Mean Field Games on Infinite Networks and the Graphon Mean Field Game Equations

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Program

- Major-Minor Agent Systems and MFG Equilibria
- LQG PO Major-Minor Agent MFG Theory
- Open Problems and Directions of Development
- Graphon Control Systems and Graphon Mean Field Games
Basic Formulation of Nonlinear Major-Minor MFG Systems

Problem Formulation:

- Notation: Subscript 0 for the major agent $\mathcal{A}_0$ and an integer valued subscript for minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$.
- The states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are $\mathbb{R}^n$ valued and denoted $z_0^N(t)$ and $z_i^N(t)$.

Dynamics of the Major and Minor Agents:

\[
\begin{align*}
    dz_0^N(t) &= \frac{1}{N} \sum_{j=1}^{N} f_0(t, z_0^N(t), u_0^N(t), z_j^N(t))dt \\
    &\quad + \frac{1}{N} \sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T, \\
    dz_i^N(t) &= \frac{1}{N} \sum_{j=1}^{N} f(t, z_i^N(t), z_0^N(t), u_i^N(t), z_j^N(t))dt \\
    &\quad + \frac{1}{N} \sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N.
\end{align*}
\]
MFG Nonlinear Major-Minor Agent Formulation

Cost Functions for Major and Minor Agents:

\[ J_0^N(u_0^N; u_{-0}^N) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] \right) dt, \]

\[ J_i^N(u_i^N; u_{-i}^N) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L[t, z_i^N(t), z_0^N(t), u_i^N(t), z_j^N(t)] \right) dt. \]

- The major agent has **non-negligible influence** on the mean field (mass) behaviour of the minor agents. (A consequence will be that the mean field is no longer a deterministic function of time.)

- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\): a complete filtered probability space

- \(\mathcal{F}_t^N := \sigma\{z_j(0), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}\).

- \(\mathcal{F}_t^{w_0} := \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}\).
Basic Formulation of Nonlinear MFG Systems

Controlled McKean-Vlasov Equations:

- Infinite population limit dynamics:
  \[ \frac{dx_t}{dt} = f[x_t, u_t, \mu_t]dt + \sigma dw_t \]
  \[ f[x, u, \mu_t] \triangleq \int_{\mathbb{R}} f(x, u, y) \mu_t(dy) \]

- Infinite population limit cost:
  \[ \inf_{u \in \mathcal{U}} J(u, \mu) \triangleq \inf_{u \in \mathcal{U}} \mathbb{E} \int_0^T L[x_t, u_t, \mu_t] dt \]
  where \( \mu_t(\cdot) = \text{measure of the population state distribution} \)
Information Patterns:

**Local to Agent i:** $\mathcal{F}_i \triangleq \sigma(x_i(\tau); \tau \leq t), \ 1 \leq i \leq N$

$\mathcal{U}_{loc,i}$: $\mathcal{F}_i$ adapted control + system parameters

**Global with respect to the Population:**

$\mathcal{F}^N \triangleq \sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)$

$\mathcal{U}$: $\mathcal{F}^N$ adapted control + system parameters

The Equilibria:

The set of controls $\mathcal{U}^a = \{u_i^a; u_i^a \text{ adapted to } \mathcal{U}_{loc,i}, 1 \leq i \leq N\}$ generates a **Nash Equilibrium** w.r.t. the costs $\{J_i; 1 \leq i \leq N\}$ if, for each $i$,

$$J_i(u_i^a, u_{-i}^a) = \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^a)$$
Saddle Point Nash Equilibrium

- Agent $y$ is a maximizer
- Agent $x$ is a minimizer
\( \epsilon \)-Nash Equilibrium

\( \epsilon \)-Nash Equilibria:

Given \( \epsilon > 0 \), the set of controls \( \mathcal{U}^0 = \{u_i^0; 1 \leq i \leq N\} \) generates an \( \epsilon \)-Nash Equilibrium w.r.t. the costs \( \{J_i; 1 \leq i \leq N\} \) if, for each \( i \),

\[
J_i(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^0) \leq J_i(u_i^0, u_{-i}^0)
\]
Fundamental Mean Field Game MV HJB-FPK Theory

- **Mean Field Game Pair (HMC, 2006, LL, 2006-07):**
  Assume the infinite population limits exist for the generic individual system dynamics equations, the generic individual cost functions, and the state distributions for the generic agent, then:
  (i) the generic agent best response is generated by an MV-HJB equation and
  (ii) the corresponding generic agent state distribution is generated by an MV-FPK equation, yielding:

  \[
  \text{[MF-HJB]} \quad - \frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x, u, \mu_t] \frac{\partial V}{\partial x} + L[x, u, \mu_t] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
  \]
  \[
  V(T, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}
  \]

  \[
  \text{[MF-FPK]} \quad \frac{\partial p(t, x)}{\partial t} = - \frac{\partial \{ f[x, u, \mu] p(t, x) \}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(t, x)}{\partial x^2}
  \]

  \[
  \text{[MF-MKV SDE] } \quad dx_t = f[x_t, \varphi(t, x|\mu_t), \mu_t] dt + \sigma dw_t
  \]

  \[
  \text{[MF-BR]} \quad u_t = \varphi(t, x|\mu_t), \quad (t, x) \in [0, T] \times \mathbb{R}
  \]
Theorem (Huang, Malhamé, PEC, CIS'06)

Subject to technical conditions:
(i) the MKV MFG Equations have a unique solution with the best response control generating a unique Nash equilibrium given by

\[ u^0_i = \varphi(t, x|u_t), \quad 1 \leq i \leq N. \]

Furthermore,
(ii) \( \forall \epsilon > 0 \ \exists N(\epsilon) \text{ s.t. } \forall N \geq N(\epsilon) \)

\[ J^N_i (u^0_i, u^0_{-i}) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J^N_i (u_i, u^0_{-i}) \leq J^N_i (u^0_i, u^0_{-i}), \]

where \( u_i \in \mathcal{U} \) is adapted to \( \mathcal{F}^N := \{ \sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N) \} \).
The Three Key Ideas of Mean Field Game Theory

Formulate Non-Cooperative Large Scale Systems Analysis in Terms of Infinite Population Stochastic Dynamic Nash Equilibria

The Three Key Aspects of Mean Field Game Theory:
Two Equilibria and One Approximation

- Equilibrium I: Nash - Equilibrium: Non-Cooperative Game Theoretic Equilibrium

- Equilibrium II: Dynamical McKean-Vlasov - Generic Agent Mean Field Equilibrium (Mean Field Regeneration)

