Arbitrage-Free Pricing in Nonlinear Market Models (or, Cooking with Adjustments)

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Outline of the talk:

- Motivation
- Trading adjustments: a unified approach
- No arbitrage for trading desk: elimination of bad market models
- Regular market models
- Fair prices and other prices
- Valuation in regular models through BSDEs

Motivations

- Major changes in the operations of financial markets:
  - Defaultability of the counterparties became one of the key problems of financial management.
  - The classical 'discounting of future cash flows using the risk-free rate' is no longer accepted; differential funding costs.
- In the presence of funding costs, counterparty credit risk, and other adjustments, the classical arbitrage pricing theory no longer applies.
- Non-uniqueness and non-linearity of one-sided prices: asymmetric pricing rules may fail to yield a mutually acceptable price for counterparties.
- Aggregation of pricing rules: non-linear effects of netting of exposures and/or margin accounts between two (or more) counterparties.
- Arbitrage opportunities: an OTC contract may introduce arbitrage opportunities to an arbitrage-free model.
- ...

Main Goal

To develop a comprehensive no-arbitrage framework for valuation of contracts in the presence of salient features of real-world trades such as trading constraints, differential funding costs, collateralization, counterparty credit risk and capital requirements.

T.R. Bielecki, Igor Cialenco, and M. Rutkowski, *Arbitrage-Free Pricing of Derivatives in Nonlinear Market Models*, PROBABILITY, UNCERTAINTY AND QUANTITATIVE RISK, 2018 vol 3, no. 2

Related works

- N. El Karoui and M. C. Quenez (1996). Non-linear pricing theory and BSDEs. In Financial Mathematics, Lecture Notes in Mathematics 1656, pages 191–246. eds. B. Biais et al., Springer, Berlin.
- Piterbarg, V. (2010) Funding beyond discounting: collateral agreements and derivatives pricing. *Risk Magazine*, February, 97–102.
- Piterbarg, V. (2012) Cooking with collateral, *Risk Magazine*, August (2012), 58-63.
- T. R. Bielecki and M. Rutkowski (2015) Valuation and hedging of contracts with funding costs and collateralization, SIAM Fin Math, 6:594–655.
- T. Nie and M. Rutkowski (2015) Fair bilateral prices in Bergman's model. IJTAF, 18:1550048.

(counter)Example 1

- Consider the Black-Scholes model (B, S^1) with zero interest rate in which, borrowing of cash is prohibited, and hedger's initial endowment is 0.

- Obviously, the model is arbitrage-free in the classical sense.

- The hedger can replicate (without borrowing cash) a short position in a put option maturing at T written on S^1. Denote by $P_t(K)$ the fair (Black-Scholes) price of this put option.

- The hedger can replicate (without borrowing cash) the contract $\mathcal{C}$ that pays

  $P_U(K)$ at time U, for some fixed $U < T$ and
  $-P_T(K) = -(K - S^1_T)^+$, at time T.

  The price of this contract is zero.

- Extend the market model by introducing the second risky asset

$$S^2_t = \mathbb{1}_{[0,T]}(t) + 2K(t-U)(P_U(K))^{-1}\mathbb{1}_{[U,T]}(t).$$

- Easy to check that (B, S^1, S^2) is still arbitrage-free if the borrowing of cash is not allowed; the only way of investing in the second asset is to sell short the first one.
- The price of the contract $\mathcal{C}$ based on the concept of replication is still equal to 0.
- However, the hedger who enters into the contract $\mathcal{C}$ at time 0 at zero price has now an arbitrage opportunity. She may now use the cash amount $P_U(K)$ received at time U to buy the asset S^2, that yields the amount $2K(T-U)$ at time T, which strictly dominates the hedger's liability $P_T(K)$, assuming that $T-U \geq 0.5$.
- Similar example can be provided with no trading constrains.

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What did go wrong and how to avoid this?

$(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered probability space, with a fixed time horizon $T > 0$. As usual, all processes are assumed to be $\mathbb{G}$-adapted, and all semimartingales, to be càdlàg.

Risky assets. We denote by $\mathcal{S} = (S^1, \dots, S^d)$ the **ex-dividend prices** of d risky assets with the corresponding **cumulative dividend streams** $\mathcal{D} = (D^1, \dots, D^d)$.

