf) is

\[
E[z^\tau] = \frac{1 - \pi}{1 - \pi z}.
\]
The pgf of $X(\tau)$

\[
\begin{align*}
E \left[ z^{X(\tau)} \right] &= E \left[ E \left[ z^{X(\tau)} | \tau \right] \right] \\
&= E \left[ E \left[ z^{X(1)} \right]^\tau \right] \\
&= E \left[ (p_1 z + p_0 + p_{-1} z^{-1})^\tau \right] \\
&= \frac{1 - \pi}{1 - \pi (p_1 z + p_0 + p_{-1} z^{-1})}.
\end{align*}
\]

This is a rational function of $z$, which we can expand by partial fractions. Let

\[0 < \alpha < 1 < \beta < \infty\]

denote the solutions of the quadratic equation

\[\pi p_1 z^2 - (1 - \pi p_0) z + \pi p_{-1} = 0.\]
The pgf of $X(\tau)$

$$E \left[ z^{X(\tau)} \right] = C \frac{\alpha}{z - \alpha} - C \frac{\beta}{z - \beta},$$

with

$$C = \frac{1 - \pi}{\pi} \frac{1}{p(\beta - \alpha)} = \frac{(1 - \alpha)(\beta - 1)}{\beta - \alpha}.$$ 

To identify the distribution of $X(\tau)$, we note that

$$E \left[ z^{X(\tau)} \right] = C \frac{\alpha/z}{1 - \alpha/z} + C \frac{1}{1 - z/\beta}.$$
The distribution of $X(\tau)$

$$\Pr\{X(\tau) = j\} = C\alpha^{-j}, \quad j = -1, -2, \ldots,$$

$$\Pr\{X(\tau) = j\} = C\beta^{-j}, \quad j = 0, 1, 2, \ldots.$$

Thus $X(\tau)$ has a two-sided geometric distribution.
The record highs and lows of the random walk

\[ M(t) = \max\{0, X(1), ..., X(t)\} \]

denote the running maximum and, similarly, \( m(t) \) the running minimum after \( t \) steps.
Joint distribution in the trinomial tree model

Suppose \( X(t) = X_1 + \ldots + X_t, \)

where \( X_i \) takes three values: \(-1, 0, 1\)

and \( P(X_i = 1) = P(X_i = -1) = p/2, \ P(X_i = 0) = q \) with \( p + q = 1. \)

We assume that \( X_1, X_2, \ldots \) is an i.i.d. sequence. Since the random walk \( X(t) \) is symmetric, the reflection principle is true (the proof is the same as that for simple symmetric random walk).

\[
P(\{X(t) = j, M(t) \geq k\}) = P(X(t) = 2k - j)
\]
Joint distribution in the trinomial tree model

Now we assume that the random walk is not symmetric,

In this case, we have

\[ P(X(t) = j, M(t) \geq k) = \left(\frac{p-1}{p_1}\right)^{k-j} P(X(t) = 2k - j). \]
Joint distribution in the trinomial tree model

Since $\tau$ is independent of $X(t)$, we have

$$P(X(\tau) = j, M(\tau) \geq k) = \left(\frac{p_{-1}}{p_1}\right)^{k-j}P(X(\tau) = 2k - j).$$

From this we can obtain the joint probability function of $X(\tau)$ and $M(\tau)$. 
Distributions of $M(\tau)$ and $m(\tau)$

Both $M(\tau)$ and $m(\tau)$ have geometric distributions:

\[
\Pr\{M(\tau) \geq k\} = \beta^{-k}, \quad k = 0, 1, 2, \ldots,
\]
\[
\Pr\{m(\tau) \leq k\} = \alpha^{-k}, \quad k = 0, -1, -2, \ldots,
\]

or,

\[
\Pr\{M(\tau) = k\} = (\beta - 1)\beta^{-k-1}, \quad k = 0, 1, 2, \ldots,
\]
\[
\Pr\{m(\tau) = k\} = (1 - \alpha)\alpha^{-k}, \quad k = 0, -1, -2, \ldots.
\]