- The Infinite to Approximate Finite Equilibrium

[ Infinite Population Control Strategies Yield Approximate Nash Equilibria for Large Finite Populations]
Program

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Recall that the fundamental observation that a Major Agent in the LQG MFG framework has the significant impact that the mean field becomes stochastic is due to:

Minyi Huang + Son Luu Nguyen (2010,2012)
Infinite Horizon Completely Observed MM MFG Problem Formulation (Huang 2010)

- **Dynamics: Completely Observed Finite Population:**
  - **Major Agent:**\[ dx_0 = [A_0 x_0 + B_0 u_0] dt + D_0 d w_0 \]
  - **Minor Agents:**\[ dx_i = [A(\theta_i) x_i + B(\theta_i) u_i + G x_0] dt + D d w_i, \quad i \in \mathbb{N} \]

- The individual infinite horizon cost for the major agent:
\[
J_0(u_0, u_{-0}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_0 - \Phi(x^N)\|_Q^2 + \|u_0\|_R^2 \right\} dt
\]

\[
\Phi(\cdot) = H_0 x^N + \eta_0 \quad x^N = (1/N) \sum_{i=1}^N x_i
\]

- The individual infinite horizon cost for a minor agent \(i, \ i \in \mathbb{N}:\)
\[
J_i(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_i - \Psi(x^N)\|_Q^2 + \|u_i\|_R^2 \right\} dt
\]

\[
\Psi(\cdot) = H_1 x_0 + H_2 x^N + \eta
\]
Major Agent and Minor Agents

- When it exists, the $L^2$ limit $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_K]$ of the states’ empirical means $x^N = [x^N_1, \ldots, x^N_K]$ constitutes the system mean field.

- Subject to time invariant local state plus mean field plus major agent state feedback control, $\bar{x}$ satisfies the mean field equations:

$$d\bar{x}_k = \sum_{j=1}^{K} \bar{A}_{k,j} \bar{x}_j dt + \bar{G}_k x_0 dt + \bar{m}_k dt, \quad 1 \leq k \leq K$$

i.e.,

$$d\bar{x}(t) = \bar{A}\bar{x}(t)dt + \bar{G}x_0(t)dt + \bar{m}(t)dt$$

where the quantities $\bar{A}$, $\bar{G}$, $\bar{m}$ are to be solved for in the tracking solution.
Major Agent and Minor Agents LQG - MFG

- **Major Agent's Extended State:**
  Major agent’s state extended by the mean field: \[
  \begin{bmatrix}
  x_0 \\
  \bar{x}
  \end{bmatrix}
  \]

- **Minor Agents' Extended States:**
  Minor agent’s state extended by major agent’s state and the mean field: \[
  \begin{bmatrix}
  x_i \\
  x_0 \\
  \bar{x}
  \end{bmatrix}
  \]

- **When MF plus \(x_0\) plus local state dependent controls are applied, the MF-dependent extended state closes the system equations into state equations.**
LQR Major and Minor Agents (Inf. Population)

- **Major Agent’s Dynamics (Infinite Population):**

\[
\begin{align*}
\begin{bmatrix}
\frac{dx_0}{dt} \\
\frac{d\bar{x}}{dt}
\end{bmatrix} &= \begin{bmatrix}
A_0 & 0_{nK \times n} \\
\bar{G} & \bar{A}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
\bar{x}
\end{bmatrix} dt \\
&+ \begin{bmatrix}
B_0 \\
0_{nK \times m}
\end{bmatrix} u_0 dt + \begin{bmatrix}
0_{n \times 1} \\
\bar{m}
\end{bmatrix} dt + \begin{bmatrix}
D_0 d\omega_0 \\
0_{nK \times 1}
\end{bmatrix}
\end{align*}
\]

\[
A_0 = \begin{bmatrix}
A_0 & 0_{nK \times n} \\
\bar{G} & \bar{A}
\end{bmatrix} \quad B_0 = \begin{bmatrix}
B_0 \\
0_{nK \times m}
\end{bmatrix}
\]

\[
M_0 = \begin{bmatrix}
0_{n \times 1} \\
\bar{m}
\end{bmatrix} \quad Q_0^\pi = \begin{bmatrix}
Q_0 \\
-H_0^\pi^T Q_0 & -Q_0 H_0^\pi
\end{bmatrix}
\]

\[
\bar{\eta}_0 = [I_{n \times n}, -H_0^\pi]^T Q_0 \eta_0 \quad H_0^\pi = \pi \otimes H_0 \triangleq [\pi_1 H_0, \pi_2 H_0, \ldots, \pi_K H_0]
\]
**Minor Agents' Dynamics (Infinite Population):**

\[
\begin{bmatrix}
\frac{dx_i}{dt} \\
\frac{dx_0}{dt} \\
\frac{d\bar{x}}{dt}
\end{bmatrix}
= \begin{bmatrix}
A_k & \begin{bmatrix} G & 0_{n \times nK} \end{bmatrix} \\
0_{(nK+n) \times n} & \tilde{A}_0
\end{bmatrix}
\begin{bmatrix}
x_i \\
x_0 \\
\bar{x}
\end{bmatrix}
dt
+ \begin{bmatrix}
B_k \\
0_{(nK+n) \times m}
\end{bmatrix}
u_i dt
+ \begin{bmatrix}
0_{n \times 1} \\
M_0
\end{bmatrix} dt
+ \begin{bmatrix}
0_{n \times m} \\
B_0 \\
0_{nK \times m}
\end{bmatrix} u_0 dt
+ \begin{bmatrix}
D d\omega_i \\
D_0 d\omega_0 \\
0_{nK \times 1}
\end{bmatrix}
\]

\[
\tilde{A}_k = \begin{bmatrix}
A_k & \begin{bmatrix} G & 0_{n \times nK} \end{bmatrix} \\
0_{(nK+n) \times n} & \tilde{A}_0 - \tilde{B}_0 R_0^{-1} \tilde{B}_0^T \Pi_0
\end{bmatrix}
\]

\[
\tilde{B}_k = \begin{bmatrix}
B_k \\
0_{(nK+n) \times m}
\end{bmatrix}
\]

\[
\tilde{\eta} = \begin{bmatrix}
I_{n \times n}, -H, -H_2^\pi
\end{bmatrix}^T Q \eta
\]

\[
H_2^\pi = \pi \otimes H_2
\]
The individual cost for the major agent:

\[ J_0^\infty(u_0, u_{-0}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_0 - \Phi(\bar{x})\|^2_{Q_0} + \|u_0\|^2_{R_0} \right\} dt \]

\[ \Phi(\cdot) = H_0^\pi \bar{x} + \eta_0 \]