Examples of 'Risky Assets': stocks, sovereign or corporate bonds, stock options, interest rate swaps, currency options, CDS, CDO ...

Funding accounts. $B^{i,l}$ will stand for the **lending funding account** associated with the i^{th} risky asset. Respectively, $B^{i,b}$ will be used to denote the **borrowing funding account**.

Cash accounts. The **riskless lending/borrowing cash account** $B^{0,l/b}$ is used for **unsecured lending/borrowing** of cash.

If $B^{i,l} = B^{i,b}$, $i = 0, 1, \dots, d$, then we simply write B^i.

We consider financial contracts between two parties - the **hedger** and the **counterparty**. All the cash flows will be viewed from the *perspective of the hedger*.

Definition

A **bilateral financial contract** is a pair $\mathcal{C} = (A, \mathcal{X})$, with A representing the **cumulative (promised) cash flows** from time 0 till maturity T, and $\mathcal{X}$ the **trading adjustments** associated to this contract.

The process A **models all contractual cash flows** of the contract, either paid out from the wealth or added to the hedger's wealth.

For example, if the contract stipulates that the hedger will 'receive' the cash flows $a_1, a_2, \dots, a_m$ at times $t_1, t_2, \dots, t_m \in (0, T]$, then A is given by

$$A_t = \sum_{l=1}^{m} \mathbb{1}_{[t_l, \infty)}(t) a_l.$$

The price of the contract $\mathcal{C}$ exchanged at its initiation is not included in A, and it is to be determined.

If the contract is initiated at time t, then the remaining cumulative cash flow is $A_u^t := A_u - A_t$.

Example: Hedger sells at time t the European call S^i, strike K, maturity T. Then $m = 1$, $t_1 = T$, $a_1 = -(S_T^i - K)^+$, and $A_u^t = -(S_T^i - K)^+ \mathbb{1}_{[T,\infty)}(u)$ for $u \in [t, T]$.

Process $\mathcal{X}$ represents the additional features or clauses of a contract, and is formally given by

$$\mathcal{X} = (X^1, \ldots, X^n; \alpha^1, \ldots, \alpha^n; \beta^1, \ldots, \beta^n).$$

- X^k is an *adjustment (stochastic) process*. It may depend on A, the hedger's trading strategy, the price of $\mathcal{C}$ (to be determined), etc and vise versa; hence, possible feedback effect.
- $\alpha^k X^k$ is an additional cash flow for the hedger either stipulated in the clauses of the contract or imposed by a third party (for instance, the regulator),
- *remuneration process* β^k used to determine the net interest payments associated with X^k.
- With proper choice of $\mathcal{X}$, we reconcile: short-selling of risky assets, repo market trading, collateralization, counterparty credit risk, regulatory capital, etc.

Example of Adjustment process: Collateralization

The hedger either receives or pledges *collateral* with value denoted by C, and represents the value of the *margin account*. We take

$$C_t = X_t^1 + X_t^2 := C_t^+ - C_t^- = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}}.$$

- *Rehypothecated* collateral (the amount received can be used by the hedger for trading). We set $\alpha_t^1 = \alpha_t^2 = 1$, and take β^1, β^2 to be the value of banking account with the interest rate paid/recieved on collateral. Consequently, the 'cash adjustment' to the wealth is

$$C_t - \int_0^t C_u^+ / \beta_u^1 d\beta_u^1 + \int_0^t C_u^- / \beta_u^2 d\beta_u^2.$$

- For *segregated* collateral: $\alpha_t^1 = 0, \alpha_t^2 = 1$, and the cash adjustment is

$$-C_t^- - \int_0^t C_u^+ / \beta_u^1 d\beta_u^1 + \int_0^t C_u^- / \beta_u^2 d\beta_u^2.$$

Composite Contract with Adjustments: Counterparty Risk I

- We denote by τ^h and τ^c the default times of the hedger and his counterparty, respectively.
- $\tau := \tau^h \wedge \tau^c$ is the moment of the first default.
- We define the random variable Υ as

$$\Upsilon = Q_\tau + \Delta A_\tau - C_\tau, \tag{5.1}$$

  where Q is the Credit Support Annex (CSA) closeout valuation process of the contract A, $\Delta A_\tau = A_\tau - A_{\tau-}$ is the jump of A at τ corresponding to a (possibly null) promised bullet dividend at τ, and C_τ is the value of the collateral process C at time τ.
- In the financial interpretation, Υ^+ is the amount the counterparty owes to the hedger at time τ, whereas Υ^- is the amount the hedger owes to the counterparty at time τ.

