Linear Quadratic Mean Field Games: asymptotic solvability and the fixed point approach

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Stochastic Control, Computational Methods, and Applications
IMA, University of Minnesota, May 2018
Outline of talk

- The $N$ player LQ Nash game
- Mean field game (MFG) theory provides a very compact specification and solution for non-cooperative decision making with a large number of players
  - The direct approach and the asymptotic solvability (AS) problem with time horizon $[0, T]$.
  - The fixed point approach
- Relation of the two approaches
- Long time behavior in the AS problem
The fundamental diagram of MFG theory

Problem $P_0$: $N$ player Nash game
states: $(x_1, x_2, \cdots, x_N)$
strategies: $(u_1, u_2, \cdots, u_N)$
costs: $(J_1, J_2, \cdots, J_N)$

A large-scale coupled equation system
Example: $N$ coupled dynamic programming equations

route 1

route 2 $N \to \infty$

Problem $P_\infty$: Optimal control of a single player
state: $x_i$ control: $u_i$
mean field $\mu_i$ is fixed and not controlled by $u_i$

MFG equation system:
1 equation of optimal control;
1 equation of mean field dynamics (for $\mu_i$)
Example: HJB PDE; FPK PDE (or McKean-Vlasov SDE)

▲ blue: direct approach  red: fixed point approach (Huang, Caines, Malhamé, 03, 07), (Huang, Malhamé, Caines, 06), (Lasry and Lions, 06); see overview of the two approaches in (Caines, Huang, and Malhamé, 17)

▲ Basic questions. Relation of the two approaches? Their respective domains of applicability?
The diagram links finite population games to an infinite population problem

Also, the blue route provides a possible way to derive the (infinite population) mean field game via solving finite population models

Question: is the blue route always feasible? We will clarify.

There is a large literature on the relation between games of finite and infinite populations (traditionally for static models).

Other references related to the direct approach:

- Huang (2003, thesis ch. 6) – mean field social optimization of $N$ players; solve a large scale algebraic Riccati equation
- Papavassilopoulos (2014) – for LQ mean field game, analyze existence for weakly coupled algebraic Riccati equations using the implicit function theorem
- Herty, Pareschi and Steffensen (2015) – $N$ agent mean field optimal control via a large Riccati equation
- Priuli (2015) – Convergence of HJB-FPKs of $N$ players to a mean field limit; start with decentralized control
- Cardaliaguet, Delarue, Lasry and Lions (2015) – Fully coupled HJBs for $N$ players; start with centralized info; always uniquely solvable due to special dynamics and costs; convergence to a master equation; no asymptotic solvability problem as we will face.

LQ Nash games are not always solvable on $[0, T]$. This is actually a useful feature for us to distinguish the direct and fixed point approaches.

By use of the diagram, we attempt to “classify” models. In the LQ setting, a model means a specification $(A, B, G, Q, Q_f, R, \Gamma, ...)$, not including the population size $N$. 
Nash game of $N$ players $A_i$, $1 \leq i \leq N$

Dynamics of player $A_i$:

$$dX_i(t) = (AX_i(t) + Bu_i(t) + GX^{(N)}(t))dt + DdW_i(t), \quad 1 \leq i \leq N,$$

where the state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^{n_1}$, $X^{(N)} = \frac{1}{N} \sum_{k=1}^{N} X_k$, $W_i \in \mathbb{R}^{n_2}$: $N$ independent Brownian motions (so, white noise).

Cost:

$$J_i = E \int_0^T \left( |X_i(t) - \Gamma X^{(N)}(t) - \eta|^2_Q + u_i^T(t)Ru_i(t) \right) dt$$
$$+ E|X_i(T) - \Gamma f X^{(N)}(T) - \eta_f|^2_{Q_f}.$$ 

The matrices (or vectors) $A, B, G, D, \Gamma, Q, R, \Gamma_f, Q_f, \eta, \eta_f$ have compatible dimensions, and $Q \geq 0, R > 0, Q_f \geq 0.$
Notation:

\[ X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{Nn}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
The $N$ player Nash game

The fixed point approach

The asymptotic solvability problem

Long time behavior

Dynamics, cost, and dynamic programming

ODE analytical tool

Dynamics: $dX(t) = \left(\hat{A}X(t) + \sum_{k=1}^{N} B_k u_k(t)\right) dt + \hat{D} dW(t)$. 