The individual cost for a minor agent \( i, i \in \mathbb{N} \):

\[ J_i^\infty(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_i - \Psi(\bar{x})\|^2_{Q} + \|u_i\|^2_{R} \right\} dt \]

\[ \Psi(\cdot) = H_1 x_0 + H_2^\pi \bar{x} + \eta \]
LQR MM MFG Feedback Control (Infinite Population)

- **Major Agent Tracking Problem Solution:**

\[
\rho \Pi_0 = \Pi_0 A_0 + A_0^T \Pi_0 - \Pi_0 B_0 R_0^{-1} B_0^T \Pi_0 + Q_0^\pi
\]

\[
\rho s_0^* = \frac{ds_0^*}{dt} + (A_0 - B_0 R_0^{-1} B_0^T \Pi_0)^T s_0^* + \Pi_0 M_0 - \bar{\eta}_0
\]

\[
u_0^o = -R_0^{-1} B_0^T [\Pi_0 (x_0^T, \bar{x}_T)^T + s_0^*]
\]

- **Minor Agent Tracking Problem Solution:**

\[
\rho \Pi_k = \Pi_k A_k + A_k^T \Pi_k - \Pi_k B_k R_k^{-1} B_k^T \Pi_k + Q
\]

\[
\rho s_k^* = \frac{ds_k^*}{dt} + (A_k - B_k R_k^{-1} B_k^T \Pi_k)^T s_k^* + \Pi_k M - \bar{\eta}
\]

\[
u_i^o = -R_k^{-1} B_k^T [\Pi_k (x_i^T, x_0^T, \bar{x}_T)^T + s_k^*]
\]
Major and Minor Agents: MF Equilibrium: Subject to H1-H4 the MF equations generate a set of stochastic control laws $\mathcal{U}_{MF}^N \triangleq \{u_i^o; 0 \leq i \leq N\}$, $1 \leq N < \infty$, such that

(i) All agent systems $S(A_i)$, $0 \leq i \leq N$, are second order stable.

(ii) $\{\mathcal{U}_{MF}^N; 1 \leq N < \infty\}$ yields an $\epsilon$-Nash equilibrium for all $\epsilon$, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$

$$J_i^N(u_i^o, u_{-i}^o) - \epsilon \leq \inf_{u_i \in \mathcal{U}_g} J_i^N(u_i, u_{-i}^o) \leq J_i^N(u_i^o, u_{-i}^o).$$
Simulation

100 minor agents and a single major agent.

\[
A \triangleq \begin{bmatrix} -0.05 & -2 \\ 1 & 0 \end{bmatrix}, \quad A_0 \triangleq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B, B_0 \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[t_{final} = 30s, \ \Delta t = 0.025s, \ \sigma_w = 0.002, \ \sigma_v = 0.05, \ \rho = 0.01, \ \eta = [0.25, 0.25]^T\]
\[\eta_0 = [0.25, 0.25]^T, \ Q = I_{2\times2}, \ Q_0 = I_{2\times2}, \ R = 1, \ R_0 = 1, \ H = 0.6 \times I_{2\times2}\]
\[H_0 = 0.6 \times I_{2\times2}, \ \hat{H} = 0.6 \times I_{2\times2}, \ G = 0_{2\times2}\]
Basic Formulation of Nonlinear Major-Minor MFG Systems

Problem Formulation:

- Notation: Subscript 0 for the major agent $A_0$ and an integer valued subscript for minor agents $\{A_i : 1 \leq i \leq N\}$.
- The states of $A_0$ and $A_i$ are $\mathbb{R}^n$ valued and denoted $z_0^N(t)$ and $z_i^N(t)$.

Dynamics of the Major and Minor Agents:

\[
\begin{align*}
\frac{dz_0^N(t)}{dt} &= \frac{1}{N} \sum_{j=1}^{N} f_0(t, z_0^N(t), u_0^N(t), z_j^N(t))dt \\
&\quad + \frac{1}{N} \sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,
\end{align*}
\]

\[
\begin{align*}
\frac{dz_i^N(t)}{dt} &= \frac{1}{N} \sum_{j=1}^{N} f_i(t, z_i^N(t), z_0^N(t), u_i^N(t), z_j^N(t))dt \\
&\quad + \frac{1}{N} \sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N.
\end{align*}
\]
Cost Functions for Major and Minor Agents:

\[
J_N^0 (u_0^N ; u_{-0}^N) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L_0 [t, z_0^N (t), u_0^N (t), z_j^N (t)] \right) dt,
\]

\[
J_i^N (u_i^N ; u_{-i}^N) := E \int_0^T \left( \frac{1}{N} \sum_{j=1}^N L [t, z_i^N (t), z_0^N (t), u_i^N (t), z_j^N (t)] \right) dt.
\]

- The major agent has non-negligible influence on the mean field (mass) behaviour of the minor agents: A consequence is that the mean field is no longer a deterministic function of time.

- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\): a complete filtered probability space

- \(\mathcal{F}_t^N := \sigma \{z_j (s), w_j (s) : 0 \leq j \leq N, 0 \leq s \leq t\}\).

- \(\mathcal{F}_t^{w_0} := \sigma \{z_0(0), w_0(s) : 0 \leq s \leq t\}\).
Major-Minor Agents’ Non-Standard SOCPs

**Major Agent’s SOCP for an Infinite Population:**

\[
    dz_0(t) = f_0[t, z_0(t), u_0(t), \mu_t(\omega_0)]dt + \sigma_0[t, z_0(t), \mu_t(\omega_0)]dw_0(t)
\]

\[
    \inf_{u_0 \in U_0} J_0(u_0) := \inf_{u_0 \in U_0} \mathbb{E} \int_0^T L[t, z_0(t), u_0(t), \mu_t(\omega_0)]dt.
\]

\[
    U_0 := \{u(\cdot) \in U_0 : u \text{ is adapted to } \mathcal{F}_t^{w_0} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty\}
\]