Composite Contract with Adjustments: Counterparty Risk II

Definition

The *CSA closeout payoff* $\mathfrak{K}$ is defined as

$$\mathfrak{K} := C_\tau + 1_{\{\tau^c<\tau^h\}}(R_c\Upsilon^+ - \Upsilon^-) + 1_{\{\tau^h<\tau^c\}}(\Upsilon^+ - R_h\Upsilon^-) + 1_{\{\tau^h=\tau^c\}}(R_c\Upsilon^+ - R_h\Upsilon^-). \tag{5.2}$$

The *counterparty risky cumulative cash flows* process $A^\sharp$ is given by

$$A^\sharp_t = 1_{\{t<\tau\}}A_t + 1_{\{t\geq\tau\}}(A_{\tau-} + \mathfrak{K}), \quad t \in [0,T]. \tag{5.3}$$

Composite Contract with Adjustments: Counterparty Risk III

Definition

By the *CCR processes*, we mean the processes CL, CG and RP where the *credit loss* CL equals

$$CL_t = -1_{\{t \geq \tau\}} 1_{\{\tau = \tau^c\}} (1 - R_c) \Upsilon^+,$$

the *credit gain* CG equals

$$CG_t = 1_{\{t \geq \tau\}} 1_{\{\tau = \tau^h\}} (1 - R_h) \Upsilon^-,$$

and the *replacement process* is given by

$$CR_t = 1_{\{t \geq \tau\}} (A_\tau - A_t + Q_\tau).$$

The *CCR cash flow* is given by $A^{\mathrm{CCR}} = CL + CG + CR$.

Composite Contract with Adjustments: Counterparty Risk IV

- Let $\mathcal{X} = (X^1, X^2) = (C^+, -C^-)$.

Proposition

The counterparty risky and collateralized contract $(A^\sharp, \mathcal{X})$ *admits the following formal decompositions*

$$(A^\sharp, \mathcal{X}) = (A, \mathcal{X}) + (A^{\mathrm{CCR}}, 0)$$

and

$$(A^\sharp, \mathcal{X}) = (A, 0) + (A^{\mathrm{CCR}}, \mathcal{X}).$$

Composite Contract with Adjustments: Counterparty Risk V

Remark

We may interpret the counterparty risky contract as the clean contract A, which is complemented by the following adjustment processes:

- *the collateral adjustment processes:* $X^1 = C^+$ *and* $X^2 = -C^-$,
- *the CCR adjustment processes:* $X^3 = CL$, $X^4 = CG$ *and* $X^5 = CR$.

Specifically, we have that

$$A_t^\sharp = A_t + \sum_{k=3}^{5} X_t^k, \quad t \in [0,T].$$

Portfolios

Definition

An **portfolio** on the time interval $[t, T]$ is an $\mathbb{R}^{3d+2}$-valued, $\mathcal{G}$-adapted process

$$\varphi^t = \Big(\underbrace{\xi^1, \dots, \xi^d}_{\text{risky assets}}; \underbrace{\psi^{0,l}, \psi^{0,b}}_{\text{cash account}}, \underbrace{\psi^{1,l}, \psi^{1,b}, \dots, \psi^{d,l}, \psi^{d,b}}_{\text{funding accounts}} \Big),$$

where the components represent the positions in risky assets (S^i, D^i), $i = 1, \dots, d$, cash accounts $B^{0,l}$, $B^{0,b}$, and funding accounts $B^{i,l}, B^{i,b}$, $i = 1, \dots, d$ for risky assets.

Throughout we assume that $\psi_t^{i,l} \geq 0$, $\psi_t^{i,b} \leq 0$ and $\psi_t^{i,l} \psi_t^{i,b} = 0$, for all $i = 0, 1, \dots, d$ and $t \in [0, T]$.

Portfolio may have some explicit constrains.