We note that $M(\tau) \geq k$ is the event that $X(n)$ reaches level $k$ before or at time $\tau$. Similarly, $m(\tau) \leq k$ is the event that $X(n)$ falls to level $k$ before or at time $\tau$.
The distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same

We note that for each $t$, the r.v.’s $M(t)$ and $X(t) - m(t)$ have the same distribution. This follows from

$$
M(t) = \max\{0, X_1, X_1 + X_2, \ldots, X_1 + X_2 + \ldots + X_t\},
$$

$$
X(t) - m(t) = \max\{0, X_t, X_t + X_{t-1}, \ldots, X_t + X_{t-1} + \ldots + X_1\}.
$$

Hence, the distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same. Similarly, the distributions of $m(\tau)$ and $X(\tau) - M(\tau)$ are the same.
The r.v.’s $M(\tau)$ and $X(\tau) - M(\tau)$ are independent

Because the conditional distribution of $X(\tau) - M(\tau)$, given $M(\tau)$, is the conditional distribution of $X(\tau)$, given $X(n) \leq 0$ for $n = 1, \ldots, \tau$, and hence the same for all values of $M(\tau)$. To see this, consider the first time $t$ when $X(t) = M(\tau)$; thus $\tau \geq t$ and $X(n) - X(t) \leq 0$ for $n = t, \ldots, \tau$. Then observe that the conditional distribution of $\tau - t$ does not depend on $t$. 
The joint distribution of $X(\tau)$ and $M(\tau)$

\[
\Pr\{X(\tau) = j, M(\tau) = h\} \\
= \Pr\{M(\tau) - X(\tau) = h - j, M(\tau) = h\} \\
= \Pr\{M(\tau) - X(\tau) = h - j\} \Pr\{M(\tau) = h\} \\
= \Pr\{m(\tau) = -(h - j)\} \Pr\{M(\tau) = h\} \\
= (1 - \alpha)\alpha^{h-j}(\beta - 1)\beta^{-h-1}.
\]

Similarly, for $h = 0, -1, -2, \ldots$ and $j \geq h$, we have

\[
\Pr\{X(\tau) = j, m(\tau) = h\} \\
= \Pr\{X(\tau) - m(\tau) = j - h, m(\tau) = h\} \\
= \Pr\{X(\tau) - m(\tau) = j - h\} \Pr\{m(\tau) = h\} \\
= \Pr\{M(\tau) = j - h\} \Pr\{m(\tau) = h\} \\
= (\beta - 1)\beta^{-(j-h)-1}(1 - \alpha)\alpha^{-h}.
\]
For $k = 0, 1, 2, \ldots$ and $j \leq k$, we find that

$$\Pr\{X(\tau) = j, M(\tau) \geq k\} = C \alpha^{-j} \left(\frac{\alpha}{\beta}\right)^k.$$ 

Of course for $j \geq k$,

$$\Pr\{X(\tau) = j, M(\tau) \geq k\} = \Pr\{X(\tau) = j\} = C \beta^{-j}.$$ 

Similarly, for $k = 0, -1, -2, \ldots$ and $j \geq k$ one shows that

$$\Pr\{X(\tau) = j, m(\tau) \leq k\} = C \beta^{-j} \left(\frac{\beta}{\alpha}\right)^k.$$ 

Of course

$$\Pr\{X(\tau) = j, m(\tau) \leq k\} = \Pr\{X(\tau) = j\} = C \alpha^{-j}$$

for $j \leq k$. 
Remark

The proofs that $M(\tau)$ and $X(\tau) - M(\tau)$ are independent, and that $X(\tau) - m(\tau)$ has the same distribution as $m(\tau)$, are valid for a general random walk. It follows that

$$P_{X(\tau)}(z) = E[z^{X(\tau)}] = E[z^{M(\tau)+X(\tau)-M(\tau)}]$$

$$= E[z^{M(\tau)}] \times E[z^{X(\tau)-M(\tau)}]$$

$$= E[z^{M(\tau)}] \times E[z^{m(\tau)}] = P_{M(\tau)}(z)P_{m(\tau)}(z).$$

This formula is a version of the Wiener-Hopf factorization.
Remark

If $X_1$ takes integer values from $-n$ to $+m$, then

\[
P_{X(\tau)}(z) = \frac{1 - \pi}{1 - \pi P_{X_1}(z)} = \frac{1 - \pi}{1 - \pi \sum_{j=-n}^{m} p_j z^j}
= \frac{(1 - \pi) z^n}{g(z)},
\]

where $g(z) = \left(1 - \pi \sum_{j=-n}^{m} p_j z^j\right) z^n$ is a polynomial of degree $m + n$. 