**Generic Minor Agent’s SOCP for an Infinite Population:**

\[
    dz_i(t) = f[t, z_i(t), u(t), \mu_t(\omega_0)]dt + \sigma[t, z_i(t), \mu_t(\omega_0)]dw_i(t)
\]

\[
    \inf_{u \in U} J_i(u) := \inf_{u \in U} \mathbb{E} \int_0^T L[t, z_i(t), z_0(t, \omega_0), u(t), \mu_t(\omega_0)]dt.
\]

\[
    U := \{u(\cdot) \in U : u \text{ is adapted to } \mathcal{F}_t^{w_0, w_i} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty\}
\]

\[
    \mu_t(\omega_0) := \mathcal{L}(z_i(t)|\mathcal{F}_t^{w_0}) \quad \omega_0\text{-dependent Mean Field (Distribution)}
\]

- SOCP with random parameters: SHJB theory by Peng, SICON'92.
Summary of the Major Agent’s Stochastic MFG (SMFG) System:

**M-SHJB**  

\[-d\phi_0(t, \omega_0, x) = \left[ \inf_{u \in U_0} H_0[t, \omega_0, x, u, D_x \phi_0(t, \omega_0, x)] \right.\]

\[+ \langle \sigma_0[t, x, \mu_t(\omega_0)], D_x \psi_0(t, \omega_0, x) \rangle + \frac{1}{2} \text{Tr}(a_0[t, \omega_0, x]D_{xx}^2 \phi_0(t, \omega_0, x)) \bigg] dt\]

\[-\psi_0^T(t, \omega_0, x)dw_0(t, \omega_0), \quad \phi_0(T, x) = 0\]

**M-SBR**  

\[u^0_0(t, \omega_0, x) = \arg \inf_{u \in U_0} H_0[t, \omega_0, x, u, D_x \phi_0(t, \omega_0, x)]\]

**M-SMV**  

\[dz_0^0(t, \omega_0) = f_0[t, z_0^0(t, \omega_0), u_0^0(t, \omega_0, z_0^0), \mu_t(\omega_0)]dt\]

\[+ \sigma_0[t, z_0^0(t, \omega_0), \mu_t(\omega_0)]dw_0(t, \omega_0)\]

where \(a_0[t, \omega_0, x] := \sigma_0[t, x, \mu_t(\omega_0)]\sigma_0^T[t, x, \mu_t(\omega_0)]\), and Hamiltonian \(H_0\) is

\[H_0[t, \omega_0, x, u, p] := \langle f_0[t, x, u, \mu_t(\omega_0)], p \rangle + L_0[t, x, u, \mu_t(\omega_0)].\]
Summary of the Minor Agents’ SMFG System:

**m-SHJB**  
\[-d\phi(t, \omega_0, x) = \left[ \inf_{u \in U} H[t, \omega_0, x, u, D_x \phi(t, \omega_0, x)] \right] \]
\[+ \frac{1}{2} \text{Tr} \left( a[t, \omega_0, x] D_{xx}^2 \phi(t, \omega_0, x) \right) \]
\[dt - \psi^T(t, \omega_0, x)dw_0(t, \omega_0), \quad \phi(T, x) = 0 \]

**m-SBR**  
\[u^o(t, \omega_0, x) = \arg \inf_{u \in U} H[t, \omega_0, x, u, D_x \phi(t, \omega_0, x)] \]

**m-SMV**  
\[dz^o(t, \omega_0) = f[t, z^o(t, \omega_0), u(t, \omega_0, z^o), \mu_t(\omega_0)]dt \]
\[+ \sigma[t, z^o(t, \omega_0), \mu_t(\omega_0)]dw(t) \]

where \(a[t, \omega_0, x] := \sigma[t, x, \mu_t(\omega_0)]\sigma^T[t, x, \mu_t(\omega_0)]\), Hamiltonian \(H\) is
\[H[t, \omega_0, x, p] := \langle f[t, x, u, \mu_t(\omega_0)], p \rangle + L[t, x, u, z_0(t, \omega_0), \mu_t(\omega_0)].\]

- Backward in time SDEs and solutions consist of a pair.
Solution Summary for the Major-Minor Agents’ SOCPs

Each infinite population SOCP gives a Mean Field Triple:

- Major SHJB, Major BR, Major SMV
- minor SHJB, minor BR, minor SMV

The functional dependence loop initiated with a nominal measure $\mu_t(\omega_0)$:

$$
\begin{align*}
\mu(\cdot)(\omega_0) \quad &\xrightarrow{\text{M-SHJB}}\quad (\phi_0(\cdot, \omega_0, x), \psi_0(\cdot, \omega_0, x)) \quad &\xrightarrow{\text{M-SBR}}\quad u_0(\cdot, \omega_0, x) \\
\uparrow \text{m-SMV} \quad &\quad \downarrow \text{M-SMV} \\
u^o(\cdot, \omega_0, x) \quad &\xleftarrow{\text{m-SBR}}\quad (\phi(\cdot, \omega_0, x), \psi(\cdot, \omega_0, x)) \quad &\xleftarrow{\text{m-SHJB}}\quad z_0^o(\cdot, \omega_0)
\end{align*}
$$

Theorem (Paraphrase) (Nourian-PEC SICOPT’13) Subject to the given conditions, a unique solution exists via fixed point argument in the space of random probability measure and the $\epsilon$-Nash property holds.
# Optimal Execution Problems in Finance

## Major trader dynamics

\[
\begin{align*}
    dQ_0(t) &= \nu_0(t)dt + \sigma_Q^Q dw^Q_0, \\
    dv_0(t) &= u_0(t)dt, \\
    dF_0(t) &= (\lambda_0 \nu_0(t) + \frac{\lambda}{N} \sum_{i=1}^{N} \nu_i(t))dt + \sigma dw^F_0(t), \\
    dS_0(t) &= dF_0(t) + a_0 dv_0(t), \\
    dZ_0(t) &= -S_0(t)dQ_0(t)
\end{align*}
\]

- \(Q\): inventory,
- \(\nu\): trading rate,
- \(u\): trading acceleration,
- \(\sigma\): positive scalar,
- \(F\): fundamental asset price,

## Minor (liquidator/acquirer) trader dynamics

\[
\begin{align*}
    dQ_i(t) &= \nu_i(t)dt + \sigma_i^Q dw^Q_i, \\
    d\nu_i(t) &= u_i(t)dt, \\
    dF_i(t) &= (\lambda_0 \nu_0(t) + \frac{\lambda}{N} \sum_{i=1}^{N} \nu_i(t))dt + \sigma dw^F_i(t), \\
    dS_i(t) &= dF_i(t) + a d\nu_i(t), \\
    dZ_i(t) &= -S_i(t)dQ_i(t)
\end{align*}
\]