- x_t represents the hedger's *endowment* of at time $t \in [0, T)$
- p_t stands for the *price* at time t of $\mathcal{C}^t = (A^t, \mathcal{X}^t)$

Definition

The **portfolio value** corresponding to the trading strategy φ is defined as

$$V_u^p(x_t, p_t, \varphi^t, \mathcal{C}^t) := \sum_{i=1}^{d} \xi_u^i S_u^i + \sum_{j=0}^{d} \left(\psi_u^{j,l} B_u^{j,l} + \psi_u^{j,b} B_u^{j,b} \right).$$

Correspondingly, the **adjusted gains process** is given by

$$G_u(x_t, p_t, \varphi^t, \mathcal{C}^t) := \sum_{i=1}^{d} \int_t^u \xi_v^i \left(dS_v^i + dD_v^i \right) + \sum_{j=0}^{d} \int_t^u \left(\psi_v^{j,l} \, dB_v^{j,l} + \psi_v^{j,b} \, dB_v^{j,b} \right) + \sum_{k=1}^{n} \alpha_u^k X_u^k - \sum_{k=1}^{n} \int_t^u X_v^k (\beta_v^k)^{-1} \, d\beta_v^k + A_u^t.$$

Definition

$(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is a **self-financing trading strategy on** $[t, T]$ if

$$V_u^p(x_t, p_t, \varphi^t, \mathcal{C}^t) = x_t + p_t + G_u(x_t, p_t, \varphi^t, \mathcal{C}^t), \quad u \in [t, T].$$

Note that

$$V_0^p(x, p, \varphi, \mathcal{C}) = \sum_{i=1}^{d} \xi_0^i S_0^i + \sum_{j=0}^{d} \left(\psi_0^{j,l} B_0^{j,l} + \psi_0^{j,b} B_0^{j,b} \right) = x + p + \sum_{k=1}^{n} \alpha_0^k X_0^k.$$

In contrast to classical theory, in nonlinear market models, the initial endowment x, the initial price p and the adjustment cash flows of a contract may all affect the dynamics of the gains process.

Definition

The *market model* is the quintuplet $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathscr{C}, \Phi(\mathscr{C}))$ where $\Phi(\mathscr{C})$ stands for the set of all self-financing trading strategies associated with the class $\mathscr{C}$ of contracts.

Discount Factor: for any $x \in \mathbb{R}$, we denote by

$$\mathcal{B}_t(x) := \mathbb{1}_{\{x \geq 0\}} B_t^{0,l} + \mathbb{1}_{\{x < 0\}} B_t^{0,b}.$$

The symbol $\sim$ will be used for discounting, e.g. $\widetilde{V} = V/\mathcal{B}$.

Assumption:

(i) for any initial endowment $x \in \mathbb{R}$ of the hedger, the *null contract* $\mathcal{N} = (0,0)$ belongs to $\mathscr{C}$.

(ii) for any $x \in \mathbb{R}$, the trading strategy $(x, 0, \widehat{\varphi}, \mathcal{N})$, where all components of $\widehat{\varphi}$ vanish except for either $\psi^{0,l}$, if $x \geq 0$, or $\psi^{0,b}$, if $x < 0$, is 'feasible' and

$$V_t^p(x, 0, \widehat{\varphi}, \mathcal{N}) = V_t(x, 0, \widehat{\varphi}, \mathcal{N}) = x\mathcal{B}_t(x), \quad t \in [0, T].$$

This assumption is needed: to insure that the zero contract has zero fair price; serves as a benchmark to assess the gains/losses.

Definition

An *arbitrage opportunity with respect to the null contract* in market model $\mathcal{M}$, or a **primary arbitrage opportunity**, for the hedger with an initial endowment x is a strategy $(x, 0, \varphi, \mathcal{N})$ such that

$$\mathbb{P}(\widetilde{V}_T(x,0,\varphi,\mathcal{N}) \geq x) = 1, \quad \mathbb{P}(\widetilde{V}_T(x,0,\varphi,\mathcal{N}) > x) > 0.$$

If no primary arbitrage opportunity exists in $\mathcal{M}$, then $\mathcal{M}$ is arbitrage free with respect to the null contract.

- This no-arbitrage property is not strong enough for developing reasonable pricing frameworks.

Definition

For a contract $\mathcal{C} = (A, \mathcal{X})$ and an initial endowment x, the *combined wealth* is defined as

$$V^{\mathrm{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) := V(x_1, 0, \varphi, A, \mathcal{X}) + V(x_2, 0, \bar{\varphi}, -A, \mathcal{Y}), \tag{7.1}$$

where $x_1 + x_2 = x$.