$V_i(t, x)$: value function of $A_i$.

HJB equation system:

$$0 = \frac{\partial V_i}{\partial t} + \frac{\partial^T V_i}{\partial x} (\hat{A}x - \sum_{k=1}^{N} \frac{1}{2} B_k R^{-1} B_k^T \frac{\partial V_k}{\partial x})$$

$$+ |x_i - \Gamma x^{(N)} - \eta|^2_Q$$

$$+ \frac{1}{4} \frac{\partial^T V_i}{\partial x} B_i R^{-1} B_i^T \frac{\partial V_i}{\partial x} + \frac{1}{2} \text{Tr} (\hat{D}^T (V_i)_{xx} \hat{D})$$.

Terminal condition: $V_i(T, x) = |x_i - \Gamma_f x^{(N)} - \eta_f|^2_{Q_f}$

Feedback Nash strategies:

$$u_i = -\frac{1}{2} R^{-1} B_i^T \frac{\partial V_i}{\partial x}, \quad 1 \leq i \leq N.$$
Write \( V_i(t, x) = x^T P_i(t) x + 2x^T S_i(t) + r_i(t) \). We derive

\[
\dot{P}_i(t) = - \left( P_i(t) \hat{A} + \hat{A}^T P_i(t) \right) + \\
\left( P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T P_k(t) \right) \\
+ \sum_{k=1}^{N} P_k(t) B_k R^{-1} B_k^T P_i(t) \\
- P_i(t) B_i R^{-1} B_i^T P_i(t) - Q_i, \quad 1 \leq i \leq N,
\]

\( P_i(T) = Q_{if}, \) (see e.g. Basar and Olsder'99)

\[
\dot{S}_i(t) = - \hat{A}^T S_i(t) - P_i(t) B_i R^{-1} B_i^T S_i(t) \\
+ P_i(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T S_k(t) \\
+ \sum_{k=1}^{N} P_k(t) B_k R^{-1} B_k^T S_i(t) + K_i^T Q\eta,
\]

\( S_i(T) = - K_{if}^T Q_f \eta_f, \)

\[
\dot{r}_i(t) = 2S_i^T(t) \sum_{k=1}^{N} B_k R^{-1} B_k^T S_k(t) \\
- S_i^T(t) B_i R^{-1} B_i^T S_i(t) - \eta^T Q\eta - \text{Tr} \left( \hat{D}^T P_i(t) \hat{D} \right),
\]

\( r_i(T) = \eta_f^T Q_f \eta_f. \)
For an $l \times m$ real matrix $Z = (z_{ij})_{i \leq l, j \leq m}$, denote the $l_1$-norm
$$\|Z\|_{l_1} = \sum_{i,j} |z_{ij}|.$$  

**Definition** The sequence of $N$ player Nash games with closed-loop perfect state information has **asymptotic solvability** if

- there exists $N_0$ such that for all $N \geq N_0$, $(P_1, \cdots, P_N)$ in the $N$ coupled Riccati ODEs has a solution on $[0, T]$ and,

$$\sup_{N \geq N_0} \sup_{1 \leq i \leq N, 0 \leq t \leq T} \|P_i(t)\|_{l_1} < \infty.$$  

The $l_1$ norm may be informally interpreted as the “total mass” of a large “pie” (the $Nn \times Nn$ matrix).
**Theorem** We assume that the Riccati ODE system has a solution \((P_1(t), \cdots, P_N(t))\) on \([0, T]\). Then the following holds.

i) \(P_1(t)\) has the representation \((N \times N \text{ blocks})\)

\[
P_1(t) = \begin{bmatrix}
\Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\
\Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\
\Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t)
\end{bmatrix},
\]

where \(\Pi_1\) and \(\Pi_3\) are \(n \times n\) symmetric matrices.

ii) For \(i > 1\), \(P_i(t) = J_1^T P_1(t) J_1\) (i.e., use simultaneous row and column exchange).
We may write an ODE system of the form

\[
\begin{bmatrix}
\dot{\Pi}_1 \\
\dot{\Pi}_2 \\
\dot{\Pi}_3
\end{bmatrix} = \Psi_N(\Pi_1, \Pi_2, \Pi_3).
\]

Main issue:

- As \( N \to \infty \), \( \Pi_2 \) and \( \Pi_3 \) will vanish.
- Directly taking \( N \to \infty \) in the ODE causes the loss of useful information; it’s overkill.