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Valuing Equity-Linked Insurance Products
Remark

Because $1 > \pi = \pi \sum_{j=-n}^{m} p_{j}$, we have

$$|z^n| > |g(z) - z^n|,$$

for $|z| = 1$.

Then by Rouché’s Theorem, $g(z)$ has the same number of zeros inside the complex disk of radius 1 as the function $z^n$. Denote these $n$ zeros of $g(z)$ as $\alpha_1, ..., \alpha_n$. Denote the other zeros of $g(z)$, those with absolute value greater than 1, as $\beta_1, ..., \beta_m$. Then, the pgf $P_{X(\tau)}(z)$ is proportional to

$$\frac{z^n}{\left(\prod_{j=1}^{n}(z - \alpha_j)\right)\left(\prod_{j=1}^{m}(z - \beta_j)\right)} = \frac{1}{\left(\prod_{j=1}^{n}(1 - \alpha_j/z)\right)\left(\prod_{j=1}^{m}(z - \beta_j)\right)}.$$
Remark

Note that $P_{M(\tau)}(1) = 1$ and $P_{m(\tau)}(1) = 1$. Because $M(\tau) \geq 0$, $P_{M(\tau)}(z)$ exists for each $z$ with $|z| < 1$. Similarly, $P_{m(\tau)}(z)$ exists for each $z$ with $|z| > 1$. Therefore

$$P_{M(\tau)}(z) = \frac{\prod_{j=1}^{m}(\beta_j - 1)}{\prod_{j=1}^{m}(\beta_j - z)}, \quad P_{m(\tau)}(z) = \frac{\prod_{j=1}^{n}(1 - \alpha_j)}{\prod_{j=1}^{m}(1 - \alpha_j/z)}.$$
Remark

In the special case of a simple random walk, we have

\[ P_{X(\tau)}(z) = C \frac{(\beta - \alpha)z}{(z - \alpha)(\beta - z)} = \frac{\beta - 1}{\beta - z} \times \frac{(1 - \alpha)z}{z - \alpha}. \]
Applications to Valuing Equity-linked Insurance Products

- To value guarantees and options in Variable Annuities

- Variable Annuities
  = Investment Funds (Mutual Funds)
  + Rider(s) : Guaranteed Minimum Benefits
Some Examples

- Guaranteed Minimum Death Benefits
- Payoff: \( \max\{S(T_x), K\} \)

where \( S(t) \) denotes the price of a stock or stock index at time \( t \), \( T_x \) is the future life time of policyholder aged \( x \).

Note that \( \max\{S(T_x), K\} = (K - S(T_x))^+ + S(T_x) \).
Some Examples

- High water mark method or low water mark method
- Payoff: \[ \max\{M_S(T_x), K\} \]

where \( M_S(t) \) denotes the running maximum of the price of a stock or stock index up to time \( t \),

- Note that \( \max\{M_S(T_x), K\} = (M_S(T_x) - K)_+ + K \).
Mathematical problem

How to calculate

\[ E[e^{-\delta T_x} b(S(T_x), M_S(T_x))]. \text{ (or)} \]
\[ E[\nu^{(K_x+1)} b(S(K_x), M_S(K_x))]. \]

where \( b(., .) \) is a benefit function, (\( \nu \) is the discount factor per unit time, \( K_x \) denotes the curtate future life time r.v.).
Index process

- \( S(t) \) price of one unit of a fund at time \( t \)
- \( S(t) = S(0)e^{X(t)}, \quad t \geq 0 \) \( X(t) \) Brownian motion, a Lévy process (in the discrete time case, a random walk)
- \( T_x \) (or \( K_x \)) is assumed to be independent of \( S(t) \)
The reduced problem