Definition

A pair $(x_1, \varphi; x_2, \bar{\varphi})$ is an **arbitrage opportunity for the trading desk** with respect to a contract $(A, \mathcal{X})$ if

$$\mathbb{P}(\widetilde{V}_T^{\mathrm{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) \geq x) = 1, \quad \mathbb{P}(\widetilde{V}_T^{\mathrm{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) > x)$$

Market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathscr{C})$ has the *no-arbitrage property for the trading desk* if there are no arbitrage opportunities for the trading desk with respect to any contract $\mathcal{C}$ from $\mathscr{C}$.

- No-arbitrage for trading desk is stronger than no-arbitrage with respect to the null contract.

- These definitions of arbitrage are meant to eliminate bad market models.

- The market model $(B, S^1, S^2, \mathcal{C})$ from (counter)Example 1 is arbitrage free with respect to the null contract, and admits arbitrage opportunity for the trading desk.

- One can derive interesting dynamics for $\widetilde{V}^{\mathrm{com}}$, and provide sufficient conditions for the trading-desk no arbitrage, in particular expressed in terms of the existence of a (super-)martingale measure.

Part II: Fair and Other Pricing

$\mathcal{C}^t$ will denote a class of contracts, and $\Psi^{t,x_t}(\mathcal{C}^t)$ the set of all 'feasible' trading strategies.

Definition

A trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t,x_t}(\mathscr{C})$ is a **hedger's pricing arbitrage opportunity on** $[t,T]$ associated with a contract $\mathcal{C}^t$ traded at p_t at time t if

$$\underbrace{\underbrace{\mathbb{P}\Big(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq x_t\Big) = 1}_{\text{superhedging}} \quad \text{and} \quad \mathbb{P}\Big(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) > x_t\Big) > 0}_{\text{strict superhedging}}.$$

$(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t,x_t}(\mathscr{C})$ is not a hedger's pricing arbitrage opportunity if either

$$\mathbb{P}\Big(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) = x_t\Big) = 1 \quad \text{or} \quad \underbrace{\mathbb{P}\Big(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) < x_t\Big) > 0}_{\text{hedger's loss condition}}.$$

Definition

We say that $p_t^f = p_t^f(x_t, \mathcal{C}^t)$ is a **fair hedger's price** at time t for $\mathcal{C}^t$ if there is no hedger's pricing arbitrage opportunity $(x_t, p_t^f, \varphi^t, \mathcal{C}^t)$.
p_t^f such that the loss condition holds for every trading strategy $(x_t, p_t^f, \varphi^t, \mathcal{C}^t) \in \Psi^{t,x_t}(\mathscr{C})$ is called a **loss-generating cost.**

$$\mathcal{H}_t^f(x_t, \mathcal{C}^t) = \left\{ p_t^f \in \mathcal{G}_t \mid p_t^f \text{ is a fair hedger's price for } \mathcal{C}^t \right\}$$

$$\overline{p}_t^f(x_t, \mathcal{C}^t) = \operatorname{ess\,sup} \mathcal{H}_t^f(x_t, \mathcal{C}^t)$$

$$\mathcal{H}_t^l(x_t, \mathcal{C}^t) = \left\{ p_t^l \in \mathcal{G}_t \mid p_t^l \text{ is a loss-generating cost for } \mathcal{C}^t \right\}$$

$$\overline{p}_t^l(x_t, \mathcal{C}^t) = \operatorname{ess\,sup} \mathcal{H}_t^l(x_t, \mathcal{C}^t)$$

$$\mathcal{H}_t^s(x_t, \mathcal{C}^t) = \left\{ p_t^s \in \mathcal{G}_t \mid p_t^s \text{ is a superhedging cost for } \mathcal{C}^t \right\}$$

$$\underline{p}_t^s(x_t, \mathcal{C}^t) = \operatorname{ess\,inf} \mathcal{H}_t^s(x_t, \mathcal{C}^t)$$

$$\mathcal{H}_t^a(x_t, \mathcal{C}^t) = \left\{ p_t^a \in \mathcal{G}_t \mid p_t^a \text{ is a strict superhedging cost for } \mathcal{C}^t \right\}$$

$$\underline{p}_t^a(x_t, \mathcal{C}^t) = \operatorname{ess\,inf} \mathcal{H}_t^a(x_t, \mathcal{C}^t)$$