Strategy: **re-scaling**

- Define \( \Lambda_1^N = \Pi_1 \), \( \Lambda_2^N = N\Pi_2 \) and \( \Lambda_3^N = N^2\Pi_3 \).
- We obtain 3 equations for \( (\Lambda_1^N, \Lambda_2^N, \Lambda_3^N) \); see next page.
- Check the limit of the **new vector field**.
The $N$ player Nash game
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\[
\begin{aligned}
\dot{\Lambda}_1^N &= \Lambda_1^N M \Lambda_1^N - (\Lambda_1^N A + A^T \Lambda_1^N) - Q + g_1(1/N, \Lambda_1^N, \Lambda_2^N), \\
\Lambda_1^N(T) &= (I - \frac{\Gamma_f^T}{N}) Q_f (I - \frac{\Gamma_f}{N}),
\end{aligned}
\]

\[
\begin{aligned}
\dot{\Lambda}_2^N &= \Lambda_1^N M \Lambda_2^N + \Lambda_2^N M \Lambda_1^N + \Lambda_2^N M \Lambda_2^N \\
&- (\Lambda_1^N G + \Lambda_2^N (G + A) + A^T \Lambda_2^N) + Q \Gamma + g_2(1/N, \Lambda_2^N, \Lambda_3^N), \\
\Lambda_2^N(T) &= -(I - \frac{\Gamma_f^T}{N}) Q_f \Gamma_f,
\end{aligned}
\]

\[
\begin{aligned}
\dot{\Lambda}_3^N &= (\Lambda_2^N)^T M \Lambda_2^N + \Lambda_3^N M \Lambda_1^N + \Lambda_1^N M \Lambda_3^N + \Lambda_3^N M \Lambda_2^N + (\Lambda_2^N)^T M \Lambda_3^N \\
&- ((\Lambda_2^N)^T G + G^T \Lambda_2^N + \Lambda_3^N G + G^T \Lambda_3 + \Lambda_3^N A + A^T \Lambda_3^N) \\
&- \Gamma^T Q \Gamma + g_3(1/N, \Lambda_2^N, \Lambda_3^N), \\
\Lambda_3^N(T) &= \Gamma_f^T Q_f \Gamma_f.
\end{aligned}
\]

$g_1, g_2, g_3$ are “small” error terms.

Taking $N \to \infty$ leads to the construction $\rightarrow$
Let $M = BR^{-1}B^T$.

The symmetric Riccati ODEs (always having a solution):

$$\begin{cases}
\dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\
\Lambda_1(T) = Q_f,
\end{cases}$$

The non-symmetric Riccati ODE:

$$\begin{cases}
\dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\
\quad - (\Lambda_1 G + \Lambda_2 (A + G) + A^T \Lambda_2) + Q \Gamma, \\
\Lambda_2(T) = -Q_f \Gamma_f.
\end{cases}$$

Finally,

$$\begin{cases}
\dot{\Lambda}_3 = \Lambda_2^T M \Lambda_2 + \Lambda_3 M \Lambda_1 + \Lambda_1 M \Lambda_3 + \Lambda_3 M \Lambda_2 + \Lambda_2^T M \Lambda_3 \\
\quad - \left( \Lambda_2^T G + G^T \Lambda_2 + \Lambda_3 (A + G) + (A^T + G^T) \Lambda_3 \right) - \Gamma^T Q \Gamma, \\
\Lambda_3(T) = \Gamma_f^T Q_f \Gamma_f.
\end{cases}$$

If $\Lambda_2$ exists on $[0, T]$, so does $\Lambda_3$. The second equation is crucial!
Theorem The sequence of $N$ player Nash games, $N \geq 2$, has asymptotic solvability if and only if $\Lambda_2$ has a unique solution on $[0, T]$.

Proof. View the ODE of $(\Lambda_1^N, \Lambda_2^N, \Lambda_3^N)$ as a slightly perturbed version of the ODE of $(\Lambda_1, \Lambda_2, \Lambda_3)$; existence in the latter is determined by that of $\Lambda_2$.