- \(\lambda\): permanent impact,
- \(\sigma\): volatility,
- \(S\): execution price,
- \(a\): temporary impact,
- \(Z\): cash process

- \(w^Q\): Wiener processes: model (i) noise in the information the major trader collects on its inventory from branches (brokers) in different locations, and (ii) the HFT's information noise,
- \(w^F\): Wiener processes which model noise = (uninformed) traders in the market. Time differences between agents in getting data from the limit order book makes the Wiener processes independent.
Optimal Execution Problems in Finance

**Major (Liquidator) Trader Cost Function: Behavioural**

\[
J_0^N(u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T)(F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right.
\]
\[
\left. \int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right]
\]
Optimal Execution Problems in Finance

Major (Liquidator) Trader Cost Function: Behavioural

final cash

\[
J^N_0(u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T) (F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \\
\int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right]
\]
Optimal Execution Problems in Finance

Major (Liquidator) Trader Cost Function: Behavioural

\[ J_0^N(\mu_0, \mu_0) = \mathbb{E}\left[ -Z_0(T) - Q_0(T)(F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right. \]

\[ \left. \int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right] \]
Major (Liquidator) Trader Cost Function: Behavioural

$$J_0^N (u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T) (F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right. \\
\left. \int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right]$$
Major (Liquidator) Trader Cost Function: Behavioural

$$J_0^N(u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T) \left( F_0(T) + \alpha Q_0(T) \right) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right.$$  

$$\int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right]$$
Optimal Execution Problems in Finance

Major (Liquidator) Trader Cost Function: Behavioural

$$J_0^N(u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T)(F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) \right. $$

$$\left. + \int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right]$$

final cash

remaining inventory penalty

large execution price avoidance

inventory cost

large trading rate avoidance

large trading acceleration avoidance

\[ \alpha, \epsilon, \beta, \phi, \delta, \theta, R_0: \text{positive scalars} \]
Major (Liquidator) Trader Cost Function: Behavioural

\[ J_0^N (u_0, u_{-0}) = \mathbb{E} \left[ -Z_0(T) - Q_0(T) (F_0(T) + \alpha Q_0(T)) + \epsilon S_0^2(T) + \beta \nu_0^2(T) + \int_0^T \left( \phi Q_0^2(s) + \delta S_0^2(s) + \theta \nu_0^2(s) + R_0 u_0^2(s) \right) ds \right] \]

\( \alpha, \epsilon, \beta, \phi, \delta, \theta, R_0 \): positive scalars
Optimal Execution Problems in Finance

**HFT (Acquirer) Cost Function: Behavioural**

- large execution price avoidance
- unacquired inventory penalty
- final cash

\[ J_i^N(u_i, u_{-i}) = \mathbb{E} \left[ Z_i(T) + (N - Q_i(T)) \left( F_i(T) + \psi(N - Q_i(T)) \right) + \xi S_i^2(T) + \mu \nu_i^2(T) \right] \]

- large trading rate avoidance
- large trading acceleration avoidance

\[ \psi, \xi, \mu, \gamma, \rho, R: \text{ positive scalars} \]

No inventory penalty! HFT does not retain inventory but trades assets rapidly to obtain even very small profit in each trade.
HFT (Liquidator) Cost Function: Targeting the Market Trading Speed

$$J_i^N(u_i, u_{-i}) = \mathbb{E}\left[ -rZ_i(T) - pQ_i(T)(F_i(T) - \psi Q_i(T)) + \xi S_i^2(T) + \mu (\nu_i(T) - \rho \nu_i^N(T))^2 \right]$$

$$\int_0^T \left( \kappa Q_i^2(s) + \gamma S_i^2(s) + \phi (\nu_i(s) - \rho \nu_i^N(s))^2 + Ru_i^2(s) \right) ds$$

Optimal Execution Problems in Finance

large execution price avoidance
remaining inventory penalty
final cash

inventory cost
market trading rate tracking
large trading acceleration avoidance

$r, p, \psi, \xi, \mu, \kappa, \gamma, \phi$ and $R$: positive scalars, $0 \leq \rho \leq 1$

No inventory penalty! HFT does not retain inventory but trades assets rapidly to obtain even very small profit in each trade.
Optimal Execution Problem: Partially Observed Major-Minor MFG Theory

Notation

\[ x_0 = \begin{bmatrix} \nu_0 \\ Q_0 \\ S_0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} \bar{\nu} \\ \bar{Q} \\ \bar{S} \end{bmatrix} \]

Major Trader’s Observation Process

\[ dy_0(t) = \mathbb{H}_0 \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} dt + dv_0(t), \]

Estimated Terms Generated by Major Trader

\( \hat{x}_0 | \mathcal{F}^y_0 \): Major agent’s estimate of its own state
\( \bar{x} | \mathcal{F}^y_0 \): Major agent’s estimate of the mean field

Major Trader’s Control Action

\[ \hat{u}_0^* = -R_0^{-1} \mathbb{B}_0 [\Pi_0 (\hat{x}_0^T | \mathcal{F}^y_0, \bar{x}^T | \mathcal{F}^y_0)^T + s_0]. \]
Optimal Execution Problem: Partially Observed Major-Minor MFG Theory

**Notation**

\[ x_i = \begin{bmatrix} \nu_i & Q_i & S_i \end{bmatrix}^T \]

**Minor Trader’s Observation Process**

\[
dy_i(t) = \mathbb{H} \left[ x_i^T x_0^T \bar{x}^T \hat{x}^T_{0|\mathcal{F}^y_i} \bar{x}^T_{0|\mathcal{F}^y_0} \right]^T dt + dv_i(t)
\]

**Estimated Terms Generated by Minor Trader**

- \( x_i|\mathcal{F}^y_i \): Minor trader’s estimate of its own state
- \( \hat{x}_0|\mathcal{F}^y_i \): Minor trader’s estimate of major trader’s state
- \( \bar{x}|\mathcal{F}^y_i \): Minor trader’s estimate of the mean field
- \( (\hat{x}_0|\mathcal{F}^y_0)|\mathcal{F}^y_i \): Minor trader’s estimate of major trader’s estimate of its own state
- \( (\bar{x}|\mathcal{F}^y_0)|\mathcal{F}^y_i \): Minor trader’s estimate of major trader’s estimate of the mean field