Clearly,

$$\overline{p}^l(x,\mathcal{C}) \le \overline{p}^f(x,\mathcal{C}), \qquad \underline{p}^s(x,\mathcal{C}) \le \underline{p}^a(x,\mathcal{C}),$$

Assumption M:

For every $\mathcal{C} \in \mathscr{C}$, $t \in [0,T)$, $x_t, p_t, q_t \in \mathcal{G}_t$, and every trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t,x_t}(\mathscr{C})$, if

$$q_t \ge p_t, \quad \text{on} \quad D \in \mathcal{G}_t, \quad \mathbb{P}(D) > 0,$$

then there exists a trading strategy $(x_t, q_t, \psi^t, \mathcal{C}^t) \in \Psi^{t,x_t}(\mathscr{C})$ such that

$$V_T(x_t, q_t, \psi^t, \mathcal{C}^t) \ge V_T(x_t, p_t, \varphi^t, \mathcal{C}^t), \quad \text{on} \quad D.$$

- This non-trivial assumption usually is not postulated in the existing literature.
- If the wealth process is governed by some simple SDE dynamics, then this assumption can be deduced from a suitable comparison theorem for ordinary integral equations.

Theorem

Under Assumption M, we have that

$$\overline{p}^l_t(x_t, \mathcal{C}^t) = \underline{p}^s_t(x_t, \mathcal{C}^t) \leq \overline{p}^f_t(x_t, \mathcal{C}^t) = \underline{p}^a_t(x_t, \mathcal{C}^t).$$

If in addition $V_T(x_t, q_t, \psi^t, \mathcal{C}^t) > V_T(x_t, p_t, \varphi^t, \mathcal{C}^t)$ *on* $D' \subset D$*, then*

$$\overline{p}^l_t(x_t, \mathcal{C}^t) = \underline{p}^s_t(x_t, \mathcal{C}^t) = \overline{p}^f_t(x_t, \mathcal{C}^t) = \underline{p}^a_t(x_t, \mathcal{C}^t).$$

Definition

A trading strategy $(x_t, p_t^r, \varphi^t, \mathcal{C}^t)$ *replicates* the contract $\mathcal{C}^t$ on $[t, T]$, if

$$\widetilde{V}_T(x_t, p_t^r, \varphi^t, \mathcal{C}^t) = x_t,$$

and the real number $p_t^r = p_t^r(x_t, \mathcal{C}^t)$ is called the *hedger's replication cost* for $\mathcal{C}^t$ at time t.

- The replication cost p_t^r is not unique, in general.
- If Assumption M holds, then any $p \in (\overline{p}_t^l, \underline{p}_t^a)$ is a replication cost and a fair price for $\mathcal{C}^t$.

- In a nonlinear market model, that meets the no-arbitrage property with respect to the null contract (or even the no-arbitrage property for the trading desk), a replication cost may fail to be a fair hedger's price.

- Thus $\Rightarrow$ **regular models**, under which the **cost of replication is never higher than the minimal cost of superhedging and, the cost of replication is a fair hedger's price**.

Definition

We say that the market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathscr{C}, \Psi^{t,x_t}(\mathscr{C}))$ is **regular on $[t,T]$ with respect to** $\mathscr{C}$ if Assumption M is met and for every replicable contract $\mathcal{C} \in \mathscr{C}$ and for any replicating strategy $(x, p_0^r(x), \varphi, \mathcal{X})$ the following properties hold:

(i) if $p_t \in \mathcal{G}_t$ and there exists $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ such that

$$\mathbb{P}\big(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq x_t\big) = 1,$$

then $p_t \geq p_t^r(x_t, \mathcal{C}^t)$;

(ii) if $p_t \in \mathcal{G}_t$ and there exists $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ such that for some $D \in \mathcal{G}_t$

$$\mathbb{P}\big(\mathbb{1}_D \widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq \mathbb{1}_D x_t\big) = 1, \quad \mathbb{P}\big(\mathbb{1}_D \widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) > \mathbb{1}_D x_t\big) > 0,$$

then $\mathbb{P}(\mathbb{1}_D\, p_t > \mathbb{1}_D p_t^r(x_t, \mathcal{C}^t)) > 0$.