Theorem If $\Lambda_2$ has a solution on $[0, T]$, then

$$\sup_{0 \leq t \leq T} (|\Pi_1 - \Lambda_1| + |N\Pi_2 - \Lambda_2| + |N^2\Pi_3 - \Lambda_3|) = O(1/N).$$

Recall:

$$P_1(t) = \begin{bmatrix} \Pi_{1}(t) & \Pi_{2}(t) & \Pi_{2}(t) & \ldots & \Pi_{2}(t) \\ \Pi_{2}^T(t) & \Pi_{3}(t) & \Pi_{3}(t) & \ldots & \Pi_{3}(t) \\ \Pi_{2}^T(t) & \Pi_{3}(t) & \Pi_{3}(t) & \ldots & \Pi_{3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_{2}^T(t) & \Pi_{3}(t) & \Pi_{3}(t) & \ldots & \Pi_{3}(t) \end{bmatrix},$$
Consider

\[ \dot{x} = f(t, x), \quad x(0) = z \in \mathbb{R}^K, \]

\[ \dot{y} = f(t, y) + g(\epsilon, t, y), \quad y(0) = z_\epsilon \in \mathbb{R}^K, \quad 0 < \epsilon \leq 1. \]

Let \( \phi(t, x) = f(t, x) \), or \( f(t, x) + g(\epsilon, t, x) \).

A1) \( \sup_{\epsilon, 0 \leq t \leq T} |f(t, 0)| + |g(\epsilon, t, 0)| \leq C_1. \)

A2) \( \phi(\cdot, x) \) is Lebesgue measurable for each fixed \( x \in \mathbb{R}^K \).

A3) For each \( t \in [0, T] \), \( \phi(t, x) : \mathbb{R}^K \to \mathbb{R}^K \) is locally Lipschitz in \( x \), uniformly with respect to \( (t, \epsilon) \), i.e., for any fixed \( r > 0 \), and \( x, y \in B_r(0) \) which is the open ball of radius \( r \) centering \( 0 \),

\[ |\phi(t, x) - \phi(t, y)| \leq \text{Lip}(r)|x - y|, \]

where \( \text{Lip}(r) \) depends only on \( r \), not on \( \epsilon \in (0, 1] \), \( t \in [0, T] \).

A4) For each fixed \( r > 0 \),

\[ \lim_{\epsilon \to 0} \sup_{0 \leq t \leq T, y \in B_r(0)} |g(\epsilon, t, y)| = 0, \quad \lim_{\epsilon \to 0} |z_\epsilon - z| = 0. \]
\[ \dot{x} = f(t, x), \quad x(0) = z \in \mathbb{R}^K, \tag{1.1} \]

\[ \dot{y} = f(t, y) + g(\epsilon, t, y), \quad y(0) = z_{\epsilon} \in \mathbb{R}^K. \tag{1.2} \]

If the solutions to (1.1) and (1.2), denoted by \( x^z(t) \) and \( y^\epsilon(t) \), exist on \([0, T]\), they are unique by the local Lipschitz condition; in this case denote \[ \delta_{\epsilon} = \int_0^T |g(\epsilon, \tau, x^z(\tau))| d\tau, \] which converges to 0 as \( \epsilon \to 0 \) due to A4).

**Theorem** i) If (1.1) has a solution \( x^z(t) \) on \([0, T]\), then there exists \( 0 < \bar{\epsilon} \leq 1 \) such that for all \( 0 < \epsilon \leq \bar{\epsilon} \), (1.2) has a solution \( y^\epsilon(t) \) on \([0, T]\) and

\[ \sup_{0 \leq t \leq T} |y^\epsilon(t) - x^z(t)| = O(|z_{\epsilon} - z| + \delta_{\epsilon}). \]

ii) Suppose there exists a sequence \( \{\epsilon_i, i \geq 1\} \) where \( 0 < \epsilon_i \leq 1 \) and \( \lim_{i \to \infty} \epsilon_i = 0 \) such that (1.2) with \( \epsilon = \epsilon_i \) has a solution \( y^{\epsilon_i}(t) \) on \([0, T]\) and \[ \sup_{i \geq 1, 0 \leq t \leq T} |y^{\epsilon_i}(t)| \leq C_2 \] for some constant \( C_2 \). Then (1.1) has a solution on \([0, T]\).
The fixed point approach