- Idea: the pdf of $T_x$ can be approximated by

$$\sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t}, \quad t > 0$$

(the $A_i$’s can be negative, as long as the sum is pdf)
The reduced problem

\[
E[e^{-\delta T_x} b(S(T_x)))] = E[E[e^{-\delta T_x} b(S(T_x))|T_x]] \\
= \int_0^\infty e^{-\delta t} f_{T_x}(t) E[b(S(t))|T_x = t] dt \\
= \int_0^\infty e^{-\delta t} f_{T_x}(t) E[b(S(t))] dt \\
\simeq \int_0^\infty e^{-\delta t} \sum_{i=1}^n A_i \lambda_i e^{-\lambda_i t} E[b(S(t))] dt \\
= \sum_{i=1}^n A_i \int_0^\infty e^{-\delta t} \lambda_i e^{-\lambda_i t} E[b(S(t))] dt.
\]
The reduced problem

So, it suffices that we know how to calculate

\[
E[e^{-\delta \tau} b(S(\tau))] = \int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} E[b(S(t))] dt
\]
The reduced problem

\( \delta \) can be eliminated

\[
\int_0^\infty e^{-\delta t} \lambda e^{-\lambda t} E[b(S(t))] dt
= \frac{\lambda}{\lambda + \delta} \int_0^\infty (\lambda + \delta)e^{-(\lambda+\delta)t} E[b(S(t))] dt
\]

rule: do the calculation without discounting
but replace \( \lambda \) by \( \lambda + \delta \) multiply by \( \frac{\lambda}{\lambda + \delta} \)
Want to calculate

\[ E[b(S(\tau), M_S(\tau))] = E[b(S(0)e^{X(\tau)}, S(0)e^{M(\tau)})] \]

where \( M(t) \) is the running maximum of \( X(s) \) up to time \( t \) and \( \tau \) is an exponential random variable.

so we need

\[ f_{X(\tau), M(\tau)}(x, y) \] the joint pdf of \( (X(\tau), M(\tau)) \)
Examples

In the following we assume that $X(t)$ is a Brownian motion.

$$E[e^{-\delta \tau} b(S(\tau))] = E[b(S(\tau^*))]$$

$$= \kappa \int_{-\infty}^{0} b(S(0)e^x)e^{-\alpha x} \, dx + \kappa \int_{0}^{\infty} b(S(0)e^x)e^{-\beta x} \, dx.$$
Examples

- (1) \( b(s) = (K - s)_+ \), \( K < S(0) \) out-of-the-money put option
  
  \[
  E[e^{-\delta \tau} (K - S(\tau))_+] = \frac{\kappa K}{-\alpha(1 - \alpha)} \left[ \frac{K}{S(0)} \right]^{-\alpha}
  \]

- (2) \( b(s) = (s - K)_+ \), \( K > S(0) \) out-of-the-money call option
  
  \[
  E[e^{-\delta \tau} (S(\tau) - K)_+] = \frac{\kappa K}{\beta(\beta - 1)} \left[ \frac{S(0)}{K} \right]^\beta
  \]
Examples

Lookback options

\[
E[e^{-\delta \tau} \left[ \max_{0 \leq t \leq \tau} S(t) - K \right]_+] \\
= E[e^{-\delta \tau} [S(0)e^{M(\tau)} - K]_+] \\
= \frac{\lambda}{\lambda + \delta} \left[ \frac{S(0)}{K} \right]^{\beta}.
\]

out-of-the-money
Examples

Barrier options

Let $L$ denote the barrier and $\ell = \ln[L/S(0)]$. Consider the up-and-in option ($S(0) < L$), the value is given by

$$\Pr(M(\tau) \geq \ell) \mathcal{E}_b(L) = \left[ \frac{S(0)}{L} \right]^\beta \mathcal{E}_b(L),$$

where

$$\mathcal{E}_b(s) = \mathbb{E}[b(S(\tau))|S(0) = s]$$
THANK YOU