(counter)Example 2

Consider market model $\mathcal{M} = (B^l, B^b, S^1, S^2, \mathscr{C})$, where

- B^l, B^b borrowing and lending accounts with constant interest rates, $0 = r^l < r^b$, and $r^b > \ln 3$;
- S^1 driven by Black-Scholes dynamics;
- $S^2 = \mathbb{1}_{[0,T]}(t) + (K(t-U))(P_U(K))^{-1}\mathbb{1}_{\{2P_U(K)>K\}}\mathbb{1}_{[U,T]}(t)$, with $T - U = 1$;
- $\mathscr{C}$ the class of long and short positions in all European put options written on the stock S^1 and maturing at T.

Proposition

The market model $\mathcal{M} = (B^l, B^b, S^1, S^2, \mathscr{C}, \Psi^{0,0}(\mathscr{C}))$ has the following properties: (a) $\mathcal{M}$ has the no-arbitrage property with respect to the null contract, and no-arbitrage property for the trading desk; (b) $\mathcal{M}$ is nonregular; (c) the extended market model $\widetilde{\mathcal{M}} = (B^l, B^b, S^1, S^2, S^3 = P(K), \mathscr{C}, \Psi^{0,0}(\mathscr{C}))$ does not have the no-arbitrage property with respect to the null contract; (d) replication cost for the put option is not unique.

Theorem

Let market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathscr{C}, \Psi^{t,x_t}(\mathscr{C}))$ *be regular on* $[t,T]$ *with respect to* $\mathscr{C}$*. Then for every contract* $\mathcal{C} \in \mathscr{C}$ *that can be replicated on* $[t,T]$ *we have:*

- *the replication cost* p_t^r *is unique.*
- $p_t^r(x_t, \mathcal{C}) = \overline{p}_t^f(x_t, \mathcal{C}) = \overline{p}_t^l(x_t, \mathcal{C}) = \underline{p}_t^s(x_t, \mathcal{C}) = \underline{p}_t^a(x_t, \mathcal{C})$.

- Even though, in regular model p_t^r is unique, one has to specify the endowment x_t.
- If the hedger starts at time $t = 0$ with the endowment x, what is a suitable choice for x_t?
- Different choices of x_t will yield different price processes:
  - A natural choice is $x_t = x_t(x) := x\mathcal{B}_t(x)$, i.e. the hedger has not been dynamically hedging the contract between time 0 and time t. Then $p_t^e := p_t^r(x_t(x), \mathcal{C}^t)$ is called the **hedger's ex-dividend price** at time t of the contract $\mathcal{C}^t$.

Instrumental prices at t

- Take $x_t = V_t(x, p_0^r(x,\mathcal{C}), \widehat{\varphi}, \mathcal{C})$, where $\widehat{\varphi}$ is a replicating strategy for $\mathcal{C}$ on $[0,T]$. Let $(-A^t, \mathcal{Y}^t)$ be the offsetting contract of $(A^t, \mathcal{X}^t)$. Then, p_t^o is the **offsetting price** if

$$\widetilde{V}_T(x_t, p_t^o, \varphi^t, 0, \mathcal{X}^t + \mathcal{Y}^t) = x.$$

- One can also define **exit price**: The *exit price* for the contract $\mathcal{C}$ entered into at time 0 by the hedger with the initial endowment x is given by the equality

$$p_t^m(x,\mathcal{C}) := -p_t^g(x,\mathcal{C})$$

  for every $t \in [0,T]$,

where

$$p_t^g(x,\mathcal{C}) := V_t(x, p_0^r(x,\mathcal{C}), \varphi, \mathcal{C}) - x\mathcal{B}_t(x) \tag{8.1}$$

is called the hedger's *gained value* associated with the replicating strategy $(x, p_0^r, \varphi, \mathcal{C})$.

Assume that $x \geq 0$, and $B^{i,l} = B^{i,b} = B^i$ for $i = 1, \dots, d$. Define

$$\widetilde{S}_t^{i,cld}(x) := (\mathcal{B}_t(x))^{-1} S_t^i + \int_0^t (\mathcal{B}_u(x))^{-1} \, dD_u^i.$$

The theorem below gives the BSDE for the gained value.