**Minor Trader’s Control Action**

\[
\hat{u}_i^*(t) = -R^{-1}B^T \left[ \Pi \left( \hat{x}_i|\mathcal{F}^y_i, \hat{x}_0|\mathcal{F}^y_i, \bar{x}|\mathcal{F}^y_i, (\hat{x}_0|\mathcal{F}^y_0)|\mathcal{F}^y_i, (\bar{x}|\mathcal{F}^y_0)|\mathcal{F}^y_i \right)^T + s \right].
\]
Simulation

The Major Liquidator’s States and Its Estimates of its Own States

![Graphs showing trading rate, inventory, and execution price over time.](image.png)
Simulation

A Generic Minor Acquirer’s States and Its Estimates of its Own States

![Graphs showing simulation results]
Simulation

A Generic Minor Liquidator’s States and Its Estimates of its Own States

![Graphs showing trading rate, inventory, and execution price over time](image-url)
Simulation

Minor Liquidator’s Estimates of Major Liquidator’s Estimates

![Graphs showing estimates of major agent states over time.](image-url)
Next on the Program

- Open Problems and Directions of Development
- Graphon Control Systems and Graphon Mean Field Games
Open Problems

- **Finite Populations**: An MFG control oriented analysis of the application of the infinite population MFG theory to "middling" or "mezzo" finite populations - it’s the usual application after all! Difficult and technical error estimation/analysis.

- **Adaptation and Learning**: Agents need the (i) the dynamical parameters and (ii) the random process parameters. An initial "classical" formulation in terms of Stochastic Adaptive Control has already been developed (IEEE TAC 2013, Kizikale, PEC).

  Adaptation via Application of Machine Learning to MFG systems and vice versa.

- **MFG** should be a good vehicle to analyze **coalition formation in games**: the problem formulation is essentially open.

- **Systems on Networks**, including Flocking and Swarming.
Motivation for a Graphon Theory of Systems and Control

Complex networks are characterized by
- Large number of nodes (millions, even billions of nodes)
- Complex connections (dense) which are predominantly *local*
- Growth in size

The recently developed mathematical theory of graphons provides a methodology for analyzing arbitrarily complex networks.
Key feature: local nodes have intrinsic states that evolve due to interactions with other nodes.

- Power grids (loads, generators and energy storage units)
- Epidemic networks
- Brain networks
- Social networks (opinions) and Fish Schooling
- Networks of computational devices

They can be freely evolving, or locally controlled, and (or) globally controlled.

In this work we introduce a theory of control of dynamical systems on arbitrarily complex networks.
**Introduction to Graphons**  
Graphs, Adjacency Matrices and Pixel Pictures

How many 4-cycles must a graph with edge density at least \( \frac{1}{2} \) have?

So, suppose \( G \) has \( n \) vertices and at least \( n(n-1)/4 \) edges, half as many as are possible. Can you avoid having many 4-cycles? It is an interesting and worthwhile exercise to try to find as many as you can; start with trying to find at least one. It is not hard to see that there are \( n^4 \) 4-cycles (in fact, there are \( 3n^4 \) possible). The following result of Erdős tells us that there must be very many 4-cycles, in fact, on the order of \( n^4 \) of them.

Theorem (Erdős)  
For any graph \( G \), \( t(\cdot, G) \).

In particular, if \( t(\cdot, G) \), then \( t(\cdot, G) \).

In light of the theorem, it would be best to reformulate our problem as follows.

Minimize \( t(\cdot, G) \) over all finite graphs \( G \) satisfying \( t(\cdot, G) \).

It is beneficial at this point to draw an analogy with a problem familiar from elementary calculus.

Minimize \( x^3 - 6x \) over all real numbers \( x \) satisfying \( x \).

The minimum here is attained at \( x = \sqrt{2} \), which, though our polynomial has rational coefficients, is irrational. The best we can do in the rational numbers is find a sequence limiting to \( \sqrt{2} \) at which the polynomial achieves values approaching the minimum. Completing the rational numbers to the real numbers allows us to objectify the limit, which algebra then allows us to realize and work with as \( \sqrt{2} \).

It turns out that we are in an analogous situation with our graph problem. Erdős' theorem tells us that the minimum of \( t(\cdot, G) \) is greater than or equal to \( 1/16 \), and with a little extra work, it can be shown that that minimum is not achieved by any finite graph. There is, however, a sequence of finite graphs \( (R_n) \) with edge density at least \( 1/2 \) and 4-cycle density approaching \( 1/16 \). Indeed, for each \( n \geq 1 \), let \( R_n \) be an instance of a random graph on \( n \) vertices where the existence of each possible edge is decided independently with probability \( 1/2 \). By throwing those \( R_n \)'s away for which \( t(\cdot, R_n) < 1/2 \), the 4-cycle density in the remaining graphs almost surely limits to \( 1/16 \).

The situation is now primed for us to seek to, in pure analogy, complete the space of graphs, realize the limit of \( (R_n) \) as workable object, and understand the way in which that object achieves the minimum of \( 1/16 \) in our problem above.

**Graphons**

Let's speculate as to the possible limits of the graph sequence \( (R_n) \), where \( R_n \) is an instance of a random graph with edge probability \( 1/2 \). One real possibility is the Rado graph, the random graph with vertex set \( \mathbb{N} \) and edge probability \( 1/2 \). (I write "the" random graph since any two instances of such a graph are almost surely isomorphic.) This and many other possible limits are explored in [1] but are not examples of graphons.

Exploring an idea that at first sight is a bit more naive, consider the following three representations of a graph.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Adjacency Matrix</th>
<th>Pixel Picture</th>
</tr>
</thead>
</table>
| ![Graph](image) | \[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{pmatrix}
\] | ![Pixel Picture] |

The whole pixel picture is presented in a unit square \([0, 1] \times [0, 1]\), so the square elements have sides of length \( 1/N \), where \( N \) is the number of nodes.
Introduction to Graphons

Graph Sequence Converging to Graphon

Graphons: bounded symmetric Lebesgue measurable functions

\[ W : [0, 1]^2 \rightarrow [0, 1] \]

interpreted as weighted graphs on the vertex set \([0, 1]\).