- Take a certain abstraction/approximation of the model from the beginning
- Then a typical agent places itself in a macroscopic environment of decision-making
Step 1. Assume $\overline{X} \in C([0, T], \mathbb{R}^n)$ were given to approximate $X^{(N)}$ in the $N$ player game and consider the optimal control problem

$$dX_i^\infty(t) = (AX_i^\infty(t) + Bu_i(t) + G\overline{X}(t))dt + DdW_i, \quad X_i^\infty(0) = X_i(0)$$

$$\bar{J}_i(u_i) = E \int_0^T (|X_i^\infty - \Gamma\overline{X} - \eta|^2_Q + u_i^TRu_i)dt$$

$$+ E|X_i^\infty(T) - \Gamma_f\overline{X}(T) - \eta_f|^2_{Q_f},$$

**Optimal control law:** \[ \hat{u}_i = -R^{-1}B^T(\Lambda_1X_i^\infty(t) + s(t)), \]

where Riccati ODE solution $\Lambda_1$ is the same as in the AS problem,

$$\begin{cases} 
\dot{s}(t) = -(A^T - \Lambda_1M)s(t) - \Lambda_1G\overline{X}(t) + Q(\Gamma\overline{X}(t) + \eta), \\
s(T) = -Q_f(\Gamma_f\overline{X}(T) + \eta_f). 
\end{cases}$$
Step 2. By the standard **consistency requirement** in MFG theory, set

$$\overline{X}(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} EX_i^\infty(t)$$

for $t \in [0, T]$, which amounts to specifying $\overline{X}$ as a **fixed point**.

Combining the ODEs of $s$ and the resulting $\overline{X}$ gives the MFG solution equation system

$$\begin{aligned}
\frac{d\overline{X}}{dt} &= (A - M \Lambda_1 + G)\overline{X} - Ms, \\
\dot{s} &= -(A^T - \Lambda_1 M)s - \Lambda_1 G\overline{X} + Q(\Gamma \overline{X} + \eta),
\end{aligned} \quad (2.1)$$

where $\overline{X}(0) = x_0$ and $s(T) = -Q_f(\Gamma_f \overline{X}(T) + \eta_f)$.

This is a two point boundary value (**TPBV**) problem.
Relation of the two approaches –

**Theorem** Asymptotic solvability implies that the TPBV problem in the fixed point approach has a unique solution.
example of no asymptotic solvability Take the parameters $A = 0.2, B = G = Q = R = 1, \Gamma = 1.2, \Gamma_f = 0, Q_f = 0$ and $T = 3$.

Figure: $\Lambda_2$ has a maximal existence interval small than $[0, T]$
Non-uniqueness example Consider the system with

\[
A = -\frac{1}{4}, \quad G = \frac{4}{5}, \quad Q = \frac{1}{16}, \quad \Gamma = \frac{4}{3}, \quad \eta = \eta_f = 1.
\]

Further take

\[
\hat{T} = 33.587095, \quad \hat{x}_0 = -0.500426.
\]

- **No asymptotic solvability** since \( A_2 \) has the maximal existence interval \((0, \hat{T}]\) (smaller than \([0, \hat{T}]\)).
- However, the TPBV problem in the fixed point approach has an infinite number of solutions.
Long time behaviour

Corresponding to the standard Riccati ODE

\[
\begin{cases}
\dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\
\Lambda_1(T) = Q_f,
\end{cases}
\]

we introduce the ARE

\[
\Lambda_{1\infty} M \Lambda_{1\infty} - (\Lambda_{1\infty} A + A^T \Lambda_{1\infty}) - Q = 0.
\]

There exists a unique solution \( \Lambda_{1\infty} \geq 0 \) under the standard stabilizability and detectability condition which now we assume.
Long time behaviour

Corresponding to the non-symmetric Riccati ODE

\[
\begin{aligned}
\dot{\Lambda}_2 &= \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\
&\quad - (\Lambda_1 G + \Lambda_2 (A + G) + A^T \Lambda_2) + Q \Gamma, \\
\Lambda_2(T) &= -Q_f \Gamma_f,
\end{aligned}
\]

we introduce the algebraic equation

\[
0 = \Lambda_{1\infty} M \Lambda_{2\infty} + \Lambda_{2\infty} M \Lambda_{1\infty} + \Lambda_{2\infty} M \Lambda_{2\infty} \\
&\quad - (\Lambda_{1\infty} G + \Lambda_{2\infty} (A + G) + A^T \Lambda_{2\infty}) + Q \Gamma,
\]

which is a non-symmetric algebraic Riccati equation (NARE).

Main issue now: If there is a real matrix solution, there may be multiple such solutions.

How to select a solution of interest?