Theorem

Assume that a trading strategy $(x, p_0^r, \widehat{\varphi}, \mathcal{C})$ replicates $\mathcal{C}$. Then $\widehat{Y} := (B^{0,l})^{-1} V(x, p_0^r, \widehat{\varphi}, \mathcal{C})$ and $\widehat{Z}^i := \widetilde{B}^{i,l} \widehat{\xi}^i$ satisfy the BSDE

$$\begin{aligned} d\widehat{Y}_t &= \sum_{i=1}^d \widehat{Z}_t^i \, d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t B_t^{0,l} + \sum_{k=1}^n \alpha_t^k X_t^k \right)^- dB_t^{0,b,l} \\ &\quad + (B_t^{0,l})^{-1} \, dA_t - \sum_{k=1}^n \widehat{X}_t^k \, d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k \, d(B_t^{0,l})^{-1} \end{aligned}$$

with the terminal condition $\widehat{Y}_T = x$.

Similar BSDEs can be derived for ex-dividend price, offsetting price, exit price ...

- Recall that the counterparty risky contract $(A^\sharp, \mathcal{X})$ admits the following decomposition

$$(A^\sharp, \mathcal{X}) = (A, \mathcal{X}) + (A^{\mathrm{CCR}}, 0), \tag{9.1}$$

- **Question:** can we disentangle the counterparty risk-free valuation of a credit risky contract from the CRR valuation?
- We have the following BSDE for the full contract $(A^\sharp, \mathcal{X})$

$$d\widehat{Y}_t = \sum_{i=1}^{d} \widehat{Z}_t^i \, d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t B_t^{0,l} + \sum_{k=1}^{n} \alpha_t^k X_t^k \right)^{-} dB_t^{0,b,l} + (B_t^{0,l})^{-1} \, \mathbf{dA_t^\sharp} - \sum_{k=1}^{n} \widehat{X}_t^k \, d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^{n} (1 - \alpha_t^k) X_t^k \, d(B_t^{0,l})^{-1} \tag{9.2}$$

with the terminal condition $\widehat{Y}_{\widehat{\tau}} = x$.

- Let $x = x_1 + x_2$ be an arbitrary split of the hedger's endowment.
- Then we obtain the following BSDE corresponding to the counterparty risk-free contract $(A, \mathcal{X})$

$$d\widehat{Y}_t^1 = \sum_{i=1}^{d} \widehat{Z}_t^{1,i}\, d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1}\left(\widehat{Y}_t^1 B_t^{0,l} + \sum_{k=1}^{n} \alpha_t^k X_t^k\right)^{-} dB_t^{0,b,l}$$
$$+ (B_t^{0,l})^{-1}\, \mathbf{dA_t} - \sum_{k=1}^{n} \widehat{X}_t^k\, d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^{n} (1-\alpha_t^k) X_t^k\, d(B_t^{0,l})^{-1} \tag{9.3}$$

with $\widehat{Y}_T^1 = x_1$.

- The BSDE associated with the CRR component $(A^{\mathrm{CCR}}, 0)$ reads

$$d\widehat{Y}_t^2 = \sum_{i=1}^{d} \widehat{Z}_t^{2,i}\, d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1}\left(\widehat{Y}_t^2 B_t^{0,l}\right)^{-} dB_t^{0,b,l} + (B_t^{0,l})^{-1}\, \mathbf{dA_t^{\mathrm{CRR}}} \tag{9.4}$$

with $\widehat{Y}_{\widehat{\tau}}^2 = x_2$.

- The question formulated above can now be restated as follows: under which conditions the equality $\widehat{Y}_0 = \widehat{Y}_0^1 + \widehat{Y}_0^2$ holds for solutions to BSDEs (9.2), (9.3) and (9.4), so that the three replication costs satisfy the following equality

$$p_0^r(x, A^\sharp, \mathcal{X}) = p_0^r(x_1, A, \mathcal{X}) + p_0^r(x_2, A^{\mathrm{CCR}}, 0),$$

which formally corresponds to decomposition (9.1) of the full contract and the split $x = x_1 + x_2$ of the hedger's initial endowment?

- Study of this question in the general non-linear framework requires further work.

Thank You !

The end of the talk ...
but not of the story