Notations of Spaces

\[ G_{0}^{sp} := \{W : [0, 1]^2 \rightarrow [0, 1]\} \]
\[ G_{1}^{sp} := \{W : [0, 1]^2 \rightarrow [-1, 1]\} \]
\[ G_{R}^{sp} := \{W : [0, 1]^2 \rightarrow \mathbb{R}\} \]
The dynamics of the $i^{th}$ agent in the network

\[ \dot{x}_i^i = \frac{1}{N} \sum_{j=1}^{N} a_{ij} x_j^i + \frac{1}{N} \sum_{j=1}^{N} b_{ij} u_j^i \]

$x_i^i \in \mathbb{R}^1$: state  
$u_i^i \in \mathbb{R}^1$: control

Consider scalar case for simplicity
Compactness of graphon space ensures the limit exists.
Infinite dimensional linear system

\[ \dot{x}_t = Ax_t + Bu_t, \quad 0 \leq t \leq T \]
\[ x_0 \in L^2[0, 1], \quad A \in G_{1}^{SP}, B \in G_{AI} \]

\[ x_t \in L^2[0, 1] : \text{system state; } u_t \in L^2[0, 1] : \text{control input} \]

(H1) \begin{align*}
(i) & \quad A \text{ generates a strongly continuous semigroup } e^{tA} \text{ on } L^2[0, 1], \\
(ii) & \quad B \in \mathcal{L}(L^2[0, 1]; L^2[0, 1]),
\end{align*}

There exists a unique solution \( x \in C([0, T]; L^2[0, 1]) \) for any \( x_0 \in L^2[0, 1] \) and any \( u \in L^2([0, T]; L^2[0, 1]) \).

The convergence to \( B \) can be defined in \( C_S([0, T]; \mathcal{L}(L^2[0, 1])) \).
Methodology for Controlling Systems on Complex Networks

Finite Dim Network System 
\((A_N; B_N)\)

Infinite Dim Network System 
\((A^N_s; B^N_s)\)

Converge 
\(N \to \infty\)

Infinite Dim Limit System 
\((A; B)\)

Synthesis (Min-Energy and LQR)

Control Law \(u_N\) for \((A_N; B_N)\)

Approximate

Control Law \(u^N_s\) for \((A^N_s; B^N_s)\)

Control Law \(u\) for \((A; B)\)

Control Design Procedure for Network Systems via Graphon Limits
Minimum energy state to state control problem:

$$\min_u J(u)$$

s.t. Initial state $x_0 \rightarrow$ Target state $x_T$,

where the control energy is given by

$$J(u) := \int_0^T ||u_\tau||_2^2 d\tau = \int_0^T \int_0^1 u_\tau(\alpha)^2 d\alpha d\tau$$
Uniform Attachment Graphon: \[ U(x, y) = 1 - \max(x, y), \]
\[ x, y \in [0, 1]. \]

Weighted Graph Generated from \( U \), its Stepfunction and Graphon Limit
Minimum Energy Graphon Control

Example I

Uniform Attachment Graphon: $U(x, y) = 1 - \max(x, y)$, $x, y \in [0, 1]$.

$$\dot{x}_t = \frac{1}{N} A_N x_t + u_t, \quad x_t \in \mathbb{R}^N, u_t \in \mathbb{R}^N$$

Simulation

Minimum Energy Target State Control on Network with 50 Nodes
Graphon Linear Quadratic Regulation

For a graphon system \((A; B)\) pose the problem of minimizing

\[
J(u) = \int_0^T \left( \|Cx_\tau\|^2 + \|u_\tau\|^2 \right) d\tau + \langle P_0 x_T, x_T \rangle
\]

over all controls \(u \in L^2(0, T; L^2(0, 1))\) subject to the system model constraints in \((A; B)\). The assumptions for \(C\) and \(P_0\) are:

\[(H2) \quad \begin{cases} 
(iii) \quad P_0 \in \mathcal{L}(L^2[0, 1]) \text{ is hermitian and non-negative,} \\
(iv) \quad C \in \mathcal{L}(L^2[0, 1]; L^2[0, 1])
\end{cases}\]
Graphon Linear Quadratic Regulation

Let \( P \) solve the following Riccati equation:

\[
\dot{P} = A^T P + PA - PBB^T P + C^T C, \quad P(0) = P_0. \tag{1}
\]

Applying the result in (Bensoussan, 2007) and specializing the Hilbert space there to be \( L^2[0, 1] \) space, we have the following:

Theorem

Assume that \((H2)\) are verified. Then problem (1) has a unique (mild) solution \( P \in C_s([0, \infty); \Sigma^+(L^2[0, 1])) \).

The closed loop equation under the optimal control over \([0, T]\) is

\[
\dot{x}_t = Ax_t - BB^*P(T - t)x_t, \tag{2}
\]

\( t \in [0, T], \ x_0 \in L^2[0, 1]. \)
Graphon Linear Quadratic Regulation

Example II

Sinusoidal Graphon:

\[ U(x, y) = \cos(\pi(x - y)), \quad x, y \in [0, 1]. \]
The Graphon Mean Field Game Equations (i)

\[
[HJB](\alpha) \quad - \frac{\partial V_\alpha(t, x)}{\partial t} = \inf_{u \in U} \left\{ \tilde{f}[x, u, \mu_G; g_\alpha] \frac{\partial V_\alpha(t, x)}{\partial x} + \tilde{l}[x, u, \mu_G; g_\alpha] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V_\alpha(t, x)}{\partial x^2},
\]

\[
V_\alpha(T, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad \alpha \in [0, 1],
\]

\[
[FPK](\alpha) \quad \frac{\partial p_\alpha(t, x)}{\partial t} = - \frac{\partial}{\partial x} \left\{ \tilde{f}[x, u^0(x_\alpha, \mu_G; g_\alpha)p_\alpha(t, x)] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 p_\alpha(t, x)}{\partial x^2},
\]

\[
[BR](\alpha) \quad u^0(x_\alpha, \mu_G; g_\alpha) = \arg \inf_u H(x_\alpha, u, \mu_G; g_\alpha),
\]

\[
= \varphi(t, x_t | \mu_G; g_\alpha)
\]
The Graphon Mean Field Game Equations (ii)

The graphon local mean field $\mu_\alpha$, the corresponding set of all the local mean fields $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$, and the graphon function $g_\alpha = \{g(\alpha, \beta); 0 \leq \beta \leq 1\}$ are inter-related by the FPK and the defining integral relation

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] := \int_{[0,1]} \int_R f(x_\alpha, u_\alpha, x_\beta) g(\alpha, \beta) \mu_\beta(dx_\beta) d\beta$$

which gives the complete graphon mean field dynamics via the sum

$$\text{[GMFGD]}(\alpha) \quad \tilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0(x_\alpha, u_\alpha) + f[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$
**Fact**

Graphons can be continuous functions which are not differentiable everywhere, e.g.

\[
g(\alpha, \beta) = 1 - \max(\alpha, \beta),
\]

\[
\alpha, \beta \in [0, 1]
\]

**Important Special Case**

The simple standard MFG framework is retrieved when the agents’ dynamics and costs are uniform, and the network is totally connected with **uniform link weights**.