Idea: impose a certain stability requirement; see e.g. Kremer and Stefan (2002)
Suppose $\Lambda_{2\infty} \in \mathbb{R}^{n \times n}$ is a solution to the NARE

$$0 = \Lambda_{1\infty} \mathcal{M} \Lambda_{2\infty} + \Lambda_{2\infty} \mathcal{M} \Lambda_{1\infty} + \Lambda_{2\infty} \mathcal{M} \Lambda_{2\infty} - (\Lambda_{1\infty} G + \Lambda_{2\infty} (A + G) + A^T \Lambda_{2\infty}) + Q \Gamma,$$

Denote

$$A_G = A - \mathcal{M}(\Lambda_{1\infty} + \Lambda_{2\infty}) + G,$$

$$A_M = A - \mathcal{M}(\Lambda_{1\infty} + \Lambda_{2\infty}^T).$$

**Definition** $\Lambda_{2\infty} \in \mathbb{R}^{n \times n}$ is called a **stabilizing solution** of the NARE if it satisfies NARE and both $A_G$ and $A_M$ are Hurwitz.

Motivation of such a stability condition $\rightarrow$
The AS problem determines –
1) Mean field state dynamics:

\[
\frac{d\tilde{X}}{dt} = (A - M(\Lambda_1 + \Lambda_2) + G)\tilde{X} - M\chi_1(t),
\]

where \(\tilde{X}(0) = x_0\) and \(\chi_1(t)\) can be explicitly specified by an ODE.

i) At \((\Lambda_1, \Lambda_2)\), want the **forward dynamics** to have stability on \([0, \infty)\).

2) The Riccati ODE:

\[
\begin{aligned}
\dot{\Lambda}_2 &= \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\
&\quad - (\Lambda_1 G + \Lambda_2 (A + G) + A^T \Lambda_2) + Q \Gamma, \\
\Lambda_2(T) &= -Q_f \Gamma_f.
\end{aligned}
\]

ii) Linearizing the vector field at \((\Lambda_{1\infty}, \Lambda_{2\infty})\), want the **backward dynamics** to have stability

i) and ii) motivate the stability requirement in the definition.
Denote

\[
\mathcal{A}_\infty = \begin{bmatrix}
A - M \Lambda_{1\infty} + G & -M \\
Q \Gamma - \Lambda_{1\infty} G & -A^T + \Lambda_{1\infty} M
\end{bmatrix} \in \mathbb{R}^{2n \times 2n},
\]

which may be viewed as a steady state form of \( \mathcal{A}(t) \) (coefficient matrix in the TPBV problem in the fixed point approach).

\( H_g \) The eigenvalues of \( \mathcal{A}_\infty \) are strong \((n, n)\) c-splitting (i.e., \( n \) eigenvalue in OLHP, and \( n \) eigenvalues in ORHP) and the associated \( n \)-dimensional stable invariant subspace is a graph subspace (i.e., spanned by columns of a matrix of the form \( \begin{bmatrix} I_n \\ X \end{bmatrix} \)).

**Theorem**

- The NARE has a stabilizing solution \( \Lambda_{2\infty} \) if and only if \( H_g \) holds.
- If \( H_g \) holds, the NARE has a unique stabilizing solution.
example of a stabilizing solution to NARE  We take

\[
A = \begin{bmatrix} 1 & 1 \\ -0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \quad \eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

and \( G = Q = I_2, \ R = 1. \) Then NARE has a stabilizing solution

\[
\Lambda_{2\infty} = \begin{bmatrix} 16.238985 & 4.099679 \\ 4.132523 & 1.570208 \end{bmatrix}.
\]

In fact, the columns of the matrix

\[
\begin{bmatrix}
-0.167388 & -0.161703 \\
0.448957 & 0.742511 \\
-0.877636 & 0.418170 \\
0.013220 & 0.497657
\end{bmatrix}
\]

span the stable invariant subspace of \( \Lambda_{\infty} \) as a graph subspace. \( \Lambda_{\infty} \) has the eigenvalues

\[-1.022350 \pm 0.730733i, \quad 2.022350 \pm 0.707903i.\]
Summary:

- There are two fundamental approaches for MFGs.
- We formulate an asymptotic solvability problem as an instance of the direct approach.
- We examine their relation in the LQ case.
- The re-scaling method can be generalized to other LQ models; in progress.

Thank you!