This gives \( \{g(\alpha, \beta) = 1; 0 \leq \alpha, \beta \leq 1\} \). In this case the FPK equations and the graphon dynamics integral equations have a solution where all the local graphon mean fields are equal, i.e.

\[
\mu_{t,\alpha} =: \mu_t, \text{ for all } \alpha \text{ hence giving the standard MFG model.}
\]
Class of Controlled Systems

To generate a theory of GMFG systems we constrain the set of systems under consideration to take the form:

$$\tilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] := f_0(x_\alpha, u_\alpha) + f[x_\alpha, u_\alpha, \mu_G; g_\alpha].$$
Graphon Mean Field Games: GMFG: System Hypotheses

(H1) \( U \) is a compact set.

(H2) \( f(x, u, y) \) and \( l(x, u, y) \) (\( f_0(x, u) \) and \( l_0(x, u) \), resp.) are continuous and bounded functions on \( \mathbb{R} \times U \times \mathbb{R} \) (\( \mathbb{R} \times U \), respect.), and are Lipschitz continuous in \( (x, y) \) (in \( x \), resp.) uniformly in \( u \).

(H3) For \( f_0, f \) and \( l_0, l \), their first and second derivatives w.r.t \( x \) are all uniformly continuous and bounded in \( \mathbb{R} \times U \times \mathbb{R} \) (or \( \mathbb{R} \times U \).

(H4) \( f(x, u, y) \) (\( f_0(x, u) \), resp.) is Lipschitz continuous in \( u \), uniformly with respect to \( (x, y) \) (to \( x \), resp.).

(H5) For any \( q \in \mathbb{R}, \alpha \in [0, 1] \) and any probability measure ensemble \( \mu_G \) satisfying M1), the set

\[
S(x, q) = \arg \min_u [q(\tilde{f}[x, u, \mu_G; g_\alpha]) + \tilde{l}[x, u, \mu_G; g_\alpha]]
\]

\[
= \arg \min_u [q(f_0(x, u) + f[x, u, \mu_G; g_\alpha])
\]

\[
+ \tilde{l}[x, u, \mu_G; g_\alpha]]
\]

is a singleton, and the resulting \( u \) as a function of \( (x, q) \), is Lipschitz continuous in \( (x, q) \), uniformly with respect to \( \mu_G \) and \( g_\alpha \).
(H6) The gain condition

- We introduce the regularity requirement

\[
\sup_{t,x,\alpha} |\phi_\alpha(t,x)|_{\mu_G} - \bar{\phi}_\alpha(t,x)|_{\bar{\mu}_G}| \leq c_1 D_T(m_G, \bar{m}_G). \tag{3}
\]

(note: this is verifiable for linear models). \(m_G\): distribution on path space \(C([0, T])\). \(\mu_G\): interpreted as marginals indexed by \([0, T]\) and \([0, 1]\) (vertices)

- We can further show

\[
D_T(m_\alpha^{\text{new}}, \bar{m}_\alpha^{\text{new}}) \leq c_2 \sup_{t,x} |\phi_\alpha(t,x)|_{\mu_G(\cdot)} - \bar{\phi}_\alpha(t,x)|_{\bar{\mu}_G(\cdot)}| \tag{4}
\]

for some constant \(c_2\)

- Now assume \(c_1 c_2 < 1\).
Theorem 1: Existence and Uniqueness of Solutions to the GMFG Equation Systems (PEC, Huang, 2017)

Subject to conditions $H(1) - H(6)$, there exists a unique solution to the graphon dynamical GMFG equations, which (i) gives the feedback control best response (BR) strategy $\varphi(t, x_t | \mu_G; g_\alpha)$ depending only upon the agent’s state and the graphon local mean fields (i.e. $(x_t, \mu_G; g_\alpha)$), and (ii) generates a Nash equilibrium.
Theorem 2: $\epsilon$-Nash Equilibria for GMFG System (PEC, Huang, 2018)

Let the conditions H(1) - H(6) hold, together with H(7) The graphon function $G = \{g(\alpha, \beta), 0 \leq \alpha, \beta \leq 1\}$ is continuous on the unit square.
Then the joint strategy \( \{u_0^i(t) = \varphi(t, x_t|\mu_G; g_\alpha) \) yields an $\epsilon$-Nash equilibrium for all $\epsilon$, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$.
Namely, $\forall \epsilon > 0 \ \exists N(\epsilon)$ s.t. $\forall N \geq N(\epsilon)$

\[
J^N_i(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in U} J^N_i(u_i, u_{-i}^0) \leq J^N_i(u_i^0, u_{-i}^0),
\]

where $u_i \in U$ is adapted to $F^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$. 
Graphon Mean Field Games: GMFG

Analysis in the Proof of $\epsilon$-Nash Equilibria for GMFG System
(PEC, Huang, 2018)

$$\left| (x^{N_g,N_i}(\tau) - x^{N_g,N_i}(0)) - (x^{\infty,\infty_i}(\tau) - x^{\infty,\infty_i}(0)) \right|$$

$$= \left| \int_0^\tau dt \left\{ \frac{1}{N_g} \sum_{j=1}^{N_g} g_{i,j} \left[ \frac{1}{N_j} \sum_{l=1}^{N_j} f(x_i, u_i, x^{[j]}_l) \right] - \int_0^1 \int_R f(x_i, u_i, x^\beta) g(\alpha, \beta) \mu(\cdot) \right\} \right|$$

$$\leq \left| \int_0^\tau dt \left\{ \frac{1}{N_g} \sum_{j=1}^{N_g} g_{i,j} \left[ \frac{1}{N_j} \sum_{l=1}^{N_j} f(x_i, u_i, x^{[j]}_l) \right] - \frac{1}{N_g} \sum_{j=1}^{N_g} g_{i,j} \left[ \int_R f(x_i, u_i, x^\beta) \mu(dx^\beta) \right] \right\} \right|$$

$$+ \left| \int_0^\tau dt \frac{1}{N_g} \sum_{j=1}^{N_g} g_{i,j} \left[ \int_R f(x_i, u_i, x^\beta) \mu(dx^\beta) \right] - \int_0^1 \int_R f(x_i, u_i, x^\beta) g(\alpha, \beta) \mu(\cdot) \right|$$